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PRECISE ASYMPTOTICS OF MOVING AVERAGE PROCESS UNDER ϕ-MIXING ASSUMPTION

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Abstract. In the paper by Liu and Lin (Statist. Probab. Lett. **76** (2006), no. 16, 1787–1799), a new kind of precise asymptotics in the law of large numbers for the sequence of i.i.d. random variables, which includes complete convergence as a special case, was studied. This paper is devoted to the study of this new kind of precise asymptotics in the law of large numbers for moving average process under ϕ-mixing assumption and some results of Liu and Lin [6] are extended to such moving average process.

1. Introduction

Let $\{\xi_i, -\infty < i < \infty\}$ be a doubly infinite sequence of identically distributed random variables with zero means and finite variances, and let $\{a_i, -\infty < i < \infty\}$ be an absolutely summable sequence of real numbers. Let

$$X_k = \sum_{i=-\infty}^{+\infty} a_{i+k}\xi_i, \quad k \geq 1, \tag{1.1}$$

be a moving average process based on $\{\xi_i, -\infty < i < \infty\}$. Denote $S_n = \sum_{k=1}^{n} X_k (n \geq 1)$ as the sequence of partial sums.

Under the assumption that $\{\xi_i, -\infty < i < \infty\}$ is a sequence of independent identically distributed random variables, many limiting results have been obtained for the moving average process $\{X_k, k \geq 1\}$. Burton and Dehling [2] got a large deviation principle; Yang [8] established the central limit theorem (CLT) and the law of the iterated logarithm (LIL); Li et al. [4] obtained the complete convergence. Under the assumption of dependence, Zhang [9] generalized the results of Li's to ϕ-mixing with slowly varying functions; Li [5] established the theorem of precise asymptotics in the law of large numbers of moving average process under ϕ-mixing as follows:

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Theorem A. *Suppose that $\{X_k, k \geq 1\}$ is defined as in* (1.1)*, where $\{a_i, -\infty < i < \infty\}$ is a sequence of real number with $\sum_{i=-\infty}^{\infty} |a_i| < \infty$, and $\{\xi_i, -\infty < i < \infty\}$ is a sequence of identically distributed ϕ-mixing random variables with $\mathsf{E}\xi_1 = 0$, $0 < \mathsf{E}\xi_1^2 < \infty$, $0 < \sigma^2 = \mathsf{E}\xi_1^2 + 2\sum_{i=2}^{n} \mathsf{E}\xi_1\xi_i < \infty$, and $\sum_{m=1}^{\infty} \phi^{1/2}(m) < \infty$. For $1 \leq p < 2$ and $r > p$, if $\mathsf{E}|\xi_1|^r < \infty$, then*

$$\lim_{\epsilon \searrow 0} \epsilon^{2(r-p)/(2-p)} \sum_{n=1}^{\infty} n^{r/p-2} \mathsf{P}\{|S_n| \geq n^{1/p}\epsilon\} = \frac{p}{r-p} \mathsf{E}|Z|^{2(r-p)/(2-p)}, \tag{1.2}$$

where $Z \sim N\big(0, \sigma^2 (\sum_{i=-\infty}^{\infty} a_{ni})^2\big)$.

Substituting $p = 1$, $r = 2$ into (1.2), we obtain

$$\lim_{\epsilon \searrow 0} \epsilon^2 \sum_{n=1}^{\infty} \mathsf{P}\{|S_n| \geq n\epsilon\} = \mathsf{E}|Z|^2. \tag{1.3}$$

Here we will discuss a new kind of complete convergence. Liu and Lin [6] have discussed the precise rates of convergence of $\sum_{n=1}^{\infty} n^{-q} \mathsf{E}|S_n|^q I\{|S_n| \geq n\epsilon\}$ for sequence of independent identically distributed random variables in the situation of $0 \leq q \leq 2$. Inspired by Liu et al. [6] and Zhang [9], here we consider precise asymptotics in the law of large numbers for moving average process under ϕ-mixing assumption. It's easy to see that $S_n = \sum_{k=1}^{n} X_k = \sum_{k=1}^{n} \xi_k$ by defining $a_{i+k} = 1$ if $i = k$, and $a_{i+k} = 0$ otherwise, then X_k is a special case of a moving average process. Results analogous to Theorems 1, 2 in Liu and Lin [6] can also be derived for mixing random variables. We mainly use the technique due to Zhang [9] to prove our results based on Theorem A.

Now we give some definitions of mixing random variables. For $m \geq 1$, define

$$\phi(m) := \sup_{k \geq 1} \{|\mathsf{P}(B|A) - \mathsf{P}(B)|,\ A \in \mathcal{F}_{-\infty}^{k},\ \mathsf{P}(A) \neq 0,\ B \in \mathcal{F}_{k+m}^{\infty}\};$$

$$\rho(m) := \sup_{k \geq 1} \{\mathrm{Corr}(U, V);\ U \in \mathrm{L}^2(\mathcal{F}_1^k),\ V \in \mathrm{L}^2(\mathcal{F}_{k+m}^{\infty})\},$$

where $\mathcal{F}_a^b = \sigma(\xi_i, a \leq i \leq b)$. $\{\xi_i, i \geq 1\}$, a sequence of random variables, is called ϕ-mixing if $\phi(m) \to 0$ and ρ-mixing if $\rho(m) \to 0$ as $m \to \infty$. It is well known that $\rho(m) \leq 2\phi^{1/2}(m)$ and hence a ϕ-mixing sequence is also ρ-mixing.

Throughout the following, C will be used to represent a positive constant although its value may change from one appearance to the next, and $[x]$ to denote the largest integer not greater than x.

2. Main results and some lemmas

Theorem 2.1. *Suppose that $\{X_k, k \geq 1\}$ is defined as in* (1.1)*, where $\{a_i, -\infty < i < \infty\}$ is a sequence of real number with $\sum_{i=-\infty}^{\infty} |a_i| < \infty$, and $\{\xi_i, -\infty < i < \infty\}$ is a sequence of identically distributed ϕ-mixing random variables with $\sum_{m=1}^{\infty} \phi^{1/2}(m) < \infty$, and*

$$\mathsf{E}\xi_1 = 0, \quad 0 < \mathsf{E}\xi_1^2 < \infty;$$

$$0 < \sigma^2 = \mathsf{E}\xi_1^2 + 2\sum_{i=2}^{n} \mathsf{E}\xi_1\xi_i < \infty,\ \mathsf{E}\xi_1^2 \log^+ |\xi_1| < \infty.$$

Then we have

$$\lim_{\epsilon\searrow 0} \frac{1}{-\log \epsilon} \sum_{n=1}^{\infty} \frac{1}{n^2} \mathsf{E}S_n^2 I\{|S_n| \geq n\epsilon\} = 2\sigma^2 \Big(\sum_{i=-\infty}^{\infty} a_{ni} \Big)^2 =: 2\tau^2. \tag{2.1}$$

Theorem 2.2. *Suppose that* $\{X_k, k \geq 1\}$ *is defined as in* (1.1)*, where* $\{a_i, -\infty < i < \infty\}$ *is a sequence of real number with* $\sum_{i=-\infty}^{\infty} |a_i| < \infty$*, and* $\{\xi_i, -\infty < i < \infty\}$ *is a sequence of identically distributed* ϕ*-mixing random variables with* $\sum_{m=1}^{\infty} \phi^{1/2}(m) < \infty$*, and*

$$\mathsf{E}\xi_1 = 0, \quad 0 < \mathsf{E}\xi_1^2 < \infty, \quad 0 < \sigma^2 = \mathsf{E}\xi_1^2 + 2\sum_{i=2}^{n} \mathsf{E}\xi_1\xi_i < \infty.$$

Then we have

$$\lim_{\epsilon\searrow 0} \epsilon^{2-p} \sum_{n=1}^{\infty} \frac{1}{n^p} \mathsf{E}|S_n|^p I\{|S_n| \geq n\epsilon\} = \frac{2}{2-p}\tau^2 \tag{2.2}$$

for $0 < p < 2$.

It needs the following lemmas to prove the above results.

Lemma 2.1 ([2])**.** *Let* $\sum_{i=-\infty}^{+\infty} a_i$ *be an absolutely convergent series of real numbers with* $a = \sum_{i=-\infty}^{+\infty} a_i$ *and* $k \geq 1$*. Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=-\infty}^{+\infty} \Big| \sum_{j=i+1}^{i+n} a_j \Big|^k = |a|^k.$$

Lemma 2.2 ([7])**.** *Let* $\{X_i, i \geq 1\}$ *be a sequence of* ϕ*-mixing random variables with zero means and finite second moments. Let* $S_n = \sum_{i=1}^{n} X_i$*. If exists* C_n *such that* $\max_{1\leq i\leq n} \mathsf{E}S_n^2 \leq C_n$*, then for all* $q \geq 2$*, there exists* $C = C(q, \phi(\cdot))$ *such that*

$$\mathsf{E} \max_{1\leq i\leq n} |S_i|^q \leq C\big(C_n^{q/2} + \mathsf{E} \max_{1<i\leq n} |X_i|^q\big).$$

3. Proof of Theorem 2.1

Without loss of generality, we assume $\tau^2 = 1$. It's easy to see

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \mathsf{E}S_n^2 I\{|S_n| \geq n\epsilon\} = \epsilon^2 \sum_{n=1}^{\infty} \mathsf{P}(|S_n| \geq n\epsilon) + \sum_{n=1}^{\infty} \frac{1}{n^2} \int_{n\epsilon}^{\infty} 2x\mathsf{P}(|S_n| \geq x)dx. \tag{3.1}$$

Proof of Theorem 2.1. Let $b(\epsilon) = [\epsilon^{-2}]$. With (1.3), we only need to prove

$$\lim_{\epsilon\searrow 0} \frac{1}{-\log \epsilon} \sum_{n=1}^{\infty} \frac{1}{n^2} \int_{n\epsilon}^{\infty} 2x\mathsf{P}(|S_n| \geq x)dx = 2. \tag{3.2}$$

By Proposition 3.1 in [6], (3.2) holds if we can prove the following propositions. □

Proposition 3.1. *One has*

(3.3)
$$\lim_{\epsilon\searrow 0}\frac{1}{-\log\epsilon}\sum_{n=1}^{b(\epsilon)}\frac{1}{n^2}\left|\int_{n\epsilon}^{\infty}2xP(|S_n|\ge x)dx-\int_{n\epsilon}^{\infty}2xP\big(|N|\ge x/\sqrt{n}\big)dx\right|=0.$$

Proof. Obviously,

$$\begin{aligned}
&\sum_{n=1}^{b(\epsilon)}\frac{1}{n^2}\left|\int_{n\epsilon}^{\infty}2x\mathsf{P}(|S_n|\ge x)dx-\int_{n\epsilon}^{\infty}2x\mathsf{P}(|N|\ge x/\sqrt{n})dx\right|\\
=&\sum_{n=1}^{b(\epsilon)}\left|\int_{0}^{\infty}2(x+\epsilon)\mathsf{P}(|S_n|\ge n(x+\epsilon))dx-\int_{0}^{\infty}2(x+\epsilon)\mathsf{P}\big(|N|\ge\sqrt{n}(x+\epsilon)\big)dx\right|\\
\le&\sum_{n=1}^{b(\epsilon)}\frac{1}{n}\int_{0}^{\infty}2n(x+\epsilon)\Big|\mathsf{P}(|S_n|\ge n(x+\epsilon))-\mathsf{P}\big(|N|\ge\sqrt{n}(x+\epsilon)\big)\Big|dx\\
\le&\sum_{n=1}^{b(\epsilon)}\frac{1}{n}(\Delta_{n1}+\Delta_{n2}+\Delta_{n3}),
\end{aligned}$$

where

$$\begin{aligned}
\Delta_{n1}&=\int_{0}^{1/\sqrt{n}\Delta_n{}^{1/4}}2n(x+\epsilon)\Big|\mathsf{P}(|S_n|\ge n(x+\epsilon))-\mathsf{P}\big(|N|\ge\sqrt{n}(x+\epsilon)\big)\Big|dx,\\
\Delta_{n2}&=\int_{1/\sqrt{n}\Delta_n{}^{1/4}}^{\infty}2n(x+\epsilon)\mathsf{P}(|S_n|\ge n(x+\epsilon))dx,\\
\Delta_{n3}&=\int_{1/\sqrt{n}\Delta_n{}^{1/4}}^{\infty}2n(x+\epsilon)\mathsf{P}(|N|\ge\sqrt{n}(x+\epsilon))dx,\\
\Delta_n&=\sup_x\Big|\mathsf{P}(|S_n|\ge\sqrt{n}x)-\mathsf{P}(|N|\ge x)\Big|.
\end{aligned}$$

By the Berry-Esseen theorem for a moving average process generated by ϕ-mixing sequence, we get $\Delta_n\to 0$ as $n\to\infty$.
Since $n\le b(\epsilon)$ implies $\sqrt{n}\epsilon\le 1$, we can get

(3.4)
$$\begin{aligned}
\Delta_{n1}&\le\int_{0}^{1/\sqrt{n}\Delta_n{}^{1/4}}2n(x+\epsilon)\Delta_n dx\\
&\le n\Delta_n\Big(\frac{1}{\sqrt{n}\Delta_n{}^{1/4}}+\epsilon\Big)^2\le\Big(\Delta_n^{1/4}+\Delta_n^{1/2}\Big)^2.
\end{aligned}$$

For Δ_{n3}, by the Markov inequality, we have

(3.5)
$$\Delta_{n3}\le Cn\int_{1/\sqrt{n}\Delta_n^{1/4}}^{\infty}\frac{1}{n^2(x+\epsilon)^3}dx\le C\Delta_n^{1/2}.$$

By (3.4) and (3.5), we obtain

$$\lim_{\epsilon \searrow 0} \frac{1}{-\log \epsilon} \sum_{n=1}^{b(\epsilon)} \Big\{ \frac{1}{n} (\Delta_{n1} + \Delta_{n3}) \Big\} = 0. \tag{3.6}$$

Let

$$\xi_i' = \xi_i I\{|\xi_i| \le \lambda\}, \quad \xi_i'' = \xi_i I\{|\xi_i| > \lambda\};$$
$$S_n' = \sum_{k=1}^{n} X_k' = \sum_{i=-\infty}^{+\infty} a_{ni}\xi_i', \quad S_n'' = \sum_{i=-\infty}^{+\infty} a_{ni}\xi_i''. \tag{3.7}$$

Therefore,

$$\begin{aligned} \sum_{n=1}^{b(\epsilon)} \frac{1}{n} \Delta_{n2} &\le \sum_{n=1}^{b(\epsilon)} \int_{1/\sqrt{n}\Delta_n{}^{1/4}}^{\infty} 2(x+\epsilon) \mathsf{P}\Big(|S_n'| \ge \frac{n(x+\epsilon)}{2} \Big) dx \\ &\quad + \sum_{n=1}^{b(\epsilon)} \int_{1/\sqrt{n}\Delta_n{}^{1/4}}^{\infty} 2(x+\epsilon) \mathsf{P}\Big(|S_n''| \ge \frac{n(x+\epsilon)}{2} \Big) dx \\ &=: I_1 + I_2, \end{aligned}$$

and

$$I_1 \le C \sum_{n=1}^{b(\epsilon)} \int_0^{\infty} \frac{\mathsf{E}|X_1'|^4}{n^2 (x+\epsilon)^3} dx \le C \mathsf{E}|X_1'|^4 \sum_{n=1}^{b(\epsilon)} \frac{1}{n^2 \epsilon} \le C \mathsf{E}|\xi_1'|^4.$$

Thus, we have

$$\lim_{\epsilon \searrow 0} \frac{1}{-\log \epsilon} I_1 = 0. \tag{3.8}$$

Observe that $\sum_{k=1}^{n} X_k = \sum_{i=-\infty}^{\infty} \sum_{k=1}^{n} a_{k+i}\xi_i = \sum_{i=-\infty}^{\infty} a_{ni}\xi_i$, where $a_{ni} = \sum_{k=1}^{n} a_{k+i}$. By Lemma 2.1, we can assume

$$\sum_{i=-\infty}^{\infty} |a_{ni}|^t \le n, \quad t \ge 1 \ \text{ and } \ \tilde{a} = \sum_{i=-\infty}^{\infty} |a_i| \le 1. \tag{3.9}$$

Let $T_n'' = \sum_{i=-\infty}^{\infty} a_{ni}\xi_i'' I\{|a_{ni}\xi_i''| \le n(x+\epsilon)\}$. Then

$$\begin{aligned} |\mathsf{E}T_n''| &= \Big| \sum_{i=-\infty}^{\infty} a_{ni} \mathsf{E}\xi_1'' I\{|a_{ni}\xi_1''| > n(x+\epsilon)\} \Big| \\ &\le \sum_{i=-\infty}^{\infty} |a_{ni}| \mathsf{E}|\xi_1''| I\{\tilde{a}|\xi_1''| > n(x+\epsilon)\} \\ &\le n \mathsf{E}|\xi_1''| I\{|\xi_1''| > n(x+\epsilon)\} \\ &\le \frac{\mathsf{E}|\xi_1''|^2 I\{|\xi_1''| > n(x+\epsilon)\}}{x+\epsilon}. \end{aligned}$$

In the situation of $x \in (1/\sqrt{n}\Delta_n^{1/4},\ \infty)$, we have

$$\frac{1}{n(x+\epsilon)}|\mathsf{E}T_n''| \le \frac{1}{n(x+\epsilon)^2}\mathsf{E}|\xi_1''|^2 I\{|\xi_1''| > n(x+\epsilon)\} < \frac{\mathsf{E}|\xi_1''|^2}{\Delta_n^{1/2}} < \epsilon.$$

Thus,

$$\begin{aligned} I_2 \le C\sum_{n=1}^{b(\epsilon)}\int_{1/\sqrt{n}\Delta_n^{1/4}}^{\infty}(x+\epsilon)\Big[\mathsf{P}\Big(\sup_i |a_{ni}\xi_i''| > n(x+\epsilon)\Big) \\ + \mathsf{P}(|T_n'' - ET_n''| \ge n(x+\epsilon)/4)\Big]dx \\ =: I_{21} + I_{22}. \end{aligned}$$

Set $I_{nj} = \{j \in \mathcal{L},\ 1/(j+1) < |a_{nj}| \le 1/j,\ j = 1, 2, \ldots\}$, then $\bigcup_{j\ge1} I_{nj} = \mathcal{L}$ (see [4]). We can get

$$\sum_{j=1}^{k} \sharp I_{nj} \le n(k+1).$$

Therefore,

$$\begin{aligned} &\mathsf{P}\Big\{\sup_i |a_{ni}\xi_i''| > n(x+\epsilon)\Big\} \\ \le &\sum_{i=-\infty}^{\infty}\mathsf{P}\{|a_{ni}\xi_i''| > n(x+\epsilon)\} \\ \le &\sum_{j=1}^{\infty}\sum_{i\in I_{nj}}\mathsf{P}\{|\xi_1''| \ge nj(x+\epsilon)\} \\ \le &\sum_{j=1}^{\infty}(\sharp I_{nj})\mathsf{P}\{|\xi_1''| \ge nj(x+\epsilon)\} \\ \le &\sum_{j=1}^{\infty}\sum_{k\ge j}(\sharp I_{nj})\mathsf{P}\{nk(x+\epsilon) \le |\xi_1''| < n(k+1)(x+\epsilon)\} \\ \le &\sum_{k=1}^{\infty}\sum_{j=1}^{k}(\sharp I_{nj})\mathsf{P}\{nk(x+\epsilon) \le |\xi_1''| < n(k+1)(x+\epsilon)\} \\ \le &\sum_{k=1}^{\infty} n(k+1)\mathsf{P}\{nk(x+\epsilon) \le |\xi_1''| < n(k+1)(x+\epsilon)\} \\ \le &\frac{\mathsf{E}|\xi_1''|I\{|\xi_1''| \ge n(x+\epsilon)\}}{x+\epsilon}. \end{aligned}$$

Then

$$I_{21} \le C\mathsf{E}|\xi_1''|\int_{1/\sqrt{n}\Delta_n^{1/4}}^{\infty}\sum_{n=1}^{b(\epsilon)}\sum_{k=n}^{\infty} I\{k(x+\epsilon) \le |\xi_1''| < (k+1)(x+\epsilon)\}dx$$

$$\le C\mathsf{E}|\xi_1''|\int_0^\infty \sum_{k=1}^\infty kI\{k(x+\epsilon)\le|\xi_1''|<(k+1)(x+\epsilon)\}dx$$

$$\le C\mathsf{E}|\xi_1''|^2\int_0^\infty (x+\epsilon)^{-1}I\{|\xi_1''|\ge(x+\epsilon)\}dx$$

$$\le C\mathsf{E}|\xi_1''|^2\log^+|\xi_1''| - C\mathsf{E}|\xi_1''|^2(\log\epsilon). \tag{3.10}$$

Hence

$$\lim_{\lambda\to\infty}\limsup_{\epsilon\searrow 0}\frac{1}{-\log\epsilon}I_{21}=0. \tag{3.11}$$

As for I_{22}, note that $\sum_{m=1}^\infty \phi^{1/2}(m)<\infty$, then

$$\sup_{-\infty<l\le m<\infty}\mathsf{E}\Big[\sum_{i=l}^m a_{ni}\xi_i''I\{|a_{ni}\xi_i''|\le n(x+\epsilon)\} - \mathsf{E}\Big(\sum_{i=l}^m a_{ni}\xi_i''I\{|a_{ni}\xi_i''|\le n(x+\epsilon)\}\Big)\Big]^2$$

$$\le C\sum_{i=-\infty}^\infty \mathsf{E}(a_{ni}\xi_1'')^2I\{|a_{ni}\xi_1''|\le n(x+\epsilon)\}.$$

By Lemma 2.2, for $q>2$, we have

$$\begin{aligned}I_{22}\le{}& \sum_{n=1}^{b(\epsilon)}\int_{1/\sqrt{n}\Delta_n^{1/4}}^\infty n^{-q}(x+\epsilon)^{1-q}\Big\{\Big(\sum_{i=-\infty}^\infty a_{ni}^2\mathsf{E}|\xi_1''|^2I\{|a_{ni}\xi_1''|\le n(x+\epsilon)\}\Big)^{q/2}\\ &+\sum_{i=-\infty}^\infty \mathsf{E}|a_{ni}\xi_1''|^qI\{|a_{ni}\xi_1''|\le n\epsilon\}\\ &+\sum_{i=-\infty}^\infty \mathsf{E}|a_{ni}\xi_1''|^qI\{n\epsilon<|a_{ni}\xi_1''|\le n(x+\epsilon)\}\Big\}dx\\ =:{}& I_{221}+I_{222}+I_{223}.\end{aligned}$$

For I_{221}, $q>2$, we have

$$I_{221}\le\sum_{n=1}^{b(\epsilon)}\int_{1/\sqrt{n}\Delta^{1/4}}^\infty n^{-q}(x+\epsilon)^{1-q}\Big(\sum_{i=-\infty}^\infty a_{ni}^2\mathsf{E}|\xi_1''|^2\Big)^{q/2}dx$$

$$\le C\sum_{n=1}^{b(\epsilon)}\frac{1}{n}\Delta_n^{\frac{q-2}{4}}. \tag{3.12}$$

First we estimate I_{222},

$$I_{222}\le\sum_{n=1}^{b(\epsilon)}\int_{1/\sqrt{n}\Delta^{1/4}}^\infty n^{-q}(x+\epsilon)^{1-q}\sum_{j=1}^\infty\sum_{i\in I_{nj}}|a_{ni}|^q\mathsf{E}|\xi_1''|^qI\{|a_{ni}\xi_1''|\le n\epsilon\}dx$$

$$\le C\epsilon^{2-q}\sum_{n=1}^{b(\epsilon)} n^{-q}\sum_{j=1}^{\infty}(\sharp I_{nj})j^{-q}\sum_{k=0}^{(j+1)n}\mathsf{E}|\xi_1''|^q I\{k\epsilon\le|\xi_1''|<(k+1)\epsilon\}dx$$
$$\le C\epsilon^{2-q}\sum_{n=1}^{b(\epsilon)} n^{-q}\sum_{j=1}^{\infty}(\sharp I_{nj})j^{-q}\Big\{\sum_{k=0}^{2n}\mathsf{E}|\xi_1''|^q I\{k\epsilon\le|\xi_1''|<(k+1)\epsilon\}$$
$$+\sum_{k=2n+1}^{(j+1)n}\mathsf{E}|\xi_1''|^q I\{k\epsilon\le|\xi_1''|<(k+1)\epsilon\}\Big\}dx$$
$$=: I_{2221}+I_{2222}.$$

Note that

$$\sum_{j=m}^{\infty}(\sharp I_{nj})(j+1)^{-q}(m+1)^{q-1}\le\sum_{j=1}^{\infty}(\sharp I_{nj})(j+1)^{-1}$$
$$\le\sum_{j=1}^{\infty}\sum_{i\in I_{nj}}|a_{ni}|=\sum_{i=-\infty}^{\infty}|a_{ni}|\le n.$$

Then

$$\sum_{j=m}^{\infty}(\sharp I_{nj})j^{-q}\le Cnm^{-(q-1)}. \tag{3.13}$$

Now we estimate I_{2221}. By (3.13), we get

$$I_{2221}\le C\epsilon^{2-q}\sum_{n=1}^{b(\epsilon)} n^{1-q}\sum_{k=0}^{2n}\mathsf{E}|\xi_1''|^q I\{k\epsilon\le|\xi_1''|<(k+1)\epsilon\}\sum_{n=[k/2]}^{b(\epsilon)} n^{1-q}$$
$$\le C\epsilon^{2-q}\sum_{k=0}^{b(\epsilon)}\mathsf{E}|\xi_1''|^q I\{k\epsilon\le|\xi_1''|<(k+1)\epsilon\}$$
$$\le C\epsilon^{2-q}\sum_{k=0}^{b(\epsilon)}k^{2-q}\mathsf{E}|\xi_1''|^q I\{k\epsilon\le|\xi_1''|<(k+1)\epsilon\}$$
$$\le C\sum_{k=0}^{\infty}\mathsf{E}|\xi_1''|^2 I\{k\epsilon\le|\xi_1''|<(k+1)\epsilon\}$$
$$\le C\mathsf{E}|\xi_1''|^2 \tag{3.14}$$

and

$$I_{2222}\le\epsilon^{2-q}\sum_{n=1}^{b(\epsilon)} n^{-q}\sum_{k=2n+1}^{\infty}\sum_{j\ge k/n-1}(\sharp I_{nj})j^{-q}\mathsf{E}|\xi_1''|^q I\{k\epsilon\le|\xi_1''|<(k+1)\epsilon\}$$
$$\le\epsilon^{2-q}\sum_{n=1}^{b(\epsilon)} n^{-q}\sum_{k=2n+1}^{\infty}n\left(\frac{k}{n}\right)^{-(q-1)}\mathsf{E}|\xi_1''|^q I\{k\epsilon\le|\xi_1''|<(k+1)\epsilon\}$$

$$\le \epsilon^{2-q}\sum_{k=2}^{\infty}\mathsf{E}|\xi_1''|^q I\{k\epsilon \le |\xi_1''| < (k+1)\epsilon\}\sum_{n=1}^{[k/2]}k^{1-q}$$

$$\le C\epsilon^{2-q}\sum_{k=2}^{\infty}k^{2-q}\mathsf{E}|\xi_1''|^q I\{k\epsilon \le |\xi_1''| < (k+1)\epsilon\}$$

$$\le C\mathsf{E}|\xi_1''|^2. \tag{3.15}$$

Now we estimate I_{223}, by (3.9), we will get

$$I_{223} \le C\sum_{n=1}^{b(\epsilon)}n^{-q}\mathsf{E}|\xi_1''|^q I\{|\tilde{a}\xi_1''| > n\epsilon\}\int_0^\infty (x+\epsilon)^{1-q}\sum_{i=-\infty}^{\infty}|a_{ni}|^q I\{|a_{ni}\xi_1''| \le n(x+\epsilon)\}dx$$

$$\le C\sum_{n=1}^{b(\epsilon)}n^{-q}\mathsf{E}|\xi_1''|^q I\{|\tilde{a}\xi_1''| > n\epsilon\}\int_0^\infty (x+\epsilon)^{1-q}\sum_{j=1}^{\infty}(\sharp I_{nj})j^{-q}\sum_{k=0}^{(j+1)n}I\{k(x+\epsilon) \le |\xi_1''| < (k+1)(x+\epsilon)\}dx$$

$$\le C\sum_{n=1}^{b(\epsilon)}n^{1-q}\mathsf{E}|\xi_1''|^q I\{|\xi_1''| > n\epsilon\}\int_0^\infty (x+\epsilon)^{1-q}\sum_{k=0}^{2n}I\{k(x+\epsilon) \le |\xi_1''| < (k+1)(x+\epsilon)\}dx$$

$$+C\sum_{n=1}^{b(\epsilon)}\mathsf{E}|\xi_1''|^q I\{|\xi_1''| > n\epsilon\}\sum_{k=2n+1}^{\infty}k^{1-q}\int_0^\infty (x+\epsilon)^{1-q}I\{k(x+\epsilon) \le |\xi_1''| < (k+1)(x+\epsilon)\}dx$$

$$\le C\sum_{n=1}^{b(\epsilon)}n^{-1}\mathsf{E}|\xi_1''|^2 I\{|\xi_1''| > n\epsilon\} + C\sum_{n=1}^{b(\epsilon)}(2n+1)^{-1}\mathsf{E}|\xi_1''|I\{|\xi_1''| > n(x+\epsilon)\}dx$$

$$\le C\mathsf{E}|\xi_1''|^2\left(\log^+|\xi_1''| - \log\epsilon\right). \tag{3.16}$$

From (3.12) to (3.16), we have

$$\lim_{\lambda\to\infty}\lim_{\epsilon\searrow 0}\frac{1}{-\log\epsilon}I_{22} = 0. \tag{3.17}$$

Combining (3.6), (3.8), (3.11), and (3.17), (3.3) is derived. □

Proposition 3.2. *One has*

(3.18)

$$\lim_{\epsilon\searrow 0}\frac{1}{-\log\epsilon}\sum_{n=b(\epsilon)+1}^{\infty}\frac{1}{n^2}\left|\int_{n\epsilon}^{\infty}2xP(|S_n| \ge x)dx - \int_{n\epsilon}^{\infty}2xP\big(|N| \ge x/\sqrt{n}\big)dx\right| = 0.$$

Proof. Consider the following:

$$\sum_{n=b(\epsilon)+1}^{\infty}\frac{1}{n^2}\left|\int_{n\epsilon}^{\infty}2x\mathsf{P}(|S_n| \ge x)dx - \int_{n\epsilon}^{\infty}2x\mathsf{P}\big(|N| \ge x/\sqrt{n}\big)dx\right|$$

$$\le \sum_{n=b(\epsilon)+1}^{\infty}\int_0^\infty 2(x+\epsilon)\mathsf{P}(|N| \ge \sqrt{n}(x+\epsilon))dx + \sum_{n=b(\epsilon)+1}^{\infty}\int_0^\infty 2(x+\epsilon)\mathsf{P}(|S_n| \ge n(x+\epsilon))dx$$

$$=: M_1 + M_2.$$

For M_1, we have

$$M_1 \le C\sum_{n=b(\epsilon)+1}^{\infty}\int_0^\infty 2(x+\epsilon)\frac{1}{n^2(x+\epsilon)^4}dx \le C\sum_{n=b(\epsilon)+1}^{\infty}\frac{1}{n^2\epsilon^2} \le C. \tag{3.19}$$

Set $U'_n = \sum_{i=-\infty}^{\infty} a_{ni}\xi_i I\{|a_{ni}\xi_i| \le n(x+\epsilon)\}$. And

$$M_2 \le \sum_{n=b(\epsilon)+1}^{\infty} \int_0^{\infty} 2(x+\epsilon)\Big[\mathsf{P}\big(\sup_i |a_{ni}\xi_i| > n(x+\epsilon)\big) + \mathsf{P}(|U'_n - EU'_n| \ge n(x+\epsilon)/2)\Big]dx$$
$$=: M_{21} + M_{22}.$$

Now we estimate M_{21}. Note that from $\mathsf{E}\xi_1^2 \log^+ |\xi_1| < \infty$, we have

$$M_{21} \le C \sum_{n=b(\epsilon)+1}^{\infty} \int_0^{\infty} \sum_{k=n}^{\infty} \mathsf{E}|\xi_1| I\{k(x+\epsilon) \le |\xi_1| < (k+1)(x+\epsilon)\}dx$$
$$\le C\mathsf{E}|\xi_1|^2 I\{|\xi_1| > \epsilon^{-1}\} \int_0^{\infty} \frac{I\{|\xi_1| > (x+\epsilon)\}}{x+\epsilon}dx$$
$$\le C\mathsf{E}|\xi_1|^2 I\{|\xi_1| > \epsilon^{-1}\}(\log^+ |\xi_1| - \log \epsilon). \tag{3.20}$$

Using the same arguments as those employed in the proof of Proposition 3.1, we have

$$M_{22} \le \sum_{n=b(\epsilon)+1}^{\infty} \int_0^{\infty} n^{-q}(x+\epsilon)^{1-q}\bigg\{\Big(\sum_{i=-\infty}^{\infty} \mathsf{E}(a_{ni}\xi_1)^2 I\{|a_{ni}\xi_1| \le n(x+\epsilon)\}\Big)^{q/2}$$
$$+ \sum_{i=-\infty}^{\infty} \mathsf{E}|a_{ni}\xi_1|^q I\{|a_{ni}\xi_1| \le n\epsilon\}$$
$$+ \sum_{i=-\infty}^{\infty} \mathsf{E}|a_{ni}\xi_1|^q I\{n\epsilon < |a_{ni}\xi_1| \le n(x+\epsilon)\}\bigg\}dx$$
$$=: M_{221} + M_{222} + M_{223}.$$

And

$$M_{221} \le \sum_{n=b(\epsilon)+1}^{\infty} \int_0^{\infty} n^{-q/2}(x+\epsilon)^{1-q}\big(\mathsf{E}\xi_1^2\big)^{q/2}dx$$
$$\le C(\mathsf{E}\xi_1^2)^{q/2} \sum_{n=b(\epsilon)+1}^{\infty} n^{-q/2}\epsilon^{2-q} \le C. \tag{3.21}$$

Then

$$M_{222} \le \epsilon^{2-q} \sum_{n=b(\epsilon)+1}^{\infty} n^{-q} \sum_{i=-\infty}^{\infty} |a_{ni}|^q \mathsf{E}|\xi_1|^q I\{|a_{ni}\xi_1| \le n\epsilon\}$$
$$\le \epsilon^{2-q} \sum_{n=b(\epsilon)+1}^{\infty} n^{-q} \sum_{j=1}^{\infty} (\sharp I_{nj}) j^{-q} \sum_{k=0}^{(j+1)n} \mathsf{E}|\xi_1|^q I\{k\epsilon \le |\xi_1| < (k+1)\epsilon\}$$

$$\le \epsilon^{2-q} \sum_{n=b(\epsilon)+1}^{\infty} n^{-q} \sum_{j=1}^{\infty} (\sharp I_{nj}) j^{-q} \Big\{ \sum_{k=0}^{2n} + \sum_{k=2n+1}^{(j+1)n} \Big\} \mathsf{E}|\xi_1|^q I\{k\epsilon \le |\xi_1| < (k+1)\epsilon\}$$

$$\le C\epsilon^{2-q} \sum_{n=b(\epsilon)+1}^{\infty} n^{1-q} \sum_{k=0}^{2n} \mathsf{E}|\xi_1|^q I\{k\epsilon \le |\xi_1| < (k+1)\epsilon\}$$

$$+C\epsilon^{2-q} \sum_{n=b(\epsilon)+1}^{\infty} n^{-q} \sum_{k=2n+1}^{\infty} n \left(\frac{k}{n}\right)^{-(q-1)} \mathsf{E}|\xi_1|^q I\{k\epsilon \le |\xi_1| < (k+1)\epsilon\}$$

$$\le C\epsilon^{2-q} \sum_{k=0}^{\infty} \mathsf{E}|\xi_1|^q I\{k\epsilon \le |\xi_1| < (k+1)\epsilon\} \sum_{n=[k/2]}^{\infty} n^{1-q}$$

$$+ C\epsilon^{2-q} \sum_{k=b(\epsilon)+1}^{\infty} k^{2-q} \mathsf{E}|\xi_1|^q I\{k\epsilon \le |\xi_1| < (k+1)\epsilon\}$$

$$\le C\mathsf{E}|\xi_1|^2. \tag{3.22}$$

Finally, by (3.20), we will get

$$M_{223} = \sum_{n=b(\epsilon)+1}^{\infty} \int_0^{\infty} n^{-q} (x+\epsilon)^{1-q} \sum_{i=-\infty}^{\infty} |a_{ni}|^q \mathsf{E}|\xi_1|^q I\{n\epsilon < |a_{ni}\xi_1| \le n(x+\epsilon)\} dx$$

$$\le C \sum_{n=b(\epsilon)+1}^{\infty} n^{-q} \mathsf{E}|\xi_1|^q I\{|\tilde{a}\xi_1| > n\epsilon\} \int_0^{\infty} (x+\epsilon)^{1-q} \sum_{i=-\infty}^{\infty} |a_{ni}|^q I\{|a_{ni}\xi_1| \le n(x+\epsilon)\} dx$$

$$\le C \sum_{n=b(\epsilon)+1}^{\infty} n^{-q} \mathsf{E}|\xi_1|^q I\{|\xi_1| > n\epsilon\} \int_0^{\infty} (x+\epsilon)^{1-q} \sum_{j=1}^{\infty} (\sharp I_{nj}) j^{-q}$$

$$\Big(\sum_{k=0}^{2n} + \sum_{k=2n+1}^{(j+1)n} \Big) I\{k(x+\epsilon) \le |\xi_1| < (k+1)(x+\epsilon)\} dx$$

$$\le C \sum_{n=b(\epsilon)+1}^{\infty} n^{-1} \mathsf{E}|\xi_1|^2 I\{|\xi_1| > n\epsilon\}$$

$$\le C\mathsf{E}|\xi_1|^2 I\{|\xi_1| > \epsilon^{-1}\} \left(\log^+ |\xi_1| - \log \epsilon\right). \tag{3.23}$$

Combining (3.19), (3.20), (3.21), (3.22), and (3.23), (3.18) is derived. □

4. Proof of Theorem 2.2

Proof of Theorem 2.2. Since $\mathsf{E}|S_n|^p I\{|S_n| \ge n\epsilon\} = \mathsf{P}(|S_n| \ge n\epsilon)$ when $p = 0$, we only discuss the case of $0 < p < 2$. Note that

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \mathsf{E}|S_n|^p I\{|S_n| \ge n\epsilon\}$$

$$= \epsilon^p \sum_{n=1}^{\infty} \mathsf{P}\{|S_n| \ge n\epsilon\} + \sum_{n=1}^{\infty} \frac{1}{n^p} \int_{n\epsilon}^{\infty} p x^{p-1} \mathsf{P}\{|S_n| \ge x\} dx. \tag{4.1}$$

By (1.3), we only need to show that

$$\lim_{\epsilon\searrow 0}\epsilon^{2-p}\sum_{n=1}^{\infty}\frac{1}{n^p}\int_{n\epsilon}^{\infty}px^{p-1}\mathsf{P}\{|S_n|\ge x\}dx=\frac{p}{2-p}. \tag{4.2}$$

By Proposition 4.1 in [6], we have

$$\lim_{\epsilon\searrow 0}\epsilon^{2-p}\sum_{n=1}^{\infty}\frac{1}{n^p}\int_{n\epsilon}^{\infty}px^{p-1}\mathsf{P}\big(|N|\ge x/\sqrt{n}\big)dx=\frac{p}{2-p}.$$

□

Hence we need show the following two propositions.

Proposition 4.1. *One has*

(4.3)

$$\lim_{\epsilon\searrow 0}\epsilon^{2-p}\sum_{n=1}^{Mb(\epsilon)}\frac{1}{n^p}\left|\int_{n\epsilon}^{\infty}px^{p-1}P(|S_n|\ge x)dx-\int_{n\epsilon}^{\infty}px^{p-1}P\big(|N|\ge x/\sqrt{n}\big)dx\right|=0.$$

Proof. It is easy to see

$$\begin{aligned}
&\sum_{n=1}^{Mb(\epsilon)}\frac{1}{n^p}\left|\int_{n\epsilon}^{\infty}px^{p-1}\mathsf{P}(|S_n|\ge x)dx-\int_{n\epsilon}^{\infty}px^{p-1}\mathsf{P}(|N|\ge x/\sqrt{n})dx\right|\\
\le&\sum_{n=1}^{Mb(\epsilon)}\frac{1}{n^{p/2}}n^{p/2}\int_0^{\infty}\mathsf{P}(x+\epsilon)^{p-1}\Big|\mathsf{P}\big(|S_n|\ge n(x+\epsilon)\big)-\mathsf{P}\big(|N|\ge\sqrt{n}(x+\epsilon)\big)\Big|dx\\
=:&\sum_{n=1}^{Mb(\epsilon)}\frac{1}{n^{p/2}}(\Delta'_{n1}+\Delta'_{n2}),
\end{aligned}$$

where

$$\begin{aligned}
\Delta'_{n1}&=n^{p/2}\int_0^{1/\sqrt{n}\Delta_n^{1/2p}}\mathsf{P}(x+\epsilon)^{p-1}\Big|\mathsf{P}(|S_n|\ge n(x+\epsilon))-\mathsf{P}\big(|N|\ge\sqrt{n}(x+\epsilon)\big)\Big|dx\\
&\le\Delta_n n^{p/2}\Big(\frac{1}{\sqrt{n}\Delta_n^{1/2p}}+\epsilon\Big)^p\le\big(\Delta_n^{1/2p}+\sqrt{M}\Delta_n^{1/p}\big)^p,
\end{aligned} \tag{4.4}$$

Since $n\le Mb(\epsilon)$ implies $\sqrt{n}\epsilon\le\sqrt{M}$, by Markov's inequality,

$$\begin{aligned}
\Delta'_{n2}&=n^{p/2}\int_{1/\sqrt{n}\Delta_n^{1/2p}}^{\infty}\mathsf{P}(x+\epsilon)^{p-1}\Big|P(|S_n|\ge n(x+\epsilon))-\mathsf{P}(|N|\ge\sqrt{n}(x+\epsilon))\Big|dx\\
&\le Cn^{p/2}\int_{1/\sqrt{n}\Delta_n^{1/2p}}^{\infty}\frac{1}{n(x+\epsilon)^{3-p}}dx\le C\Delta_n^{1/p-1/2}.
\end{aligned} \tag{4.5}$$

Combining (4.4) and (4.5), (4.3) is derived. □

Proposition 4.2. *One has*

(4.6)

$$\lim_{M\to\infty}\limsup_{\epsilon\searrow 0}\epsilon^{2-p}\sum_{n=Mb(\epsilon)+1}^{\infty}\frac{1}{n^p}\left|\int_{n\epsilon}^{\infty}px^{p-1}P(|S_n|\ge x)dx-\int_{n\epsilon}^{\infty}px^{p-1}P(|N|\ge x/\sqrt{n})dx\right|=0.$$

Proof. Define U'_n as in Proposition 3.2. We have

$$\sum_{n=Mb(\epsilon)+1}^{\infty}\frac{1}{n^p}\Big|\int_{n\epsilon}^{\infty}px^{p-1}\mathsf{P}(|S_n|\geq x)dx-\int_{n\epsilon}^{\infty}px^{p-1}\mathsf{P}\big(|N|\geq x/\sqrt{n}\big)dx\Big|$$
$$\leq \sum_{n=Mb(\epsilon)+1}^{\infty}\int_0^{\infty}p(x+\epsilon)^{p-1}\mathsf{P}(|N|\geq\sqrt{n}(x+\epsilon))dx$$
$$+\sum_{n=Mb(\epsilon)+1}^{\infty}\int_0^{\infty}p(x+\epsilon)^{p-1}\mathsf{P}(|S_n|\geq n(x+\epsilon))dx$$
$$\leq K_1+K_2.$$

For K_1, by the Markov inequality,

$$\begin{aligned}K_1&\leq C\sum_{n=Mb(\epsilon)+1}^{\infty}\int_0^{\infty}p(x+\epsilon)^{p-1}\frac{\mathsf{E}|N|^4}{n^2(x+\epsilon)^4}dx\\&\leq C\sum_{n=Mb(\epsilon)+1}^{\infty}n^{-2}\epsilon^{p-4}\\&\leq CM^{-1}\epsilon^{p-2}.\end{aligned}\tag{4.7}$$

Therefore,

$$\lim_{M\to\infty}\limsup_{\epsilon\searrow 0}\epsilon^{2-p}K_1=0.\tag{4.8}$$

It is easy to see

$$\begin{aligned}K_2\leq&\sum_{n=Mb(\epsilon)+1}^{\infty}\int_0^{\infty}p(x+\epsilon)^{p-1}\Big[\mathsf{P}(\sup_i|a_{ni}\xi_i|>n(x+\epsilon))\\&\qquad+\mathsf{P}(|U'_n-EU'_n|\geq n(x+\epsilon)/2)\Big]dx\\=:&\ K_{21}+K_{22}.\end{aligned}$$

Similar with the proof of (3.10), we can get

$$\begin{aligned}K_{21}&\leq C\mathsf{E}|\xi_1|\int_0^{\infty}\sum_{n=Mb(\epsilon)+1}^{\infty}(x+\epsilon)^{p-2}\sum_{k=n}^{\infty}I\{k(x+\epsilon)\leq|\xi_1|<(k+1)(x+\epsilon)\}dx\\&\leq C\mathsf{E}\xi_1^2\int_0^{\infty}(x+\epsilon)^{p-3}\sum_{k=Mb(\epsilon)+1}^{\infty}I\{k(x+\epsilon)\leq|\xi_1|<(k+1)(x+\epsilon)\}dx\\&\leq C\mathsf{E}|\xi_1|^2I\{|\xi_1|\geq M\epsilon^{-1}\}\int_0^{\infty}(x+\epsilon)^{p-3}dx\\&\leq C\epsilon^{p-2}\mathsf{E}|\xi_1|^2I\{|\xi_1|\geq M\epsilon^{-1}\}.\end{aligned}$$

So we have

$$\lim_{M\to\infty}\limsup_{\epsilon\searrow 0}\epsilon^{2-p}K_{21}=0. \tag{4.9}$$

Note that $\sum_{m=1}^{\infty}\phi^{1/2}(m)<\infty$, by Lemma 2.2, for $q>2$,

$$\begin{aligned}K_{22}\le&\sum_{n=Mb(\epsilon)+1}^{\infty}\int_0^{\infty}pn^{-q}(x+\epsilon)^{p-1-q}\Big\{\Big(\sum_{i=-\infty}^{\infty}\mathsf{E}(a_{ni}\xi_1)^2I\{|a_{ni}\xi_1|\le n(x+\epsilon)\}\Big)^{q/2}\\&+\sum_{i=-\infty}^{\infty}\mathsf{E}|a_{ni}\xi_1|^qI\{|a_{ni}\xi_1|\le n\epsilon\}\\&+\sum_{i=-\infty}^{\infty}\mathsf{E}|a_{ni}\xi_1|^qI\{n\epsilon<|a_{ni}\xi_1|\le n(x+\epsilon)\}\Big\}dx\\=:&\,K_{221}+K_{222}+K_{223}.\end{aligned}$$

Here we omit the proof of K_{221} because it is routine. Next we will estimate K_{222},K_{223}.

$$\begin{aligned}K_{222}\le&\,\epsilon^{p-q}\sum_{n=Mb(\epsilon)+1}^{\infty}n^{-q}\sum_{i=-\infty}^{\infty}|a_{ni}|^q\mathsf{E}|\xi_1|^qI\{|a_{ni}\xi_1|\le n\epsilon\}\\\le&\,\epsilon^{p-q}\sum_{n=Mb(\epsilon)+1}^{\infty}n^{-q}\sum_{j=1}^{\infty}(\sharp I_{nj})j^{-q}\sum_{k=0}^{(j+1)n}\mathsf{E}|\xi_1|^qI\{k\epsilon\le|\xi_1|<(k+1)\epsilon\}\\\le&\,\epsilon^{p-q}\sum_{n=Mb(\epsilon)+1}^{\infty}n^{-q}\sum_{j=1}^{\infty}(\sharp I_{nj})j^{-q}\Big\{\sum_{k=0}^{2n}+\sum_{k=2n+1}^{(j+1)n}\Big\}\mathsf{E}|\xi_1|^qI\{k\epsilon\le|\xi_1|<(k+1)\epsilon\}\\\le&\,C\epsilon^{p-q}\sum_{n=Mb(\epsilon)+1}^{\infty}n^{1-q}\sum_{k=0}^{2n}\mathsf{E}|\xi_1|^qI\{k\epsilon\le|\xi_1|<(k+1)\epsilon\}\\&+C\epsilon^{p-q}\sum_{n=Mb(\epsilon)+1}^{\infty}n^{-q}\sum_{k=2n+1}^{\infty}n\left(\frac{k}{n}\right)^{-(q-1)}\mathsf{E}|\xi_1|^qI\{k\epsilon\le|\xi_1|<(k+1)\epsilon\}\\\le&\,C\epsilon^{p-q}\sum_{k=0}^{\infty}\mathsf{E}|\xi_1|^qI\{k\epsilon\le|\xi_1|<(k+1)\epsilon\}\sum_{n=[k/2]}^{\infty}n^{1-q}\\&+C\epsilon^{p-q}\sum_{k=Mb(\epsilon)+1}^{\infty}k^{2-q}\mathsf{E}|\xi_1|^qI\{k\epsilon\le|\xi_1|<(k+1)\epsilon\}\\\le&\,C\epsilon^{p-2}\mathsf{E}|\xi_1|^2I\{|\xi_1|>M\epsilon^{-1}\}.\end{aligned}\tag{4.10}$$

Finally, we will get

$$K_{223}=\sum_{n=Mb(\epsilon)+1}^{\infty}\int_0^{\infty}pn^{-q}(x+\epsilon)^{p-1-q}\sum_{i=-\infty}^{\infty}\mathsf{E}|a_{ni}\xi_1|^qI\{n\epsilon<|a_{ni}\xi_1|\le n(x+\epsilon)\}dx$$

$$\leq C \sum_{n=Mb(\epsilon)+1}^{\infty} n^{-q}\mathsf{E}|\xi_1|^q I\{|\tilde{a}\xi_1| > n\epsilon\}$$
$$\int_0^{\infty} (x+\epsilon)^{p-1-q} \sum_{j=1}^{\infty} (\sharp I_{nj}) j^{-q} \sum_{k=0}^{(j+1)n} I\{k(x+\epsilon) \leq |\xi_1| < n(x+\epsilon)\}dx$$
$$\leq C \sum_{n=Mb(\epsilon)+1}^{\infty} n^{-q}\mathsf{E}|\xi_1|^q I\{|\xi_1| > n\epsilon\} \int_0^{\infty} (x+\epsilon)^{p-1-q} \sum_{j=1}^{\infty} (\sharp I_{nj}) j^{-q}$$
$$\Big(\sum_{k=0}^{2n} + \sum_{k=2n+1}^{(j+1)n}\Big) I\{k(x+\epsilon) \leq |\xi_1| < (k+1)(x+\epsilon)\}dx$$
$$\leq C \sum_{n=Mb(\epsilon)+1}^{\infty} n^{1-p}\mathsf{E}|\xi_1|^p I\{|\xi_1| > n\epsilon\}$$
$$\leq \epsilon^{p-2}\mathsf{E}|\xi_1|^2 I\{|\xi_1| > M\epsilon^{-1}\}. \tag{4.11}$$

Then we obtain

$$\lim_{M\to\infty} \limsup_{\epsilon\searrow 0} \epsilon^{2-p} K_{22} = 0. \tag{4.12}$$

Combining (4.8), (4.9), and (4.12), then (4.6) is derived. □

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