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THE CHIRAL SUPERSTRING SIEGEL FORM IN DEGREE TWO IS A LIFT

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Abstract. We prove that the Siegel modular form of D'Hoker and Phong that gives the chiral superstring measure in degree two is a lift. This gives a fast algorithm for computing its Fourier coefficients. We prove a general lifting from Jacobi cusp forms of half integral index $t/2$ over the theta group $\Gamma_1(1,2)$ to Siegel modular cusp forms over certain subgroups $\Gamma^{\mathrm{para}}(t;1,2)$ of paramodular groups. The theta group lift given here is a modification of the Gritsenko lift.

1. Introduction

We construct a lifting L from Jacobi cusp forms of index $t/2$ for the theta group $\Gamma_1(1,2)$ to Siegel modular forms on subgroups $\Gamma^{\mathrm{para}}(t;1,2)$ of the paramodular groups $\Gamma^{\mathrm{para}}(t)$:

$$L : J^{\mathrm{cusp}}_{k,t/2}(\Gamma_1(1,2)^J) \to M_k\left(\Gamma^{\mathrm{para}}(t;1,2)\right).$$

Our construction imitates the construction of the lift due to Gritsenko, Grit : $J_{k,m}(\mathrm{SL}_2(\mathbb{Z})^J) \to M_k(\Gamma^{\mathrm{para}}(m))$, which sends Jacobi forms of index m on $\mathrm{SL}_2(\mathbb{Z})$ to Siegel modular forms on the paramodular group $\Gamma^{\mathrm{para}}(m)$, see [9]. Although we proceed in greater generality, our main interest is the case where t is odd. In order to properly call L a lift, we should really discuss the L-series of the lifted Siegel modular form but here we content ourselves with giving the Fourier coefficients.

Theorem 1. *Let $t \in \mathbb{N}$ and $k \in \mathbb{Z}$. There is a monomorphism*

$$L : J^{\mathrm{cusp}}_{k,t/2}(\Gamma_1(1,2)^J) \to M_k\left(\Gamma^{\mathrm{para}}(t;1,2)\right)$$

such that if $\phi \in J^{\mathrm{cusp}}_{k,t/2}(\Gamma_1(1,2)^J)$ has the Fourier expansion

$$\phi(\tau,z) = \sum_{n,r\in\mathbb{Z}:\ tn-r^2>0,\ n>0} c(n,r)e(\tfrac{1}{2}n\tau + rz),$$

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for $\tau \in \mathcal{H}_1$ *and* $z \in \mathbb{C}$, *then* $L(\phi) \in M_k\left(\Gamma^{\mathrm{para}}(t;1,2)\right)$ *has the Fourier expansion*

$$L(\phi)(\Omega) = \sum_{\substack{T=\left(\begin{smallmatrix} n & r \\ r & m\end{smallmatrix}\right):\ t|m, \\ mn-r^2>0,\ n>0,\ m>0}} a(T)e(\tfrac{1}{2}\mathrm{tr}(T\Omega)),$$

for $\Omega \in \mathcal{H}_2$, *where*

$$a\left(\begin{pmatrix} n & r \\ r & m \end{pmatrix}\right) = (-1)^{(m/t+1)(n+1)} \sum_{\substack{a|(n,r,m/t) \\ a \text{ odd}}} a^{k-1}\, c\left(\frac{mn}{ta^2}, \frac{r}{a}\right).$$

If $t \not\equiv 0 \mod 4$, *then* $L(\phi)$ *is a cusp form.*

Although the lifting L is adequately described as an imitation of the Gritsenko lift, the choice of Hecke operators used to construct L was not obvious to us. The special case $t = 1$ is a lifting from Jacobi cusp forms of index $1/2$ for the Jacobi theta group $\Gamma_1(1,2)^J$ to Siegel modular cusp forms for the theta group $\Gamma_2(1,2)$,

$$L : J^{\mathrm{cusp}}_{k,1/2}(\Gamma_1(1,2)^J) \to S_k\left(\Gamma_2(1,2)\right).$$

We can also connect the lift L with elliptic modular forms on the theta group $\Gamma_1(1,2)$ if we make use of multiplication by the theta function $\theta[0] \in J_{1/2,1/2}(\Gamma_1(1,2)^J, v_\theta)$. Here, v_θ is the multiplier of the theta function and takes values in the eighth roots of unity.

Corollary 2. *For* $k \in \mathbb{N}$, *with* $4|k$, *there are monomorphisms*

$$S_{k-\frac{1}{2}}\left(\Gamma_1(1,2), v_\theta^{2k-1}\right) \xrightarrow{\cdot\theta[0](z,\tau)} J^{\mathrm{cusp}}_{k,1/2}(\Gamma_1(1,2)^J) \xrightarrow{L} S_k\left(\Gamma_2(1,2)\right).$$

Corollary 2 shows that L, which is defined with respect to the theta group, is just as fundamental as any member of the Saito-Kurokawa family of lifts, to which L belongs. For a general context and for an extended family of lifts, see the thesis of F. Clery [3].

An application

D'Hoker and Phong [5] have computed the chiral superstring measure $d\nu^{(g)}[e] = \Xi_g[e]\, d\mu^{(g)}$ in $g = 2$ and it is determined by $\Xi_2[0] \in S_8\left(\Gamma_2(1,2)\right)$, which can be defined, for example, as a polynomial of degree 16 in the thetanullwerte, see [11]:

$$\Xi_2[0] = \frac{1}{1024}\left(2\,\theta\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)^{16} - \theta\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)^{8} \sum_{\zeta \text{ even}}^{10} \theta[\zeta]^8 + 2\,\theta\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)^{4} F\right), \tag{1}$$

with

$$\begin{aligned} F = {} & \theta\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)^4\theta\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)^4\theta\left(\begin{smallmatrix}0&0\\1&1\end{smallmatrix}\right)^4 + \theta\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)^4\theta\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)^4\theta\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)^4 + \theta\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)^4\theta\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)^4\theta\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)^4 \\ & + \theta\left(\begin{smallmatrix}0&0\\1&1\end{smallmatrix}\right)^4\theta\left(\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right)^4\theta\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)^4 + \theta\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)^4\theta\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)^4\theta\left(\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right)^4 + \theta\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)^4\theta\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)^4\theta\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)^4. \end{aligned}$$

The solution $\Xi_2[0]$ may be variously viewed as a Siegel modular form, a Teichmuller modular form or as a binary invariant depending upon whether it is viewed as a section of a sheaf over the moduli space of abelian varieties, curves or hyperelliptic curves. At the request of the referee, we identify these sheaves and sections in some detail. Consider the morphisms over $\mathbb{C}$:

$$W/\operatorname{SL}_2(\mathbb{C}) \overset{\bar{B}}{\backsimeq} h_g(2) \xrightarrow{\imath} \mathcal{M}_g(2) \xrightarrow{T} \mathcal{X}_g(2).$$

Here $\mathcal{X}_g(\ell)$, $\mathcal{M}_g(\ell)$ and $h_g(\ell)$ are the moduli stacks with level ℓ structures of principally polarized abelian schemes of relative dimension g, smooth proper curves of genus g and hyperelliptic smooth proper curves of genus g, respectively. The quasi-projective variety W is the subset of $\prod^{2g+2}\mathbb{P}^1(\mathbb{C})$ with distinct coordinates. The quotient, $W/\operatorname{SL}_2(\mathbb{C})$, given by the diagonal action of $\operatorname{SL}_2(\mathbb{C})$, is quasi-affine, as can be seen by normalizing three coordinates. The Torelli map T associates a curve with level ℓ structure to its Jacobian with the corresponding symplectic level ℓ structure. For $a \in W$, the normalization of the algebraic curve $y^2 = \prod_{i:a_i\neq\infty}(x-a_i)$ defines a hyperelliptic curve C_a. Let $B : W \to h_g(2)$ be the morphism given by sending $a \in W$ to the equivalence class of C_a in $h_g(2)$, inheriting the level two structure from the traditional marking of a hyperelliptic curve with ordered branch points, see [17]. This surjection descends to an isomorphism $\bar{B} : W/\operatorname{SL}_2(\mathbb{C}) \backsimeq h_g(2)$.

We construct sheaves over these moduli spaces. Let $r : \mathcal{A} \to \mathcal{X}_g(\ell)$ be the universal abelian scheme with symplectic level ℓ structure with sheaf of relative differentials $\Omega_{\mathcal{A}/\mathcal{X}_g(\ell)}$ and note $\mu = \wedge^g r_*\left(\Omega_{\mathcal{A}/\mathcal{X}_g(\ell)}\right)$ is an invertible sheaf on $\mathcal{X}_g(\ell)$. Similarly, let $\pi : \mathcal{C} \to \mathcal{M}_g(\ell)$ be the universal curve with level ℓ structure, with sheaf of relative differentials $\Omega_{\mathcal{C}/\mathcal{M}_g(\ell)}$ and note that $\lambda = \wedge^g \pi_*\left(\Omega_{\mathcal{C}/\mathcal{M}_g(\ell)}\right)$ is an invertible sheaf on $\mathcal{M}_g(\ell)$. One has $T^*\mu = \lambda$. The general definition of a Siegel modular form of degree g, level ℓ and weight k, defined over a $\mathbb{Z}$-algebra M is a global section of $\mu^{\otimes k}\otimes_{\mathbb{Z}} M$ over $\mathcal{X}_g(\ell)$. Following T. Ichikawa [13], a Teichmüller modular form of degree g, level ℓ and weight k, defined over a $\mathbb{Z}$-algebra M, is a global section of $\lambda^{\otimes k}\otimes_{\mathbb{Z}} M$ over $\mathcal{M}_g(\ell)$. The functorial homomorphisms

$$\begin{aligned}&H^0\left(\mathcal{X}_g(2), \mu^{\otimes k}\otimes_{\mathbb{Z}}\mathbb{C}\right) \xrightarrow{T^*} H^0\left(\mathcal{M}_g(2), \lambda^{\otimes k}\otimes_{\mathbb{Z}}\mathbb{C}\right) \xrightarrow{\imath^*}\\ &H^0\left(h_g(2), (\imath^*\lambda)^{\otimes k}\otimes_{\mathbb{Z}}\mathbb{C}\right) \xrightarrow{\bar{B}^*} H^0\left(W/\operatorname{SL}_2(\mathbb{C}), (\bar{B}^*\imath^*\lambda)^{\otimes k}\otimes_{\mathbb{Z}}\mathbb{C}\right)\end{aligned}$$

show that every Siegel modular form has a T^*-image that is a Teichmüller modular form and a $(T\circ\imath)^*$-image that is a section over the hyperelliptic moduli stack $h_g(2)$. It remains to reconcile this general notion of Siegel modular form with the classical case and to explain how $H^0\left(W/\operatorname{SL}_2(\mathbb{C}), (\bar{B}^*\imath^*\lambda)^{\otimes k}\otimes_{\mathbb{Z}}\mathbb{C}\right)$ $\backsimeq H^0\left(W, (B^*\imath^*\lambda)^{\otimes k}\otimes_{\mathbb{Z}}\mathbb{C}\right)^{\operatorname{SL}_2(\mathbb{C})}$ may be said to contain binary forms. We refer the reader to [7] for the first point and to [14] for the second.

In the setting of Siegel modular forms, the Ansatz of D'Hoker and Phong [2], [5], [11], [12] asks for a family of Siegel modular forms satisfying:

1) $\Xi_{g_1+g_2}[0]\left(\left(\begin{smallmatrix}\Omega_1 & 0\\ 0 & \Omega_2\end{smallmatrix}\right)\right) = \Xi_{g_1}[0](\Omega_1)\Xi_{g_2}[0](\Omega_2)$ for Ω_1, Ω_2 in the Jacobian loci.

2) $\mathrm{tr}\,(\Xi_g[0])$ vanishes on the Jacobian locus.

3) The family $\{\Xi_g[0]\}$ is uniquely determined on the Jacobian loci by the genus one solution $\Xi_1[0] = \theta_{00}^4\eta^{12}$. This Ansatz can be satisfied through $g \le 5$ but is thought unlikely to extend further [18]. Over the hyperelliptic locus, however, the corresponding conditions are solved for all g by a family of binary invariants, see [19]. As of this writing it remains an open question whether the corresponding conditions can be satisfied by a Teichmuller modular form beyond $g = 5$. See [16] for an entry to the physics literature. In Table 1, we give some Fourier coefficients for $a\,(T; \Xi_2[0])$ using the above polynomial in the thetanullwerte (1).

Table 1. Fourier coefficients for $\Xi_2[0]$.

$a\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = 1;$	$a\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix} = 6;$	$a\begin{pmatrix}1 & 0\\ 0 & 3\end{pmatrix} = 0;$	$a\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix} = 64;$
$a\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix} = 0;$	$a\begin{pmatrix}1 & 0\\ 0 & 4\end{pmatrix} = -64;$	$a\begin{pmatrix}2 & 0\\ 0 & 3\end{pmatrix} = 252;$	$a\begin{pmatrix}2 & 1\\ 1 & 3\end{pmatrix} = -84;$
$a\begin{pmatrix}1 & 0\\ 0 & 5\end{pmatrix} = -84;$	$a\begin{pmatrix}2 & 0\\ 0 & 4\end{pmatrix} = 384;$	$a\begin{pmatrix}2 & 1\\ 1 & 4\end{pmatrix} = -512;$	$a\begin{pmatrix}3 & 0\\ 0 & 3\end{pmatrix} = 1080;$
$a\begin{pmatrix}3 & 1\\ 1 & 3\end{pmatrix} = -384;$	$a\begin{pmatrix}1 & 0\\ 0 & 6\end{pmatrix} = 252;$	$a\begin{pmatrix}2 & 0\\ 0 & 5\end{pmatrix} = 28;$	$a\begin{pmatrix}2 & 1\\ 1 & 5\end{pmatrix} = -1107.$

A rapid method exists for computing these Fourier coefficients because $\Xi_2[0]$ is a lift. Consider $\Psi = \theta_{00}^3\eta^{12} \in S_{15/2}\left(\Gamma_1(1,2), v_\theta^{15}\right)$. Its Fourier expansion $\Psi(\tau) = \sum_{n\in\mathbb{N}} c(n;\Psi)q^{n/2}$, with $q = e^{2\pi i\tau} = e(\tau)$, is given by:

$$\begin{aligned}\Psi(2\tau) = q &+ 6q^2 - 64q^4 - 84q^5 + 252q^6 + 512q^7 - 384q^8 - 1107q^9\\ &+ 28q^{10} + 3724q^{13} + 792q^{14} - 4608q^{15} + 4096q^{16} - 168q^{17} + O(q^{18}).\end{aligned}$$

Use Corollary 2 to define a Jacobi form $\phi \in J^{\mathrm{cusp}}_{8,1/2}(\Gamma_1(1,2)^J)$ by $\phi(\tau, z) = \theta_{00}(z,\tau)\,\Psi(\tau)$. The lift $L(\phi)$ is then in the one dimensional space $S_8\,(\Gamma_2(1,2))$. By checking agreement on one Fourier coefficient we conclude $\Xi_2[0] = L(\phi)$ and obtain the formula

$$a\left(\begin{pmatrix}n & r\\ r & m\end{pmatrix}; \Xi_2[0]\right) = (-1)^{(m+1)(n+1)} \sum_{\substack{a\mid(n,r,m)\\ a \text{ odd}}} a^7 c\left(\frac{mn-r^2}{a^2}; \Psi\right).$$

Thus, the entries in Table 1 can be easily verified from the q-expansion of the elliptic modular form Ψ.

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2. Groups

Beyond interest in the chiral superstring form $\Xi_2[0] \in S_8(\Gamma_2(1,2))$, the justifications for working out a variant of the Saito-Kurokawa lift are the precise specification of the group of automorphy and the cuspidality of the lift. For index 1/2, the lift $L(\phi)$ is automorphic with respect to the theta group $\Gamma_2(1,2)$. For index $t/2$, this role is played by $\Gamma^{\mathrm{para}}(t;1,2)$, a subgroup of the paramodular group $\Gamma^{\mathrm{para}}(t)$. In order to determine the group of automorphy for the lift we will need to know generators of $\Gamma^{\mathrm{para}}(t;1,2)$. The thesis of Delzeith [4] shows that $\Gamma^{\mathrm{para}}(t)$ is generated by its translations and by $J(t) = \left(\begin{smallmatrix} 0 & T^{-1} \\ -T & 0 \end{smallmatrix}\right)$ for $T = \left(\begin{smallmatrix} 1 & 0 \\ 0 & t \end{smallmatrix}\right)$. In order to show the cuspidality of the lift for $t \not\equiv 0 \mod 4$, we require coset decompositions of $\mathrm{Sp}_2(\mathbb{Q})$ with respect to $\Gamma^{\mathrm{para}}(t)$ and $\Gamma^{\mathrm{para}}(t;1,2)$. Let $J = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right) \in \mathrm{GL}_{2g}(\mathbb{Z})$. For $\mathbb{F} \subseteq \mathbb{R}$, set $\mathbb{F}^+ = \{x \in \mathbb{F} : x > 0\}$. We write the transpose of a matrix γ as γ'.

Definition 3. For $\mathbb{F} = \mathbb{R}, \mathbb{Q}$ or $\mathbb{Z}$, define groups of matrices:

$$\mathrm{Sp}_g(\mathbb{F}) = \{\gamma \in M_{2g\times 2g}(\mathbb{F}) : \gamma J \gamma' = J\},$$
$$\mathrm{GSp}_g^+(\mathbb{F}) = \{\gamma \in M_{2g\times 2g}(\mathbb{F}) : \exists \mu(\gamma) \in \mathbb{F}^+ : \gamma J \gamma' = \mu(\gamma) J\}.$$

The theta group of genus g is

$$\Gamma_g(1,2) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_g(\mathbb{Z}) : A'C, B'D \text{ have even diagonal entries} \right\}.$$

For a domain $\mathbb{D} \subseteq \mathbb{C}$, let $V_g(\mathbb{D})$ be the g-by-g symmetric matrices with coefficients in $\mathbb{D}$. For $\mathbb{D} \subseteq \mathbb{R}$, let $\mathcal{P}_g(\mathbb{D})^{\mathrm{semi}} \subseteq V_g(\mathbb{D})$ be the semidefinite elements and let $\mathcal{P}_g(\mathbb{D})$ be the definite elements. Let $\mathcal{H}_g$ be the Siegel upper half space of degree g, the subset of $V_g(\mathbb{C})$ with positive definite imaginary part. The symplectic group $\mathrm{Sp}_g(\mathbb{R})$ acts on $\Omega \in \mathcal{H}_g$ via

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \circ \Omega := (A\Omega + B)(C\Omega + D)^{-1}.$$

Definition 4. Define the following subgroup of $\mathrm{GSp}_2^+(\mathbb{F})$:

$$G\Gamma_\infty(\mathbb{F}) = \left\{ \begin{pmatrix} * & 0 & * & * \\ * & * & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \in \mathrm{GSp}_2^+(\mathbb{F}) \right\}.$$

Denote $\Gamma_\infty(\mathbb{F}) = G\Gamma_\infty(\mathbb{F}) \cap \mathrm{Sp}_2(\mathbb{F})$ and $\Gamma_2(1,2)_\infty = \Gamma_2(1,2) \cap \Gamma_\infty(\mathbb{Z})$.

For an element $\gamma \in \mathrm{GSp}_2^+(\mathbb{F})$ to be in $G\Gamma_\infty(\mathbb{F})$, it suffices that the second column be of the correct form. Introduce the notation $(\gamma)_2$ to mean the second column of γ written as a row 4-tuple for typesetting convenience.

Lemma 5. *$G\Gamma_\infty(\mathbb{F}) = \{\gamma \in \mathrm{GSp}_2^+(\mathbb{F}) : (\gamma)_2 = (0,*,0,0)$ for $* \in \mathbb{F}\}$.*

Proof. Elementary. □

The intersection of this subgroup with the theta group may be constructed in terms of more elementary groups as follows: Consider the Heisenberg group $H(\mathbb{F}) = \mathbb{F}^3 = \{(\lambda, v, k) \in \mathbb{F}^3\}$ with the multiplication $(\lambda_1, v_1, k_1)(\lambda_2, v_2, k_2) = (\lambda_1 + \lambda_2, v_1 + v_2, k_1 + k_2 + (\lambda_1 v_2 - \lambda_2 v_1))$. The theta group $\Gamma_1(1,2)$ produces two orbits in $H(\mathbb{Z})$ under the action that sends (λ, v, k) to $(\lambda, v, k)(\sigma \oplus 1)$ for $\sigma \in \Gamma_1(1,2)$:

$$\begin{aligned} H_e(\mathbb{Z}) &= \{(\lambda, v, k) \in \mathbb{Z}^3 : \lambda v + k \text{ is even}\}, \\ H_o(\mathbb{Z}) &= \{(\lambda, v, k) \in \mathbb{Z}^3 : \lambda v + k \text{ is odd}\}. \end{aligned}$$

We denote by $\Gamma_1(1,2)^J$ the semidirect product $\Gamma_1(1,2) \ltimes H_e(\mathbb{Z})$.

Lemma 6. *The following multiplicative homomorphisms are injective:*
$i : \mathrm{GL}_2(\mathbb{R}) \to G\Gamma_\infty(\mathbb{R})$, given by

$$i\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & 0 & b & 0 \\ 0 & ad-bc & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and $w : H(\mathbb{R}) \to \Gamma_\infty(\mathbb{R})$ given by

$$w(\lambda, v, k) = \begin{pmatrix} 1 & 0 & 0 & v \\ \lambda & 1 & v & k \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The map $\Gamma_1(1,2)^J = \Gamma_1(1,2) \ltimes H_e(\mathbb{Z}) \to \Gamma_2(1,2)_\infty/\{\pm I\}$, defined by sending $(g,h) \mapsto \pm\, i(g)w(h)$, is a group isomorphism. Define the element $\epsilon = \mathrm{diag}(1,-1,1,-1)$. We have an exact sequence

$$\{I\} \to \langle w\left(H_e(\mathbb{Z})\right), \epsilon \rangle \to \Gamma_2(1,2)_\infty \to \Gamma_1(1,2) \to \{I\}$$

given by sending $\gamma \in \Gamma_2(1,2)_\infty$ to $\left(\begin{smallmatrix} \gamma_{11} & \gamma_{13} \\ \gamma_{31} & \gamma_{33} \end{smallmatrix}\right) \in \Gamma_1(1,2)$.

Proof. Left to the reader. □

Definition 7. For $t \in \mathbb{N}$, define the paramodular group to be

$$\Gamma^{\mathrm{para}}(t) = \left\{ \begin{pmatrix} * & t* & * & * \\ * & * & * & */t \\ * & t* & * & * \\ t* & t* & t* & * \end{pmatrix} : * \in \mathbb{Z} \right\} \cap \mathrm{Sp}_2(\mathbb{Q}).$$

Define the paramodular theta group:

$$\Gamma^{\mathrm{para}}(t;1,2) = \{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma^{\mathrm{para}}(t) : A'C = \left(\begin{smallmatrix} a & * \\ * & bt \end{smallmatrix}\right), B'D = \left(\begin{smallmatrix} c & * \\ * & d/t \end{smallmatrix}\right), a,b,c,d \in 2\mathbb{Z}\}.$$

We note that this definition and notation for the paramodular group differ from those in [20]. The groups $\Gamma^{\rm para}(t)$ and $\Gamma^{\rm para}(t;1,2)$ have a common normalizer $\mu_t \in \mathrm{Sp}_2(\mathbb{R})$ with the property that $\mu_t^2 = -I$:

$$\mu_t = \begin{pmatrix} 0 & \sqrt{t} & 0 & 0 \\ \frac{-1}{\sqrt{t}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{t}} \\ 0 & 0 & -\sqrt{t} & 0 \end{pmatrix}.$$

We now determine the parabolic part of the paramodular groups. For $t \in \mathbb{N}$, define γ_t as below and set $\Gamma_2(1,2)_\infty[t] = \langle \Gamma_2(1,2)_\infty, \gamma_t \rangle$, the group generated by $\Gamma_2(1,2)_\infty$ and γ_t inside $\mathrm{Sp}_2(\mathbb{Q})$, where

$$\gamma_t = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2/t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Lemma 8. *We have* $\Gamma_\infty(\mathbb{Q}) \cap \Gamma^{\rm para}(t;1,2) = \Gamma_2(1,2)_\infty[t]$.

Proof. The "$\supseteq$" inclusion is easy. Take $\delta \in \Gamma_\infty(\mathbb{Q}) \cap \Gamma^{\rm para}(t;1,2)$:

$$\delta = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} a & 0 & b & c \\ d & \epsilon_1 & e & f/t \\ g & 0 & h & i \\ 0 & 0 & 0 & \epsilon_2 \end{pmatrix}$$

for some $a, b, c, d, e, f, g, h, i \in \mathbb{Z}$ and $\epsilon_1, \epsilon_2 \in \{-1, +1\}$. The diagonal of $A'C$ and the upper left entry of $B'D$ are even integers because $\delta \in \Gamma^{\rm para}(t;1,2)$; the lower right entry of $B'D$ is an even multiple of $1/t$. So $ci + \epsilon_2 f/t = 2z/t$ for some $z \in \mathbb{Z}$. In particular, $tci + \epsilon_2 f$ is even. We proceed by cases.

If t is odd, then let $\beta = \gamma_t^{\epsilon_2 f(t-1)/2}\delta$ and we see that its $(2,4)$ entry is $\epsilon_2^2 f = f$ and so $\beta \in \Gamma^{\rm para}(t;1,2) \cap \Gamma_\infty(\mathbb{Z})$. We now show that the lower right entry of its "$B'D$" is even; it is $ci + \epsilon_2 f \equiv tci + \epsilon_2 f \equiv 0 \mod 2$ because t is odd. Thus $\beta \in \Gamma_2(1,2)_\infty$. Then $\delta = \gamma_t^{-\epsilon_2 f(t-1)/2}\beta \in \Gamma_2(1,2)_\infty[t]$.

If t is even, then the condition that $tci + \epsilon_2 f$ is an even integer forces f to be even. Then let $\beta = \gamma_t^{-\epsilon_2 f/2 - cit/2}\delta$ and note that its $(2,4)$ entry is $-\epsilon_2 ci$ and so $\beta \in \Gamma^{\rm para}(t;1,2) \cap \Gamma_\infty(\mathbb{Z})$. We now show that the lower right entry of its "$B'D$" is an even integer; it is $ci - \epsilon_2^2 ci = 0$. Thus $\beta \in \Gamma_2(1,2)_\infty$ and $\delta = \gamma_t^{\epsilon_2 f/2 + cit/2}\beta \in \Gamma_2(1,2)_\infty[t]$. $\square$

Proofs about generators are best done inside an integral version of the paramodular group. To this end, denote $T = \left(\begin{smallmatrix} 1 & 0 \\ 0 & t \end{smallmatrix}\right)$, $I_t = \left(\begin{smallmatrix} I & 0 \\ 0 & T \end{smallmatrix}\right)$, and $E_t = \left(\begin{smallmatrix} 0 & T \\ -T & 0 \end{smallmatrix}\right)$. For any group $G \subseteq \mathrm{GL}_4(\mathbb{R})$, denote $G^I = I_t^{-1} G I_t$. Then

$$\text{(2)}\quad \Gamma^{\rm para}(t;1,2)^I = \{g = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{GL}_4(\mathbb{Z}) : g' E_t g = E_t \text{ and } \left(T^{-1}\alpha' T \gamma\right)_0 \equiv 0 \mod 2,\ \left(T^{-1}\beta' T \delta\right)_0 \equiv 0 \mod 2\}.$$

The presentation (2) makes it clear that the integral version of the paramodular theta group $\Gamma^{\mathrm{para}}(t;1,2)^I$ is a natural generalization of the theta group to nonprincipal polarizations and that when $t=1$, we have the equalities $\Gamma^{\mathrm{para}}(t;1,2)=\Gamma^{\mathrm{para}}(t;1,2)^I=\Gamma_2(1,2)$.

Definition 9. For $t\in\mathbb{N}$, define the group

$$G_t=\langle \Gamma_2(1,2)_\infty[t],\ \mu_t\Gamma_2(1,2)_\infty[t]\mu_t\rangle.$$

We will in due course show $G_t=\Gamma^{\mathrm{para}}(t;1,2)$. To compare this with the generators given by Gritsenko for $\Gamma^{\mathrm{para}}(t)$ see [10].

Lemma 10. *The following matrices are elements of* G_t^I : $J=\left(\begin{smallmatrix}0&I\\-I&0\end{smallmatrix}\right)$,

$$E_1=\begin{pmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&-1&0&0\end{pmatrix},\ g(\lambda,v,k,\ell)=\begin{pmatrix}1&0&0&tv\\\lambda&1&v&kt+2\ell\\0&0&1&-\lambda t\\0&0&0&1\end{pmatrix},$$

$$Jg(\lambda,v,k,\ell)J^{-1}=\begin{pmatrix}1&-\lambda t&0&0\\0&1&0&0\\0&-vt&1&0\\-v&-kt-2\ell&\lambda&1\end{pmatrix},$$

whenever $k+\lambda v\in 2\mathbb{Z}$ *and* $k,\lambda,v,\ell\in\mathbb{Z}$.

Also, $i(A)$ *and* $j(A)\in G_t^I$ *for* $A\in\Gamma_1(1,2)$, *where*

$$j\left(\begin{pmatrix}a&b\\c&d\end{pmatrix}\right)=\begin{pmatrix}1&0&0&0\\0&a&0&b\\0&0&1&0\\0&c&0&d\end{pmatrix}.$$

Proof. We have

$$J=I_t^{-1}\mu_t i\left(\begin{pmatrix}0&1\\-1&0\end{pmatrix}\right)\mu_t^{-1}i\left(\begin{pmatrix}0&1\\-1&0\end{pmatrix}\right)I_t\in G_t^I,$$
$$E_1=I_t^{-1}\mu_t i\left(\begin{pmatrix}0&1\\-1&0\end{pmatrix}\right)\mu_t^{-1}I_t\in G_t^I.$$

The element $g(k,\lambda,v,\ell)$ is in the Heisenberg part $w(H_e(\mathbb{Z}))^I\subseteq \Gamma_2(1,2)_\infty{}^I$ and the conjugate $Jg(k,\lambda,v,\ell)J^{-1}$ is therefore in G_t^I. Since $\left(\begin{smallmatrix}1&0\\2&1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$ generate $\Gamma_1(1,2)$, we have $\forall A\in\Gamma_1(1,2)$, $j(A)\in G_t^I$. We know $\forall A\in\Gamma_1(1,2)$, $i(A)\in G_t^I$. □

Proposition 11. *For* $t\in\mathbb{N}$, $\Gamma^{\mathrm{para}}(t;1,2)=G_t$.

Proof. Since $G_t\subseteq\Gamma^{\mathrm{para}}(t;1,2)$ is easily checked, we only prove the inclusion $\Gamma^{\mathrm{para}}(t;1,2)^I\subseteq G_t^I$. Take any $\gamma\in\Gamma^{\mathrm{para}}(t;1,2)^I$. Recall the notation $(\gamma)_2=(u_1,u_2,u_3,u_4)$ to mean the second column of γ written as a row 4-tuple. Since γ is integral of determinant one, we know that $\gcd(u_1,u_2,u_3,u_4)=1$.

STEP 1: $\exists \beta_0 \in G_t^I : (\beta_0\gamma)_2 = (x_1, x_2, x_3, x_4)$ and $x_4 \neq 0$.
Note at least one of the u_i must be nonzero. If $u_4 \neq 0$, then $\beta_0 = I$ works. If $u_4 = 0$ and $u_3 \neq 0$, then $\beta_0 = Jg(1,0,0,0)J^{-1}$ works. If $u_4 = 0$ and $u_1 \neq 0$, then $\beta_0 = Jg(1,0,0,0)J^{-1}\, i(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right))$ works. Finally, if $u_4 = 0$ and $u_2 \neq 0$, then $\beta_0 = j(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right))$ works. Note that we always have $\gcd(x_1, x_2, x_3, x_4) = 1$.

STEP 2: $\exists \beta_1 \in G_t^I : (\beta_1\beta_0\gamma)_2 = (y_1, y_2, y_3, y_4)$ and $\gcd(y_2, y_4) = 1$.
Set $w = \gcd(x_2, x_4)$ and $w_2 = \gcd(x_1, x_3)$ and $w_3 = \gcd(x_4/w, w^{|x_4|})$. We make the following comments: $w \neq 0$ because $x_4 \neq 0$. There are $a, b \in \mathbb{Z}$ such that $w_2 = ax_1 + bx_3$. Finally, $\gcd(x_4/(ww_3), w) = 1$ and for any prime p, $p|w_3$ implies $p|w$.

Let $\beta_1 = g(\lambda, v, k, \ell)$ with $\lambda = ax_4/(ww_3)$, $v = bx_4/(ww_3)$, $k = -\lambda v$ and $\ell = 0$ so that

$$\beta_1 = g(\lambda, v, k, \ell) = \begin{pmatrix} 1 & 0 & 0 & \frac{bx_4t}{ww_3} \\ \frac{ax_4}{ww_3} & 1 & \frac{bx_4}{ww_3} & kt \\ 0 & 0 & 1 & -\frac{ax_4t}{ww_3} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then $(\beta_1\beta_0\gamma)_2 = (y_1, y_2, y_3, y_4)$ where

$$y_4 = x_4 \text{ and } y_2 = x_2 + \gcd(x_1, x_3)\frac{x_4}{ww_3} + ktx_4.$$

It is already the case that $\gcd(y_2, y_4) = 1$. Consider any prime $p|y_4$.
Case $p|w$: If $p|w$, then $p|x_2$, and $p \nmid \gcd(x_1, x_3)$ since $\gcd(x_1, x_2, x_3, x_4) = 1$. But also $p \nmid \frac{x_4}{ww_3}$ since $\gcd(x_4/(ww_3), w) = 1$ and so $p \nmid y_2$.
Case $p \nmid w$: If $p \nmid w$, then $p \nmid x_2$ and $p|\frac{x_4}{w}$ since $p|x_4$ and $p \nmid w$. Furthermore $p \nmid w_3$ (else $p|w$) so that $p|\frac{x_4}{ww_3}$. Then $p \nmid y_2$. In either case $p \nmid \gcd(y_2, y_4)$ and thus $\gcd(y_2, y_4) = 1$.

STEP 3: $\exists \beta_2 \in G_t^I$ such that $(\beta_2\beta_1\beta_0\gamma)_2 = (z_1, 1, z_3, 0)$.
Note that if one of y_2, y_4 is even (and hence the other is odd), then we can find $A \in \Gamma_1(1,2)$ such that $(j(A)\beta_1\beta_0\gamma)_2 = (z_1, 1, z_3, 0)$. In this case we may take $\beta_2 = j(A)$. If both y_2, y_4 are odd, then (y_1, y_2, y_3, y_4) being the second column of a $\Gamma^{\text{para}}(t; 1, 2)^I$ matrix forces $t|y_1$, $t|y_3$ and $y_1y_3/t + y_2y_4 \equiv 0 \mod 2$ which forces y_1, y_3 to be odd as well. Then $g(1,0,0,0)\beta_1\beta_0\gamma$ satisfies $(g(1,0,0,0)\beta_1\beta_0\gamma)_2 = (y_1, y_1 + y_2, y_3 - ty_4, y_4)$. Then $y_1 + y_2$ is even and y_4 is still odd, so that $\beta_2 = j(A)g(1,0,0,0)$ suffices by the first argument.

STEP 4: $\exists \beta_3 \in G_t^I$ such that $(\beta_3\beta_2\beta_1\beta_0\gamma)_2 = (0, 1, z_3, z_1z_3/t)$.
From $\beta_2\beta_1\beta_0\gamma \in \Gamma^{\text{para}}(t; 1, 2)^I$ we see that $z_1z_3/t \equiv 0 \mod 2$, and that $t|z_1$ and $t|z_3$. Then $\beta_3 = Jg(z_1/t, 0, 0, 0)J^{-1}$ gives $(\beta_3\beta_2\beta_1\beta_0\gamma)_2 = (0, 1, z_3, z_1z_3/t)$.

STEP 5: $\exists \beta_4 \in G_t^I$ such that $(\beta_4\beta_3\beta_2\beta_1\beta_0\gamma)_2 = (z_3, 1, 0, 0)$.
Note that z_1z_3/t is even and $\beta_4 = i(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right))j(\left(\begin{smallmatrix} 1 & 0 \\ -z_1z_3/t & 1 \end{smallmatrix}\right)) \in G_t^I$ works.

STEP 6: $\exists \beta_5 \in G_t^I$ such that $(\beta_5\beta_4\beta_3\beta_2\beta_1\beta_0\gamma)_2 = (0, 1, 0, 0)$.
Use $\beta_5 = Jg(z_3/t, 0, 0, 0)J^{-1} \in G_t^I$.

By Lemma 5, we have $\beta_5\beta_4\beta_3\beta_2\beta_1\beta_0\gamma \in G\Gamma_\infty(\mathbb{Z}) \cap \Gamma^{\mathrm{para}}(t;1,2)^I$. Now, $G\Gamma_\infty(\mathbb{Z}) \cap \Gamma^{\mathrm{para}}(t;1,2)^I \subseteq \left(G\Gamma_\infty(\mathbb{Q}) \cap \Gamma^{\mathrm{para}}(t;1,2)\right)^I$ and we have $G\Gamma_\infty(\mathbb{Q}) \cap \Gamma^{\mathrm{para}}(t;1,2) = \Gamma_\infty(\mathbb{Q}) \cap \Gamma^{\mathrm{para}}(t;1,2) = \Gamma_2(1,2)_\infty[t]$, where the last equality is by Lemma 8. Thus $\beta_5\beta_4\beta_3\beta_2\beta_1\beta_0\gamma \in \Gamma_2(1,2)_\infty[t]^I \subseteq G_t^I$ and $\gamma \in G_t^I$. □

Lemma 12. *We have* $\mathrm{Sp}_2(\mathbb{Z}) \subseteq \Gamma^{\mathrm{para}}(t)\, U\, \Gamma_\infty(\mathbb{Q})$, *where*

$$U = \left\{ \begin{pmatrix} 1 & c & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -c & 1 \end{pmatrix} : c \in \mathbb{Z} \right\}.$$

Proof. Take any $\alpha \in \mathrm{Sp}_2(\mathbb{Z})$. Since the second column of α must have relatively prime entries, by a similar argument to Steps 1 and 2 of the proof to Proposition 11 we can find a $\beta_1 = \left(\begin{smallmatrix} 1 & 0 & 0 & v \\ \lambda & 1 & v & k \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right) \in \Gamma^{\mathrm{para}}(t) \cap \mathrm{Sp}_2(\mathbb{Z})$ such that $(\beta_1\alpha)_2 = (y_1, y_2, y_3, y_4)$ where $\gcd(y_2, y_4) = 1$. Let $g = \gcd(ty_2, y_4) = aty_2 + by_4$ for some $a, b \in \mathbb{Z}$. Note $g|t$. Then let

$$\beta_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b/t \\ 0 & 0 & 1 & 0 \\ 0 & -y_4t/g & 0 & ty_2/g \end{pmatrix} \in \Gamma^{\mathrm{para}}(t)$$

so that $(\beta_2\beta_1\alpha)_2 = (y_1, g/t, y_3, 0)$. Next let $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ such that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} y_1 \\ y_3 \end{smallmatrix}\right) = \left(\begin{smallmatrix} z \\ 0 \end{smallmatrix}\right)$ where $z = \gcd(y_1, y_3)$. Let

$$\beta_3 = \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \Gamma^{\mathrm{para}}(t)$$

so that $(\beta_3\beta_2\beta_1\alpha)_2 = (z, g/t, 0, 0)$. Finally, let

$$u = \begin{pmatrix} 1 & zt/g & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -zt/g & 1 \end{pmatrix}$$

so that $(u^{-1}\beta_3\beta_2\beta_1\alpha)_2 = (0, g/t, 0, 0)$. The $\frac{zt}{g}$ are the integers c in the statement of the lemma. Calling $\gamma = u^{-1}\beta_3\beta_2\beta_1\alpha$, then $\gamma \in \mathrm{Sp}_2(\mathbb{Q})$ and $(\gamma)_2 = (0, g/t, 0, 0)$ forces $\gamma \in \Gamma_\infty(\mathbb{Q})$ by Lemma 5. Then $\alpha = \beta_1^{-1}\beta_2^{-1}\beta_3^{-1}u\gamma$ says that $\alpha \in \Gamma^{\mathrm{para}}(t)\, U\, \Gamma_\infty(\mathbb{Q})$. □

Lemma 13. *For $\Gamma^{\rm para}(t;1,2)\backslash\Gamma^{\rm para}(t)$, a complete list of right coset representatives can be taken to be*

$$C_1 = \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix},\quad C_2 = \begin{pmatrix} 1&0&1&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix},\ C_3 = \begin{pmatrix} 1&0&0&0\\0&1&0&1/t\\0&0&1&0\\0&0&0&1 \end{pmatrix},$$

$$C_4 = \begin{pmatrix} 1&0&1&0\\0&1&0&1/t\\0&0&1&0\\0&0&0&1 \end{pmatrix},\ C_5 = \begin{pmatrix} 1&0&0&0\\0&1&0&0\\1&0&1&0\\0&0&0&1 \end{pmatrix},\ C_6 = \begin{pmatrix} 1&0&0&0\\0&1&0&1/t\\1&0&1&0\\0&0&0&1 \end{pmatrix},$$

$$C_7 = \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&t&0&1 \end{pmatrix},\quad C_8 = \begin{pmatrix} 1&0&1&0\\0&1&0&0\\0&0&1&0\\0&t&0&1 \end{pmatrix},\ C_9 = \begin{pmatrix} 1&0&0&0\\0&1&0&0\\1&0&1&0\\0&t&0&1 \end{pmatrix},$$

$$C_{10} = \begin{pmatrix} 1&0&0&1\\0&1&1&0\\1&0&1&1\\0&t&t&1 \end{pmatrix}$$

for t odd, and we can omit C_{10} for t even.

Proof. It is a straightforward calculation to check that the set of cosets $\{\Gamma^{\rm para}(t;1,2)C_i\}_{i=1}^{10}$ is stable under right multiplication by the following set of generators for $\Gamma^{\rm para}(t)$:

$$\alpha := \begin{pmatrix} 1&0&1&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix},\ \beta := \begin{pmatrix} 1&0&0&0\\0&1&0&1/t\\0&0&1&0\\0&0&0&1 \end{pmatrix},$$

$$\gamma := \begin{pmatrix} 1&0&0&1\\0&1&1&0\\0&0&1&0\\0&0&0&1 \end{pmatrix},\ \delta := \begin{pmatrix} 0&0&1&0\\0&0&0&1/t\\-1&0&0&0\\0&-t&0&0 \end{pmatrix},$$

which shows that $\Gamma^{\rm para}(t) = \bigcup_{i=1}^{10} \Gamma^{\rm para}(t;1,2)C_i$. It is another simple calculation to see that $C_iC_j^{-1} \notin \Gamma^{\rm para}(t;1,2)$ when $i \neq j$ except in the case when t is even and $\{i,j\} = \{9,10\}$; this shows that the coset representatives are nonredundant except that we can omit C_{10} when t is even. $\square$

One can check that the permutations of cosets induced by the right multiplication of these generators as cycles in S_{10} (or in S_9 for even t) are: $\bar\alpha = (12)(34)(78)$ and $\bar\beta = (13)(24)(56)$ and $\bar\delta = (25)(37)(49)(68)$ in either case, whereas $\bar\gamma$ is the identity for even t and $(56)(78)(9\,10)$ for odd t.

We use the following notation for the subgroup of the symplectic group with "$C = 0$":

$$\Delta_2(\mathbb{F}) = \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in \mathrm{Sp}_2(\mathbb{F}) \right\}.$$

Proposition 14. *Let $t \in \mathbb{N}$.*

(1) *For t odd, we have*

$$\mathrm{Sp}_2(\mathbb{Z}) \subseteq \Gamma^{\mathrm{para}}(t;1,2)\Gamma_\infty(\mathbb{Z})\Delta_2(\mathbb{Q})\Gamma_\infty(\mathbb{Q}).$$

(2) *For t even but with $t/2$ odd, we have*

$$\begin{aligned}\mathrm{Sp}_2(\mathbb{Z}) \subseteq\ & \Gamma^{\mathrm{para}}(t;1,2)\Gamma_\infty(\mathbb{Z})\Delta_2(\mathbb{Q})\Gamma_\infty(\mathbb{Q}) \\ & \cup \Gamma^{\mathrm{para}}(t;1,2)\mu_t^{-1}\Gamma_\infty(\mathbb{Z})\mu_t\Delta_2(\mathbb{Q})\Gamma_\infty(\mathbb{Q}).\end{aligned}$$

Proof. From Lemma 12 and Lemma 13, we have that

$$\mathrm{Sp}_2(\mathbb{Z}) \subseteq \bigcup_{i=1}^{10} \Gamma^{\mathrm{para}}(t;1,2)C_i U \Gamma_\infty(\mathbb{Q}),$$

where U is as defined in Lemma 12. It is clear that $C_i \subseteq \Gamma_\infty(\mathbb{Z})\Delta_2(\mathbb{Q})$ for $i = 1, \dots, 6$, and so we have the inclusion

$$\Gamma^{\mathrm{para}}(t;1,2)C_i U\Gamma_\infty(\mathbb{Q}) \subseteq \Gamma^{\mathrm{para}}(t;1,2)\Gamma_\infty(\mathbb{Z})\Delta_2(\mathbb{Q})\Gamma_\infty(\mathbb{Q}) \text{ for these } i.$$

For the case where t is odd, we have

$$C_7 = \begin{pmatrix} 1 & -t & 0 & 0 \\ -1 & 1+t & 0 & 0 \\ -1-t & 0 & 1+t & 1 \\ -2t & t(t+1) & t & 1 \end{pmatrix} \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 1&0&1&0 \\ 0&0&0&1 \end{pmatrix} \begin{pmatrix} t+1 & t & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -t & t+1 \end{pmatrix}$$

and $C_8 = C_7 \begin{pmatrix} 1&0&1&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix}$ so that $C_7, C_8 \in \Gamma^{\mathrm{para}}(t;1,2)\Gamma_\infty(\mathbb{Z})\Delta_2(\mathbb{Q})$. And we have

$$C_9 = \begin{pmatrix} 1 & -t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -t & 1 & 0 \\ -t & t & t & 1 \end{pmatrix} \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 1&0&1&0 \\ 0&0&0&1 \end{pmatrix} \begin{pmatrix} 1 & t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -t & 1 \end{pmatrix}$$

and $C_{10} = C_9 \begin{pmatrix} 1&0&0&1 \\ 0&1&1&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix}$ so that $C_9, C_{10} \in \Gamma^{\mathrm{para}}(t;1,2)\Gamma_\infty(\mathbb{Z})\Delta_2(\mathbb{Q})$. Then $\Gamma^{\mathrm{para}}(t;1,2)C_iU\Gamma_\infty(\mathbb{Q}) \subseteq \Gamma^{\mathrm{para}}(t;1,2)\Gamma_\infty(\mathbb{Z})\Delta_2(\mathbb{Q})\Gamma_\infty(\mathbb{Q})$ for $i = 7, 8, 9, 10$ as well, which proves item (1).

For t even (and $t/2$ odd), we use that $C_7 = \mu_t^{-1}C_5\mu_t$ so that $C_7, C_8 \in \mu_t^{-1}\Gamma_\infty(\mathbb{Z})\mu_t\Delta_2(\mathbb{Q})$. The final case is C_9 when t is even. We will manipulate C_9u for any $u \in U$. Any $u \in U$ is of the form $u = \begin{pmatrix} 1 & c & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -c & 1 \end{pmatrix}$ with $c \in \mathbb{Z}$. So $C_9u = \begin{pmatrix} 1 & c & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & c & 1 & 0 \\ 0 & t & -c & 1 \end{pmatrix}$. For the case where c is odd, let $g = \gcd(c-t, c) = a(c-t)+bc$ for some $a, b \in \mathbb{Z}$. We can verify that

$$C_9u = \begin{pmatrix} 1 & t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -t & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & t(c+1) & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{c-t}{g} & 0 & -b & 0 \\ 0 & 1 & 0 & 0 \\ \frac{c}{g} & 0 & a & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & g & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -g & 1 \end{pmatrix} \begin{pmatrix} a+b & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ \frac{-t}{g} & 0 & \frac{c-t}{g} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and this proves $C_9u \in \Gamma^{\mathrm{para}}(t;1,2)\Gamma_\infty(\mathbb{Z})\Delta_2(\mathbb{Q})\Gamma_\infty(\mathbb{Q})$ when c is odd. For the case where c is even, we can verify that

$$\begin{aligned} C_9u = &\begin{pmatrix} 1 & tc/2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -tc/2 & 1 \end{pmatrix} \begin{pmatrix} 1-t/2 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &\cdot \mu_t^{-1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1+c^2/2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mu_t \begin{pmatrix} 1 & c & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -c & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -t/2 & 0 & 1-t/2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

One important note is that we are assuming that $t/2$ is odd so that $\begin{pmatrix} 1-t/2 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \Gamma^{\mathrm{para}}(t;1,2)$, and so the above proves

$$C_9u \in \Gamma^{\mathrm{para}}(t;1,2)\mu_t^{-1}\Gamma_\infty(\mathbb{Z})\mu_t\Delta_2(\mathbb{Q})\Gamma_\infty(\mathbb{Q})$$

when c is even. Thus

$$\begin{aligned} \Gamma^{\mathrm{para}}(t;1,2)C_9U\Gamma_\infty(\mathbb{Q}) \subseteq \; &\Gamma^{\mathrm{para}}(t;1,2)\Gamma_\infty(\mathbb{Z})\Delta_2(\mathbb{Q})\Gamma_\infty(\mathbb{Q}) \\ &\cup \Gamma^{\mathrm{para}}(t;1,2)\mu_t^{-1}\Gamma_\infty(\mathbb{Z})\mu_t\Delta_2(\mathbb{Q})\Gamma_\infty(\mathbb{Q}). \end{aligned}$$

The proof of item (2) is now complete. □

3. Hecke algebras

We recall the abstract Hecke algebra $\mathcal{H}_R(U,S)$. Let $U \subseteq S$ be a group contained in a semigroup inside of some larger group. For a ring R, let $\mathcal{L}_R(U,S)$ be the free R-module of finite linear combinations of the basis $U\backslash S$. A right action of U on $\mathcal{L}_R(U,S)$ is defined by $(Us)u \mapsto U(su)$, extended R-linearly. The invariant R-module is denoted

$$\mathcal{H}_R(U,S) = \{T \in \mathcal{L}_R(U,S) : \forall u \in U,\, Tu = T\}.$$

The right invariance of $\mathcal{H}_R(U,S)$ under U allows us to define a product $\mathcal{H}_R(U,S) \times\mathcal{L}_R(U,S) \to \mathcal{L}_R(U,S)$ by $(\sum_\alpha c_\alpha U s_\alpha)Us = \sum_\alpha c_\alpha U s_\alpha s$ for $c_\alpha \in R$ and $s_\alpha \in S$. The restriction of this product to $\mathcal{H}_R(U,S) \times \mathcal{H}_R(U,S) \to \mathcal{H}_R(U,S)$ makes $\mathcal{H}_R(U,S)$ an associative R-algebra and $\mathcal{H}_R(U,S)$ also acts on $\mathcal{L}_R(U,S)$ from the left. Here is a useful lemma that does not seem to be in [1].

Lemma 15. *Let $U_0 \subseteq S_0$ and $U \subseteq S$ be groups contained in semigroups inside of some larger groups. Let $i : (U_0,S_0) \to (U,S)$ be a relative homomorphism. Let R be a ring. If*

(L1) *There exists a subgroup $H \subseteq U$ such that $U = i(U_0)H$,*
(L2) *For all $s \in i(S_0)$, we have $sHs^{-1} \subseteq U$,*

then there is an R-algebra homomorphism $j : \mathcal{H}_R(U_0,S_0) \to \mathcal{H}_R(U,S)$ such that $j(\sum_\alpha c_\alpha U_0 x_\alpha) = \sum_\alpha c_\alpha U i(x_\alpha)$ for $c_\alpha \in R$ and $x_\alpha \in S_0$. Furthermore, if i is injective and

(L3) $i(S_0)i(S_0)^{-1} \cap U \subseteq i(U_0)$,

then j is injective.

Proof. Since $i(U_0) \subseteq U$ we may define a R-linear map $j : \mathcal{L}_R(U_0, S_0) \to \mathcal{L}_R(U, S)$ by $\sum_\alpha c_\alpha U_0 x_\alpha \mapsto \sum_\alpha c_\alpha U i(x_\alpha)$. To show that j restricts to an R-linear map on the Hecke algebras, select $T = \sum_\alpha c_\alpha U_0 x_\alpha \in \mathcal{H}_R(U_0, S_0)$. The right invariance of T under U_0 implies that $j(T)$ is right invariant under $i(U_0)$: $j(T)i(u_0) = j(Tu_0) = j(T)$. The right invariance of $j(T)$ under $h \in H$ follows from $(L2)$: $j(T)h = \sum_\alpha c_\alpha U i(x_\alpha)h = \sum_\alpha c_\alpha U \left(i(x_\alpha)h\, i(x_\alpha)^{-1}\right) i(x_\alpha) = \sum_\alpha c_\alpha U i(x_\alpha) = j(T)$. Since $U = i(U_0)H$ by (L1), we have $j(T) \in \mathcal{H}_R(U, S)$.

To show that $j : \mathcal{H}_R(U_0, S_0) \to \mathcal{H}_R(U, S)$ is a homomorphism it suffices to prove the commutativity of the following diagram:

$$\begin{array}{ccc} \mathcal{H}_R(U_0,S_0)\times\mathcal{L}_R(U_0,S_0) & \xrightarrow{\times} & \mathcal{L}_R(U_0,S_0) \\ j\times j\downarrow & & \downarrow j \\ \mathcal{H}_R(U,S)\times\mathcal{L}_R(U,S) & \xrightarrow{\times} & \mathcal{L}_R(U,S). \end{array}$$

We have

$$j(T(U_0x)) = \sum_\alpha c_\alpha U i(x_\alpha x) = \left(\sum_\alpha c_\alpha U i(x_\alpha)\right)(Ui(x)) = j(T)j(U_0x).$$

To show the injectivity of j given the injectivity of i and (L3), write $T = \sum_\alpha c_\alpha U_0 x_\alpha \in \mathcal{H}_R(U_0, S_0)$ with distinct cosets $U_0 x_\alpha$. It suffices to show that the cosets $j(U_0 x_\alpha) = U i(x_\alpha)$ are distinct. If $Ui(x_1) = Ui(x_2)$, then $i(x_1)i(x_2)^{-1} \in i(S_0)i(S_0)^{-1} \cap U \subseteq i(U_0)$ by (L3) and we conclude $U_0x_1 = U_0x_2$ by the injectivity of i. □

We will apply Lemma 15 with the following choices:

Lemma 16. *Consider the Hecke pairs (U_0, S_0) and (U, S) :*

$$\begin{aligned} U_0 &= \Gamma_1(1,2), \\ S_0 &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G\,\mathrm{Sp}_1^+(\mathbb{Z}) : ac, bd \text{ even}, ad - bc \in \mathbb{N} \right\}; \\ U &= \Gamma_2(1,2)_\infty, \\ S &= \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G\,\mathrm{Sp}_2^+(\mathbb{Z}) : A'C, B'D \text{ even matrices}, AD' - BC' \in \mathbb{N}I \right\} \end{aligned}$$

and the relative injection $i : (U_0, S_0) \to (U, S)$ given in Lemma 6. We have $i(S_0)i(S_0)^{-1} \cap U = i(U_0)$. Let $H \subseteq U$ be the subgroup given by $H = \pm w(H_e(\mathbb{Z}))$. H is a normal subgroup with $U = i(U_0)H$. In fact, for all $s \in i(S_0)$, we have $sHs^{-1} \subseteq H$. Therefore, (U_0, S_0) and (U, S) satisfy (L1), (L2) *and* (L3) *of Lemma 15.*

Proof. That H is a normal subgroup of $U = \Gamma_2(1,2)_\infty$ with $U = i(U_0)H$ follows from Lemma 6. We have

$$H = \left\{ \rho \begin{pmatrix} 1 & 0 & 0 & v \\ \lambda & 1 & v & k \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix} : v, k, \lambda \in \mathbb{Z}, \rho = \pm 1, k + v\lambda \text{ even} \right\}. \tag{3}$$

For condition (L2), take any $s = i(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)) \in i(S_0)$. So $ad - bc \in \mathbb{N}$ and ac, bd are even. Take a general $h \in H$ as in (3). Then

$$sHs^{-1} = \rho \begin{pmatrix} 1 & 0 & 0 & -b\lambda + av \\ d\lambda - cv & 1 & -b\lambda + av & (ad - bc)k \\ 0 & 0 & 1 & -d\lambda + cv \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

For this to be in H, we need that the following is even:

$$(ad - bc)k + (d\lambda - cv)(-b\lambda + av).$$

But this can be rearranged to

$$(ad - bc)(k + v\lambda) + 2bcv\lambda - acv^2 - bd\lambda^2$$

which is even because $(k + v\lambda), ac, bd$ are all even. Thus $sHs^{-1} \subseteq H$, and condition (L2) of Lemma 15 is also satisfied. The verification of $i(S_0)i(S_0)^{-1} \cap U = i(U_0)$ is easy. $\square$

Definition 17. For each $m \in \mathbb{N}$, consider the operator

$$T_m^{(1)} = \sum U_0 \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathcal{L}_{\mathbb{Z}}(U_0, S_0),$$

where the sum is over $a, b, d \in \mathbb{N}$ with $ad = m$, $0 \le b < 2d$, and $a, (b + d)$ both odd.

Lemma 18. *For each* $m \in \mathbb{N}$, $T_m^{(1)} \in \mathcal{H}_{\mathbb{Z}}(U_0, S_0)$.

Proof. First note that the left cosets in the above sum are disjoint because

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ 0 & d_2 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{a}{a_2} & \frac{b}{d_2} - \frac{b_2}{d} \\ 0 & \frac{a_2}{a} \end{pmatrix},$$

and the only way that this could be in U_0 is if $a = a_2$, hence $d = d_2$, and $\frac{b - b_2}{d}$ is even, which means that $b - b_2$ would have to be a multiple of $2d$. Next, we will show that $T_m^{(1)}$ is right invariant by elements from U_0. Since $U_0 = \Gamma_1(1,2)$ is generated as a group by the two elements $\left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, we only need to show right invariance by these two elements. In fact, because the above cosets are disjoint, we only need to show that a coset representative multiplied on the right by these generators always lands in another of the cosets above.

First, we can easily calculate that

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2\ell \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b + 2a - 2\ell d \\ 0 & d \end{pmatrix}.$$

By picking $\ell \in \mathbb{Z}$ such that $0 \le b+2a-2\ell d < 2d$ and noting that $(b+2a-2\ell d)$ has the same parity as b, then $\left(\begin{smallmatrix} a & b+2a-2\ell d \\ 0 & d \end{smallmatrix}\right)$ is one of the coset representatives.

Second, let $u = \gcd(b,d)$, so that u is odd. Let $x, y \in \mathbb{Z}$ such that $bx+dy = u$. Since u is odd, we can choose x, y such that b, x have the same parity and d, y have the same parity (Just replace by $b(x+d)+d(y-b) = u$ if necessary). Let $A = \begin{pmatrix} -b/u & y \\ -d/u & -x \end{pmatrix}$. Note $A \in U_0$. One can easily verify that

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = A\begin{pmatrix} u & -ax \\ 0 & ad/u \end{pmatrix} = A\begin{pmatrix} 1 & 2\ell \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u & -ax-2\ell ad/u \\ 0 & ad/u \end{pmatrix},$$

where we choose $\ell \in \mathbb{Z}$ so that $0 \le -ax - 2\ell ad/u < 2ad/u$. Since ax has the same parity as b, and since $\frac{ad}{u}$ has the same parity as d, then $\begin{pmatrix} u & -ax-2\ell ad/u \\ 0 & ad/u \end{pmatrix}$ is one of the coset representatives, and we have shown that $T_m^{(1)}$ is right invariant by U_0. $\square$

Corollary 19. *We may define*

$$T_m = j(T_m^{(1)}) = \sum \Gamma_2(1,2)_\infty i\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right) \in \mathcal{H}_{\mathbb{Z}}(U,S),$$

where the sum is over $a, b, d \in \mathbb{N}$ with $ad = m$, $0 \le b < 2d$, and $a, (b+d)$ both odd.

Proof. Lemmas 15, 16 and 18 imply that $j(T_m^{(1)})$ is in $\mathcal{H}_{\mathbb{Z}}(U,S)$. $\square$

4. Jacobi forms and Siegel forms and the lift

For $r \in \mathbb{Q}$ and $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_g(\mathbb{R})$ and $\Omega \in \mathcal{H}_g$, we set

$$(f|_r\gamma)(\Omega) = \det(C\Omega + D)^{-r} f(\gamma \circ \Omega),$$

for the choice of holomorphic root on $\mathcal{H}_g$ determined by the condition that $\det(\Omega/i)^r > 0$ for $\Omega = iY$ with $Y \in \mathcal{P}_g(\mathbb{R})$. Let Γ be a subgroup commensurable with Γ_g. A holomorphic function $f : \mathcal{H}_g \to \mathbb{C}$ is a modular form of weight r with respect to Γ and a map $v : \Gamma \to \mathbb{C}^*$ if

$$\forall \gamma \in \Gamma, \forall \Omega \in \mathcal{H}_g, \quad (f|_r\gamma)(\Omega) = v(\gamma)\, f(\Omega),$$

and if additionally, for all $\gamma \in \Gamma_g$ and for all $Y_0 \in \mathcal{P}_g(\mathbb{R})$, $f|\gamma$ is bounded on domains of type $\{\Omega \in \mathcal{H}_g : \operatorname{Im}\Omega > Y_0\}$. By a result of Koecher, this boundedness condition is redundant for $g \ge 2$. We denote by $M_r(\Gamma, v)$ the vector space of such functions and use the notation $M_r(\Gamma)$ when the map v is identically 1. The space $M_r(\Gamma, v)$ is trivial unless $\mu(\gamma, \Omega) = \det(C\Omega + D)^r v(\gamma)$ is a factor of automorphy; that is, $\mu : \Gamma \times \mathcal{H}_g \to \mathbb{C}^*$ satisfies the cocycle condition: $\mu(\gamma_1\gamma_2, \Omega) = \mu(\gamma_1, \gamma_2 \circ \Omega)\mu(\gamma_2, \Omega)$. For integral weights k, $\det(C\Omega + D)^k$ is already a factor of automorphy and hence $v : \Gamma \to \mathbb{C}^*$ is a character.

The definition of the theta function is in Section 5. The transformation formula for the theta function, see pages 176 and 182 of [15],

$$\exists v_\theta^{(g)} : \Gamma_g(1,2) \to e\,(1/8) : \forall \gamma \in \Gamma_g(1,2), \quad \theta[0]|_{1/2}\gamma = v_\theta^{(g)}(\gamma)\theta[0],$$

gives an example of a Siegel modular form of weight 1/2; the standard thetanull $\theta[0](0,\Omega)$ gives an element of $M_{1/2}(\Gamma_g(1,2), v_\theta^{(g)})$. We write $v_\theta = v_\theta^{(g)}$ when the degree g is clear from the context.

For holomorphic $f : \mathcal{H}_g \to \mathbb{C}$ we define the Siegel operator

$$\Phi(f)(\Omega_1) = \lim_{\lambda\to+\infty} f\begin{pmatrix}\Omega_1 & 0\\ 0 & i\lambda\end{pmatrix}$$

when this limit exists for all $\Omega_1 \in \mathcal{H}_{g-1}$. In particular, this operator maps $M_r(\Gamma_g)$ to $M_r(\Gamma_{g-1})$ and $M_r(\Gamma_g(1,2))$ to $M_r(\Gamma_{g-1}(1,2))$, see [8] for details. A modular form is a cusp form if $\forall\gamma \in \Gamma_g$, $\Phi(f|_r\gamma) = 0$. We shall denote by $S_r(\Gamma, v)$ the subspace of cusp forms and use the notation $S_r(\Gamma)$ when v is identically 1. We let $e(z) = e^{2\pi i z}$ for $z \in \mathbb{C}$.

Definition 20. Let $k, m \in \mathbb{Q}$. Let $\Gamma \subseteq \Gamma_\infty(\mathbb{Z})$ and fix a map $v : \Gamma \to \mathbb{C}^*$. The Jacobi forms with respect to Γ and v, denoted $J_{k,m}(\Gamma, v)$, are the vector space of holomorphic $\phi : \mathcal{H}_1 \times \mathbb{C} \to \mathbb{C}$ such that for all $\gamma \in \Gamma$, we have $\tilde\phi|_k\gamma = v(\gamma)\,\tilde\phi$, where we define

$$\tilde\phi\left(\begin{pmatrix}\tau & z\\ z & w\end{pmatrix}\right) = \phi(\tau, z)e(mw),$$

and for all $\gamma \in \Gamma_\infty(\mathbb{Z})$, we have that the Fourier expansion for $\tilde\phi|_k\gamma$ is supported on semidefinite index matrices, namely $(\tilde\phi|_k\gamma)((\begin{smallmatrix}\tau & z\\ z & w\end{smallmatrix})) = \sum_{s\geq 0} c(s)e(\mathrm{tr}(s(\begin{smallmatrix}\tau & z\\ z & w\end{smallmatrix})))$, where $s \geq 0$ indicates s is summed over only semidefinite 2×2 matrices. Furthermore, we say ϕ is a Jacobi cusp form and write $\phi \in J^{\mathrm{cusp}}_{k,m}(\Gamma, v)$ if for all $\gamma \in \Gamma_\infty(\mathbb{Z})$, we have that the Fourier expansion for $\tilde\phi|_k\gamma$ has no nonzero coefficients at indefinite index matrices, namely $(\tilde\phi|_k\gamma)((\begin{smallmatrix}\tau & z\\ z & w\end{smallmatrix})) = \sum_{s>0} c(s)e(\mathrm{tr}(s(\begin{smallmatrix}\tau & z\\ z & w\end{smallmatrix})))$, where $s > 0$ indicates s is summed over only positive definite 2×2 matrices. When v is identically 1, we write $J_{k,m}(\Gamma) = J_{k,m}(\Gamma, v)$ and similarly for cusp forms.

We study $J_{k,t/2}(\Gamma_2(1,2)_\infty, v_\theta^{2k})$ for $2k, t \in \mathbb{N}$ in this article. Note that $\Gamma_2(1,2)_\infty$ contains translation matrices of the form $(\begin{smallmatrix}I & S\\ 0 & I\end{smallmatrix})$ where S is symmetric integral with even diagonal entries. This implies that $\phi \in J^{\mathrm{cusp}}_{k,t/2}(\Gamma_2(1,2)_\infty)$ has a Fourier expansion of the form

$$\phi(\tau, z) = \sum_{\substack{n,r\in\mathbb{Z}:\\ tn-r^2>0,\, n>0}} c(n, r)e(\tfrac{1}{2}n\tau + rz).$$

For $g = 2$, the Fourier Jacobi expansion of $\theta[0] \in M_{1/2}(\Gamma_2(1,2), v_\theta)$,

$$\theta[0]\begin{pmatrix}\tau & z\\ z & \omega\end{pmatrix} = \theta[0](0,\tau) + 2\theta[0](z,\tau)e(\omega/2) + \cdots,$$

shows that $\theta[0](z,\tau)$ is automorphic with respect to $\Gamma_2(1,2)\cap\Gamma_\infty(\mathbb{Z}) = \Gamma_2(1,2)_\infty$ of weight 1/2 and index 1/2. Thus $\theta[0](z,\tau)$ gives an element of $J_{\frac{1}{2},\frac{1}{2}}(\Gamma_2(1,2)_\infty, v_\theta)$.

The definition of Jacobi form above is equivalent to that in [6]. The group $\Gamma_1(1,2)^J = \Gamma_1(1,2) \ltimes H_e(\mathbb{Z})$ is isomorphic to $\Gamma_2(1,2)_\infty/\{\pm I\}$ by Lemma 6, and this shows the equivalence to the usual definition by taking generators of $\Gamma_1(1,2)$ and $H_e(\mathbb{Z})$. These transformations are

$$\forall \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(1,2),$$
$$\phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = v\left(i\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)(c\tau+d)^k e\left(\frac{cmz^2}{c\tau+d}\right)\phi(\tau,z),$$
$$\forall (\lambda, v, \kappa) \in H_e(\mathbb{Z}),$$
$$\phi(\tau, z+\lambda\tau+v) = v\left(w(\lambda,v,\kappa)\right)e\left(m(\lambda^2\tau+2\lambda z+(\lambda v+\kappa))\right)\phi(\tau,z).$$

The first equation shows that if $\phi \in J_{k,m}(\Gamma, v)$, then $\phi(\tau, 0)$ gives an element of $J_{k,0}(\Gamma, v)$. Using the isomorphism $M_k(i^{-1}(\Gamma), i^*v) = J_{k,0}(\Gamma, v)$ we have $M_{k_1}(i^{-1}(\Gamma), i^*v_1)J_{k_2,m}(\Gamma, v_2) \subseteq J_{k_1+k_2,m}(\Gamma, v_1v_2)$. We use this containment in the statement of Corollary 2 to write

$$S_{k-\frac{1}{2}}(\Gamma_1(1,2), (v_\theta^{(1)})^{2k-1})\theta[0](z,\tau) \subseteq J^{\mathrm{cusp}}_{k,\frac{1}{2}}(\Gamma_1(1,2)^J, (v_\theta^{(2)})^{2k}).$$

Here one needs to check that $i^*(v_\theta^{(2)}) = v_\theta^{(1)}$ on $\Gamma_1(1,2)$. This can be done by restricting the theta function to diagonal $\left(\begin{smallmatrix} \tau & 0 \\ 0 & \omega \end{smallmatrix}\right) \in \mathcal{H}_2$.

Definition 21. Fix $t \in \mathbb{N}$ and $k \in \mathbb{Z}$. For $\phi \in J^{\mathrm{cusp}}_{k,\frac{1}{2}t}(\Gamma_2(1,2)_\infty)$, define

$$\tilde{\phi}\left(\begin{pmatrix} \tau & z \\ z & w \end{pmatrix}\right) = \phi(\tau,z)e(\tfrac{1}{2}tw).$$

Define a formal series F_ϕ by

$$F_\phi = \sum_{m=1}^{\infty} m^{2-k}(-1)^{m+1}\tilde{\phi}\,|_k\, T_m = \sum_{m=1}^{\infty}\sum_{a,b,d} m^{2-k}(-1)^{m+1}\tilde{\phi}\,|_k\, i\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right),$$

where the inner sum is over $a, b, d \in \mathbb{N}$ with $ad = m$, $0 \le b < 2d$, and $a, (b+d)$ both odd.

Proposition 22. *Let $\phi \in J^{\mathrm{cusp}}_{k,\frac{1}{2}t}(\Gamma_2(1,2)_\infty)$ have expansion*

$$\phi(\tau,z) = \sum_{n,r\in\mathbb{Z}:\, tn-r^2>0,\, n>0} c(n,r)e(\tfrac{1}{2}n\tau + rz).$$

Then the formal series $F_\phi(\Omega)$ may be rearranged to

$$F_\phi(\Omega) = \sum_{\substack{T=\left(\begin{smallmatrix} n & r \\ r & m \end{smallmatrix}\right):\, t|m,\\ mn-r^2>0,\, n>0, m>0}} a(T)e(\tfrac{1}{2}\mathrm{tr}(T\Omega)),$$

where the coefficients $a(T)$ are given by

$$a\left(\begin{pmatrix} n & r \\ r & m \end{pmatrix}\right) = (-1)^{(m/t+1)(n+1)} \sum_{\substack{a|(n,r,m/t)\\ a \text{ odd}}} a^{k-1}c\left(\frac{mn}{ta^2}, \frac{r}{a}\right). \tag{4}$$

Proof. This is a standard type of calculation, see [10] or [3], page 104. One requires only the formula: for $n, m, d \in \mathbb{N}$ with m/d odd,

$$\sum_{\substack{b \in \mathbb{Z}:\ 0 \le b < 2d, \\ b+d \text{ odd}}} e\left(\frac{nb}{2d}\right) = \begin{cases} d(-1)^{(m+1)\frac{n}{d}}, & \text{if } d|n, \\ 0, & \text{otherwise.} \end{cases}$$

□

Proposition 23. *Fix $t \in \mathbb{N}$ and $k \in \mathbb{Z}$. Let $\phi \in J^{\mathrm{cusp}}_{k,\frac{1}{2}t}(\Gamma_2(1,2)_\infty)$. The series $F_\phi(\Omega)$ converges absolutely for all $\Omega \in \mathcal{H}_2$ and $F_\phi : \mathcal{H}_2 \to \mathbb{C}$ defines a holomorphic function. Also, for $\left(\begin{smallmatrix} \tau & z \\ z & w \end{smallmatrix}\right) \in \mathcal{H}_2$ we have*

$$F_\phi\left(\begin{pmatrix} tw & z \\ z & \frac{1}{t}\tau \end{pmatrix}\right) = F_\phi\left(\begin{pmatrix} \tau & z \\ z & w \end{pmatrix}\right).$$

Proof. Since ϕ has its Fourier coefficients $c(n, r; \phi)$ bounded by polynomial growth, F_ϕ also has its Fourier coefficients $a(\left(\begin{smallmatrix} n & r \\ r & m \end{smallmatrix}\right))$ bounded by polynomial growth. This suffices to show the absolute convergence of F_ϕ on compact subsets of $\mathcal{H}_2$. The stated transformation of F_ϕ is equivalent to the identity on the Fourier coefficients, for $t|m$,

$$a\left(\begin{pmatrix} m/t & r \\ r & nt \end{pmatrix}\right) = a\left(\begin{pmatrix} n & r \\ r & m \end{pmatrix}\right)$$

which follows from the equation (4). □

Proposition 24. *Fix $t \in \mathbb{N}$ and let ϕ and F_ϕ be as in Proposition 23. Then $F_\phi\,|_k\,\mu_t = (-1)^k F_\phi$.*

Proof. Note that $\tilde{\phi}\,|_k\, i(\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)) = \tilde{\phi}$ implies $\phi(\tau, -z) = (-1)^k \phi(\tau, z)$ and so $c(n, -r; \phi) = (-1)^k c(n, r; \phi)$. We have

$$\begin{aligned} (F_\phi\,|_k\,\mu)\left(\begin{pmatrix} \tau & z \\ z & w \end{pmatrix}\right) &= F_\phi\left(\begin{pmatrix} tw & -z \\ -z & \frac{1}{t}\tau \end{pmatrix}\right) \\ &= (-1)^k F_\phi\left(\begin{pmatrix} tw & z \\ z & \frac{1}{t}\tau \end{pmatrix}\right) = (-1)^k F_\phi\left(\begin{pmatrix} \tau & z \\ z & w \end{pmatrix}\right). \end{aligned}$$

□

The following theorem completes the proof of Theorem 1 from Introduction. The form of the Fourier coefficients has already been given in Proposition 22.

Theorem 25. *Let $t \in \mathbb{N}$ and $k \in \mathbb{Z}$. For $\phi \in J^{\mathrm{cusp}}_{k,\frac{1}{2}t}(\Gamma_2(1,2)_\infty)$, we have $F_\phi \in M_k(\Gamma^{\mathrm{para}}(t; 1, 2))$ and $F_\phi|\mu_t = (-1)^k F_\phi$. If $t \not\equiv 0 \mod 4$, then we have $F_\phi \in S_k(\Gamma^{\mathrm{para}}(t; 1, 2))$.*

Proof. We know that F_ϕ is holomorphic from Proposition 23. From Definition 21, we know that F_ϕ is invariant under $\Gamma_2(1,2)_\infty$ because the series defining it is term by term invariant. From the form of F_ϕ in Proposition 22, it is clear that F_ϕ is invariant under γ_t and so F_ϕ is invariant under $\Gamma_2(1,2)_\infty[t]$. From Proposition 24, we know $F_\phi\,|_k\,\mu_t = (-1)^k F_\phi$ and therefore F_ϕ is invariant under $G_t = \langle \Gamma_2(1,2)_\infty[t],\, \mu_t \Gamma_2(1,2)_\infty[t] \mu_t \rangle = \Gamma^{\mathrm{para}}(t; 1, 2)$ by Proposition 11.

We only need to prove that F_ϕ is a cusp form when $t \not\equiv 0 \mod 4$. Take any $\beta \in \mathrm{Sp}_2(\mathbb{Z})$. Since $t \not\equiv 0 \mod 4$, by Proposition 14, we have that $\beta = \alpha\gamma_1\delta\gamma_2$, or $\beta = \alpha\mu_t^{-1}\gamma_1\mu_t\delta\gamma_2$, where $\alpha \in \Gamma^{\mathrm{para}}(t;1,2)$, $\delta \in \Delta_2(\mathbb{Q})$ and $\gamma_1 \in \Gamma_\infty(\mathbb{Z})$ and $\gamma_2 \in \Gamma_\infty(\mathbb{Q})$. Then $F_\phi|\beta = F_\phi|\gamma_1\delta\gamma_2$ or $F_\phi|\beta = (-1)^k F_\phi|\gamma_1\mu_t\delta\gamma_2$.

Since F_ϕ has no nonzero indefinite coefficients in its Fourier expansion, and since $\gamma_1 \in \Gamma_\infty(\mathbb{Z})$, we have that $F_\phi|\gamma_1$ has no nonzero indefinite coefficients. Since δ and $\mu_t\delta$ are upper triangular, then $F_\phi|\gamma_1\delta$ and $F_\phi|\gamma_1\mu_t\delta$ have no nonzero indefinite coefficients either; these two cases can be unified together by saying that $F_\phi|\beta\gamma_2^{-1}$ has no nonzero indefinite coefficients.

Consider the Siegel operator $(\Phi_2 f)(\tau) = \lim_{s\to\infty} f(\left(\begin{smallmatrix} \tau & 0 \\ 0 & is \end{smallmatrix}\right))$ for a modular form f. Since $\gamma_2 \in \Gamma_\infty(\mathbb{Q})$ and $(f|\gamma_2)(\left(\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right)) = (*)f(\left(\begin{smallmatrix} * & * \\ * & \omega+* \end{smallmatrix}\right))$ where the $*$ depend only on τ, z and not on ω, then $\Phi_2 f = 0$ would imply $\Phi_2(f|\gamma_2) = 0$. Thus

$$\Phi_2(F_\phi|\beta) = \Phi_2((F_\phi|\beta\gamma_2^{-1})|\gamma_2) = 0.$$

Since this is true for all $\beta \in \mathrm{Sp}_2(\mathbb{Z})$, F_ϕ is a cusp form. $\square$

When $t = 1$ and k is even, we get the following corollary which we state as a theorem because it is of particular interest for the degree two chiral superstring Siegel modular form of D'Hoker and Phong.

Theorem 26 (Lifting to Degree Two Theta Group for even k)**.** *Let $k \in \mathbb{N}$ be even and $\phi \in J^{\mathrm{cusp}}_{k,\frac{1}{2}}(\Gamma_2(1,2)_\infty)$. Then $F_\phi \in S_k(\Gamma_2(1,2))$.*

Corollary 27. *For $t = 1$, if $k \in \mathbb{N}$ is odd and $\phi \in J^{\mathrm{cusp}}_{k,\frac{1}{2}}(\Gamma_2(1,2)_\infty)$, then $F_\phi = 0$.*

Proof. Since k is odd, then by Proposition 24, $F_\phi|\mu_t = -F_\phi$. Let

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \epsilon = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Note that both $g, \epsilon \in \Gamma_2(1,2)_\infty$ (see Lemma 6) and so $F_\phi|g = F_\phi$ and $F_\phi|\epsilon = F_\phi$. But it is straightforward to check that

$$\mu_1 g\epsilon\mu_1 g^{-1}\epsilon\mu_1 g = I$$

is the identity matrix. But $F_\phi|(\mu_1 g\epsilon\mu_1 g^{-1}\epsilon\mu_1 g) = (-1)^3 F_\phi = -F_\phi$ and $F_\phi|I = F_\phi$. This forces $F_\phi = 0$. $\square$

5. The chiral string modular form in genus 2

Now we discuss the weight 15/2 form that lifts to $\Xi_2[0]$. For $\Omega \in \mathcal{H}_g$, $z \in \mathbb{C}^g$ and $a, b \in \mathbb{R}^g$, define the theta function with characteristics a and b as a holomorphic function on $\mathbb{C}^g \times \mathcal{H}_g$ given by the series

$$\theta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](z,\Omega) = \sum_{n\in\mathbb{Z}^g} e\left(\frac{1}{2}(n+a)'\Omega(n+a) + (n+a)'(n+z+b)\right).$$

In $g = 1$, we use the standard abbreviations $\theta_{ab}(\tau) = \theta\begin{bmatrix} a/2 \\ b/2 \end{bmatrix}(0, \tau)$ for $a, b \in \{0, 1\}$. In $g = 2$, we use $\theta\left(\begin{smallmatrix} a_1 & a_2 \\ b_1 & b_2 \end{smallmatrix}\right)(\Omega) = \theta\begin{bmatrix} a_1/2 & a_2/2 \\ b_1/2 & b_2/2 \end{bmatrix}(0, \Omega)$. The Dedekind eta function is the standard one and we define

$$\Psi = \theta_{00}^3\, \eta^{12} \in S_{15/2}(\Gamma_1(1,2), v_\theta^{15}).$$

Consider a form $g \in S_{k-\frac{1}{2}}(\Gamma_1(1,2), v_\theta^{2k-1})$ whose Fourier expansion is $g(\tau) = \sum_{n\in\mathbb{N}} c(n; g) q^{n/2}$. Multiplication by $\theta[0] \in J_{1/2,1/2}(\Gamma_1(1,2)^J, v_\theta)$ whose Fourier expansion is $\theta[0](z,\tau) = \sum_{n\in\mathbb{Z}} q^{n^2/2}\zeta^n$ produces a Jacobi form $\phi \in J^{\mathrm{cusp}}_{k,1/2}(\Gamma_1(1,2)^J, v_\theta^{2k})$ whose Fourier expansion is $\phi(\tau, z) = g(\tau)\theta[0](z,\tau) = \sum_{n\in\mathbb{N}, r\in\mathbb{Z}} c(n - r^2; g) q^{n/2}\zeta^r$. Note that when $4|k$, ϕ has trivial multiplier. In this case we have $c(n, r; \phi) = c(n - r^2; g)$ and the formula for the Fourier coefficients of the lift $L(\phi)$ is

$$a\left(\begin{pmatrix} n & r \\ r & m \end{pmatrix}; L(\phi)\right) = (-1)^{(m+1)(n+1)} \sum_{\substack{a|(n,r,m) \\ a \text{ odd}}} a^{k-1} c\left(\frac{mn - r^2}{a^2}; g\right).$$

This proves the formula for the Fourier coefficients of the chiral superstring form $\Xi_2[0]$ that was given at the conclusion of the Introduction.

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