

NEYMAN-PEARSON THEORY AND ITS APPLICATION TO SHORTFALL RISK IN FINANCE

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ABSTRACT. Shortfall risk is considered by taking some exposed risks because the superhedging price is too expensive to be used in practice. Minimizing shortfall risk can be reduced to the problem of finding a randomized test ψ in the static problem. The optimization problem can be solved via the classical Neyman-Pearson theory, and can be also explained in terms of hypothesis testing. We introduce the classical Neyman-Pearson lemma expressed in terms of mathematics and see how it is applied to shortfall risk in finance.

1. INTRODUCTION

It is not possible to replicate every contingent claim in incomplete markets, in which the equivalent martingale measures are not unique. With the super-hedging price, an agent or an investor could eliminate the shortfall risk completely by choosing a suitable hedging strategy. But the prices derived by super-replication are too high and not acceptable in practice.

With the initial capital less than the super-hedging price, i.e., under the capital constraint an agent or an investor is unable to eliminate all exposed risk associated to the contingent claim completely and so wants to find optimal strategies which minimize the shortfall risk. They are seeking optimal partial hedging strategies with the initial capital less than the super-hedging price by taking some risks [1, 5, 6, 10, 11, 17, 18].

The optimal problem which minimizes the shortfall risk becomes the max-min optimal one and can be explained in terms of statistical hypothesis testing.

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The hypothesis testing is to decide whether or not some hypothesis that has been formulated is correct. Only two decisions lie between accepting or rejecting the hypothesis.

Suppose that there are two probability measures, the null hypothesis Q against the alternative hypothesis P on a measurable space (Ω, \mathcal{F}) . Let $X : \Omega \rightarrow \{0, 1\}$ be a random variable.

When performing a test one may arrive at the correct decision, or one may commit one of two errors. It is the error of the first kind if the null hypothesis is rejected despite of the fact that Q is the true probability. Similarly, it is the error of the second kind when the null hypothesis is not rejected, although Q is not the true probability.

For example, a doctor tests whether or not a patient takes some disease, and let $X = 1$ be the event that a patient takes some disease actually. The error of the first kind occur when a doctor makes a wrong diagnosis of the presence of some disease which may cause the patient discomfort and financial loss. The probability of the error of the first kind is given by $Q[X = 1]$. The error of the second kind occurs with probability $P[X = 0] = 1 - P[X = 1]$ when a doctor fails to makes a diagnosis of the presence of some disease which may lead to the patient's death.

It is desirable to carry out the test in a way to minimize the probabilities of these two types of errors simultaneously. In fact, it is not possible to control both probabilities simultaneously.

For another example [7], and let $Y = 1$ be the event that the enemy aircraft appear actually. A more sensitive radar decreases the chance of letting enemy aircraft go undetected, but also makes false alarms more likely. The probability of type 1 error is the probability, $Q[Y = 1]$, of neglecting the radar alarms even though the enemy aircraft appears actually, and the probability of type 2 error is the probability, $P[Y = 0] = 1 - P[Y = 1]$ which is called the power of the test $[Y = 1]$, of accepting the radar alarms even though the enemy aircraft did not appear actually.

It is necessary to assign a bound to the possibility of incorrectly rejecting null hypothesis when it is true and to attempt to minimize the other probability under this condition.

So for a given acceptable significance level $\alpha \in (0, 1)$ one should maximize $P[Y = 1]$ to minimize the type 2 error whereas keeping the probability of type 1 error below α , i.e., $Q[Y = 1] \leq \alpha$. This is the classical Neyman-Pearson theory of hypothesis testing [16].

The contributions of this paper are on the systematic explanation of relationship between hypothesis testing and shortfall risk in finance, which minimizing shortfall risk becomes the max-min optimal problem.

This paper is constructed as follows. The classical Neyman-Pearson lemma is introduced in Section 2. The generalized Neyman-Pearson lemma is explained and the relation to hypothesis testing is shown in Section 3. It is shown how hypothesis testing theory is applied to shortfall risk in finance in Section 4. It is shown how the optimal hedging is actually calculated in a complete market in Section 5.

2. NEYMAN-PEARSON LEMMA

In this section, we give the well-known results of the classical Neyman-Pearson lemma [12].

Theorem 2.1. *Let P and Q be probability measures on (Ω, \mathcal{F}) . Then there exists $N \in \mathcal{F}$ with $Q[N] = 0$ and \mathcal{F} -measurable function $\varphi \geq 0$ such that for \mathcal{F} -measurable function $f \geq 0$*

$$\int f dP = \int_N f dP + \int_{N^c} f\varphi dQ.$$

If f is given by $f = I_A$ for $A \in \mathcal{F}$, then we have

$$P[A] = P[A \cap N] + \int_A \varphi dQ.$$

Here the φ is defined as

$$\frac{dP}{dQ} = \begin{cases} \varphi & \text{on } N^c, \\ +\infty & \text{on } N \end{cases}.$$

Proof. Let $R = \frac{1}{2}(Q + P)$. Then both Q and P are absolutely continuous with respect to R with densities dQ/dR and dP/dR , respectively. Define N as

$$(2.1) \quad N = \left\{ \frac{dQ}{dR} = 0 \right\}.$$

Then $Q[N] = 0$ by definition. φ can be expressed as

$$(2.2) \quad \frac{dP}{dQ} = \varphi = \begin{cases} \frac{dP}{dR} \cdot \left(\frac{dQ}{dR}\right)^{-1} & \text{on } N^c, \\ +\infty & \text{on } N \end{cases}.$$

Then, for \mathcal{F} -measurable $f \geq 0$,

$$\begin{aligned} \int f dP &= \int_N f dP + \int_{N^c} f \frac{dP}{dR} dR \\ &= \int_N f dP + \int_{N^c} f \frac{dP}{dR} \cdot \left(\frac{dQ}{dR}\right)^{-1} dQ = \int_N f dP + \int_{N^c} f \varphi dQ. \end{aligned}$$

□

Let $c \geq 0$ be fixed and let A^0 be

$$(2.3) \quad A^0 = \left\{ \frac{dP}{dQ} > c \right\},$$

where dP/dQ is defined as (2.2).

Proposition 2.2. *Let A^0 be defined in (2.3). If $A \in \mathcal{F}$ is such that $Q[A] \leq Q[A^0]$, then $P[A] \leq P[A^0]$.*

Proof. Let $F = I_{A^0} - I_A$. Let N be the set as in (2.1). By the definition of $\frac{dP}{dQ}$, $N \subset A^0$. So $F \geq 0$ on N . Moreover, on $N^c \cap (A^0)^c$, $F \leq 0$ and $dP \leq c \cdot dQ$. Therefore we have $F \cdot (dP/dQ - c) \geq 0$, i.e., $F \cdot dP \geq cF \cdot dQ$. Hence we have

$$P[A^0] - P[A] = \int F dP \geq c \int F dQ = c(Q[A^0] - Q[A]). \quad \square$$

In statistics, A^0 is thought of as the likelihood-ratio test of the null hypothesis Q against the alternative hypothesis P . If the outcome ω of a statistical experiment is in A^0 , then the null hypothesis is rejected. The probability of a type 1 error is given by $Q[A^0]$, which is called the significance level of the statistical test A^0 . A type 2 error occurs with probability $P[(A^0)^c]$. The probability $P[A^0] = 1 - P[(A^0)^c]$ is called the power of the test A^0 . The Neyman-Pearson lemma states that this likelihood ratio test is the most powerful among all level tests for this problem.

Define \mathcal{R} as

$$\mathcal{R} = \{\psi \mid \psi : \Omega \rightarrow [0, 1], \psi \text{ is } \mathcal{F}\text{-measurable}\}.$$

For $\alpha \in (0, 1)$, a α -quantile of a random variable X on (Ω, \mathcal{F}, P) is a real number q such that

$$P[X < q] \leq \alpha \leq P[X \leq q].$$

The *upper* and the *lower* quantiles functions of X are defined as

$$\begin{aligned} q_X^+(\alpha) &= \inf\{x \in R \mid P[X \leq x] > \alpha\} = \sup\{x \in R \mid P[X < x] \leq \alpha\}, \\ q_X^-(\alpha) &= \sup\{x \in R \mid P[X < x] < \alpha\} = \inf\{x \in R \mid P[X \leq x] \geq \alpha\}, \end{aligned}$$

respectively.

The upper or lower quantile functions are a right-continuous or left-continuous inverse function of the distribution function F_X of X , respectively.

Theorem 2.3 ([12]). *Let P and Q be probability measures on (Ω, \mathcal{F}) . Let $R = \frac{1}{2}(Q + P)$. Let the density $\varphi = dP/dQ$ be defined as in (2.2). Then the followings hold.*

(1) *Let $c \geq 0$ be fixed. Suppose that $\psi^0 \in \mathcal{R}$ is a function satisfying*

$$(2.4) \quad \psi^0 = \begin{cases} 1 & \text{on } \{\varphi > c\} \\ 0 & \text{on } \{\varphi < c\} \end{cases} .$$

Then for any $\psi \in \mathcal{R}$

$$(2.5) \quad \int \psi dQ \leq \int \psi^0 dQ \implies \int \psi dP \leq \int \psi^0 dP.$$

(2) *For any $\alpha \in (0, 1)$ there is some $\psi^0 \in \mathcal{R}$ of the form (2.4) such that $\int \psi^0 dQ = \alpha$. More precisely, if c is an $(1 - \alpha)$ -quantile of φ under Q , then ψ^0 can be expressed by*

$$(2.6) \quad \psi^0 = I_{\{\varphi > c\}} + \kappa I_{\{\varphi = c\}},$$

where κ is defined as

$$(2.7) \quad \kappa = \begin{cases} 0 & \text{if } Q[\varphi = c] = 0 \\ \frac{\alpha - Q[\varphi > c]}{Q[\varphi = c]} & \text{otherwise} \end{cases} .$$

(3) *Any $\psi^0 \in \mathcal{R}$ satisfying (2.5) is of the form (2.4) for some $c \geq 0$.*

3. GENERALIZED NEYMAN-PEARSON LEMMA

On the measurable space $(\Omega, \mathcal{F}, \nu)$, suppose that there are an entire family \mathbb{Q} of probability measures which are composite hypothesis, which the family \mathbb{Q} should be tested against another family \mathbb{P} of probability measures which are composite alternative. Assume

$$\begin{aligned} \mathbb{P} \cap \mathbb{Q} &= \emptyset, \\ P \ll \nu, \quad Q \ll \nu & \quad \text{for all } P \in \mathbb{P}, Q \in \mathbb{Q}, \end{aligned}$$

and for each $P \in \mathbb{P}, Q \in \mathbb{Q}$ define Z_P and Z_Q as

$$Z_P = \frac{dP}{d\nu}, \quad Z_Q = \frac{dQ}{d\nu},$$

respectively.

Define \mathcal{R}_α as

$$\mathcal{R}_\alpha := \{\psi : \Omega \rightarrow [0, 1] \mid E^Q[\psi] \leq \alpha \quad \forall Q \in \mathbb{Q}\}.$$

The hypothesis testing can be expressed as to find a randomized test $\tilde{\psi} \in \mathcal{R}_\alpha$ that maximizes the smallest power

$$(3.8) \quad \inf_{P \in \mathbb{P}} E^P[\tilde{\psi}]$$

over all randomized tests ψ satisfying

$$(3.9) \quad \sup_{Q \in \mathbb{Q}} E^Q[\psi] \leq \alpha.$$

If $\alpha \in (0, 1)$, then from the Theorem 2.3 the solution of (3.8) and (3.9) is given by

$$\tilde{\psi} = I_{\{c < dP/dQ\}} + \kappa \cdot I_{\{c = dP/dQ\}},$$

where c and κ are defined as

$$\begin{aligned} c &= \inf\{a \mid Q(a < dP/dQ) \leq \alpha\} \\ \kappa &= \begin{cases} 0, & Q(c = dP/dQ) = 0 \\ \frac{\alpha - Q(c < dP/dQ)}{Q(c = dP/dQ)}, & Q(c = dP/dQ) \neq 0 \end{cases} \end{aligned}$$

respectively.

Definition 3.1. If such a randomized test $\psi \in \mathcal{R}_\alpha$ exists, it will be called *max-min-optimal* for testing the (composite) hypothesis \mathbb{Q} against the (composite) alternative \mathbb{P} , at the given level of significance $\alpha \in (0, 1)$.

Under more general conditions, Cvitanic and Karatzas [7] generalized the Neyman-Pearson lemma and solved it by setting the hypothesis testing into the max-min-optimal problem. The main results of Cvitanic and Karatzas [7] are expressed as follows.

Define $\mathcal{H}, \mathcal{G} \subset L^1(\nu)$ which are two subspaces of ν -integrable random variables as

$$\begin{aligned} \mathcal{H} &= \{H \in L^1(\nu) \mid H \geq 0 \quad \nu - a.e. \text{ and } E^\nu[\psi H] \leq \alpha \quad \forall \psi \in \mathcal{R}_\alpha\}, \\ \mathcal{G} &= \{Z_P\}_{P \in \mathbb{P}}, \end{aligned}$$

where it is assumed that \mathcal{H} is convex and closed under ν -a.e. Then it can be easily shown that $\mathcal{G} \subset L^1(\nu)$ is convex and closed under ν -a.e.

Note that

$$(3.10) \quad \begin{aligned} E^P[\psi] = E^\nu[\psi G] &= E^\nu[\psi(G - zH)] + z \cdot E^\nu[\psi H] \\ &\leq E^\nu[\psi(G - zH)^+] + z\alpha \quad \forall z > 0, \forall \psi \in \mathcal{R}_\alpha. \end{aligned}$$

There exist saddle points $\tilde{G} \in \mathcal{G}$ and $\tilde{\psi} \in \mathcal{R}_\alpha$ satisfying

$$E[\psi \tilde{G}] \leq E[\tilde{\psi} \tilde{G}] \leq E[\tilde{\psi} G] \quad \forall \psi \in \mathcal{R}_\alpha, G \in \mathcal{G}.$$

This implies that

$$V(\alpha) := E[\tilde{\psi} \tilde{G}] = \sup_{\psi \in \mathcal{R}_\alpha} \inf_{G \in \mathcal{G}} E[\psi G] = \inf_{G \in \mathcal{G}} \sup_{\psi \in \mathcal{R}_\alpha} E[\psi G].$$

If the value function \tilde{V} of z is defined as

$$\tilde{V}(z) = \inf_{(G,H) \in \mathcal{G} \times \mathcal{H}} E^\nu[(G - zH)^+], \quad 0 < z < \infty,$$

then the equality in (3.10) holds, i.e., there exists $\tilde{z} > 0$ satisfying

$$V(\alpha) = \inf_{z > 0} \{\alpha z + \tilde{V}(z)\} = \alpha \tilde{z} + \tilde{V}(\tilde{z}).$$

In this case,

$$E^\nu[\tilde{\psi} \tilde{H}] = \alpha,$$

where $\tilde{\psi}$ is given by

$$\tilde{\psi} = I_{\{\tilde{z}\tilde{H} < \tilde{G}\}} + B \cdot I_{\{\tilde{z}\tilde{H} = \tilde{G}\}} \quad \nu - a.e.$$

for some random variable $B : \Omega \rightarrow [0, 1]$.

4. APPLICATION TO SHORTFALL RISK IN FINANCE

In this section, we introduce and analyze the solutions of the static problem (4.17) in terms of hypothesis testing which are found in the papers [17, 18, 20, 15]. The solutions depend on the chosen risk measures.

Nakano [17, 18] and Rudloff [20] consider coherent risk measures to measure the shortfall risk. Kim [15] extends the random variable spaces to the Orlicz hearts on which risk measures are defined. The Orlicz hearts setting allows us to treat various loss functions and various claims in a unified framework.

4.1. Reduction of the dynamic problem to the static one. In this subsection, we show how the dynamic hedging problem can be reduced to the static problem. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, P)$ be a complete filtered probability space. Let $S = (S_t)_{0 \leq t \leq T}$ be an adapted positive process which is a semimartingale. It is assumed that the riskless interest rate is zero for simplicity and $\mathcal{M} = \{Q \mid Q \sim P, S \text{ is a local martingale under } Q\} \neq \emptyset$ to avoid the arbitrage opportunities [9].

Define \mathcal{Q} as

$$\mathcal{Q} = \{Q \ll P \mid Q \text{ is a probability measure on } (\Omega, \mathcal{F})\}.$$

For each $Q \in \mathcal{Q}$, let $Z_Q = \frac{dQ}{dP}$.

Definition 4.1. A *self-financing* strategy (x, ξ) is defined as an initial capital $x \geq 0$ and a predictable process ξ_t such that the value process (value of the current holdings)

$$X_t = x + \int_0^t \xi_u dS_u, \quad t \in [0, T]$$

is P -a.s. well defined.

The self-financing strategy (x, ξ) is called *admissible* if the corresponding value process X_t satisfies

$$X_t = x + \int_0^t \xi_u dS_u \geq -c \quad \text{for some } c \in \mathbb{R}^+, \quad \forall t \in [0, T].$$

Define the admissible set $\mathcal{X}(\alpha)$ as

$$\mathcal{X}(\alpha) = \left\{ (x, \xi) \mid (x, \xi) \text{ is an admissible strategy and } x \leq \alpha \right\}.$$

Definition 4.2. A contingent claim H is called *attainable* (or *replicable*, *redundant*) if there exists admissible strategy (x_0, ξ) such that

$$H = x_0 + \int_0^T \xi_u dS_u.$$

See the book [12] or the paper [9] for the following Theorems 4.3 and 4.4.

Theorem 4.3. *Any attainable claim H is integrable with respect to each equivalent martingale measure (or pricing measure),*

$$E^Q[H] < \infty \quad \forall Q \in \mathcal{M}.$$

Moreover, $\forall Q \in \mathcal{M}$

$$X_t = E^Q[H \mid \mathcal{F}_t] \quad Q - a.s.$$

is a non-negative Q -martingale.

Theorem 4.4. *The market model is arbitrage-free if and only if the \mathcal{M} of all equivalent martingale measure is non-empty.*

Lemma 4.5. *Let $H \geq 0$ be a \mathcal{F}_T -measurable contingent claim. Then there exists an admissible strategy $(x_0, \xi) \in \mathcal{X}(\alpha)$ for some $\alpha > 0$ such that*

$$(4.11) \quad H \leq x_0 + \int_0^T \xi_u dS_u \quad P - a.s.$$

if and only if

$$(4.12) \quad H \in \left\{ X \geq 0 \mid X \text{ is } \mathcal{F}_T - \text{measurable, } \sup_{Q \in \mathcal{M}} E^Q[X] \leq x_0 \right\}.$$

Definition 4.6. The superhedge price V_0 for H is defined as

$$V_0 = \inf \left\{ x \mid \exists \text{ admissible strategy } (x, \xi) \text{ s.t. } H \leq x + \int_0^T \xi_u dS_u \quad P - a.s. \right\}.$$

By the Lemma 4.5 we can see the superhedge price is $V_0 = \sup_{Q \in \mathcal{M}} E^Q[H]$. That is, V_0 is the smallest initial capital eliminating all shortfall risk. The seller of H can cover almost any possible obligation from the sale of H and thus eliminate completely the corresponding risk. The following example in the book [12] shows that the superhedge price of H is the same as the price of underlying asset. So the hedging price of the seller is too high and can't be used in practice.

When the seller is unwilling to invest the superhedge price in a hedging strategy, the seller looks for the optimal partial hedging strategy minimizing problem

$$(4.13) \quad \min_{(x, \xi) \in \mathcal{X}(\alpha)} \left[\rho \left(\left(H - x - \int_0^T \xi_u dS_u \right)^+ \right) \right]$$

with the initial capital constraint

$$(4.14) \quad 0 < \alpha < V_0 = \sup_{Q \in \mathcal{M}} E^Q[H].$$

Here ρ in (4.13) is a risk measure.

Definition 4.7. An admissible strategy $(x^*, \xi^*) \in \mathcal{X}(\alpha)$ is called *robust-efficient* if it is the optimal solution:

$$(x^*, \xi^*) \in \arg \min_{(x, \xi) \in \mathcal{X}(\alpha)} \left[\rho \left(\left(H - x - \int_0^T \xi_u dS_u \right)^+ \right) \right].$$

Definition 4.8. A \mathcal{F}_T -measurable random variable X^* is called *maxmin-optimal* if it is the optimal solution:

$$(4.15) \quad X^* \in \arg \min_{X \in \mathcal{K}} \rho(H - X),$$

where \mathcal{K} is defined as $\mathcal{K} = \{X \mid 0 \leq X \leq H, E^Q[X] \leq \alpha, Q \in \mathcal{M}\}$.

Theorem 4.9. *Let $H \geq 0$ be a \mathcal{F}_T -measurable contingent claim. If the claim X^* with initial capital α is a maxmin-optimal solution, then the super-hedging strategy $(x^*, \xi^*) \in \mathcal{X}(\alpha)$ for the claim X^* is robust-efficient. Conversely, if $(\tilde{x}, \tilde{\xi})$ is a robust-efficient strategy, then the following claim*

$$(4.16) \quad \tilde{X} := \left(\tilde{x} + \int_0^T \tilde{\xi}_u dS_u \right) \wedge H$$

is maxmin-optimal.

Assume that

$$0 < \alpha < V_0 = \sup_{Q \in \mathcal{M}} E^Q[H].$$

Let ρ be a coherent measure of risk [2, 3, 8] defined on some chosen spaces.

Define \mathcal{R}_0 as

$$\mathcal{R}_0 := \left\{ \psi \mid \psi : \Omega \rightarrow [0, 1] \text{ is } \mathcal{F}_T\text{-measurable, } \sup_{Q \in \mathcal{M}} E^Q[\psi H] \leq \alpha \right\}.$$

Theorem 4.9 states that the optimal hedging strategy can be constructed as two steps. The first step is to find the maxmin-optimal solution X^* in the static problem (4.15) and the second step is to fit the terminal value X_T of an admissible strategy to the claim X^* .

Let X^* be a maxmin-optimal solution in the static problem (4.15) and $\tilde{X} := H \wedge X^*$. Then we can conclude that \tilde{X} is also the maxmin-optimal solution, since $0 \leq \tilde{X} \leq H$, $E^Q[\tilde{X}] \leq \alpha$ and $H - \tilde{X} = H - H \wedge X^* = (H - X^*)^+$. So it may be assumed that $0 \leq X^* \leq H$, or equivalently, that $X^* = H\psi^*$ for $\psi^* \in \mathcal{R}_0$. So the dynamic optimization problem (4.13) with the constraint (4.14) can be restated as two steps. The first one is to find an optimal modified claim $\tilde{\psi}H$ where $\tilde{\psi}$ is the solution of the static problem

$$(4.17) \quad \min_{\psi \in \mathcal{R}_0} \rho((1 - \psi)H) = \rho((1 - \tilde{\psi})H).$$

The second one is to find a superhedging strategy for the modified claim $\tilde{\psi}H$.

Now we try to find the solution of (4.17) when coherent risk measure is given by $\rho(X) = E^Q[X]$ for $Q \in \mathcal{Q}$. Then the static problem (4.17) is reduced to the problem

$$(4.18) \quad \max_{\psi \in \mathcal{R}_0} E^Q[\psi H]$$

with the constraint

$$(4.19) \quad \sup_{P^* \in \mathcal{M}} E^{P^*}[\psi H] \leq \alpha, \quad \psi \in \mathcal{R}.$$

Assume that $E[H] > 0$.

Define the measures R and R^* as

$$(4.20) \quad \frac{dR}{dQ} = \frac{H}{E^Q[H]} \quad \text{and} \quad \frac{dR^*}{dP^*} = \frac{H}{E^{P^*}[H]} \quad \text{for } P^* \in \mathcal{M}.$$

respectively. Substituting (4.20) into the static problem (4.18) with the constraint (4.19), it becomes the problem of maximizing

$$(4.21) \quad E^R[\psi] = \int \psi dR$$

with the constraint

$$(4.22) \quad E^{R^*}[\psi] \leq \frac{\alpha}{E^{P^*}[H]}.$$

The above maximization problem (4.21) with the constraint (4.22) is to find the optimal test $\tilde{\psi}$ in testing of simple null hypothesis $\{R^*(P^*)\}$, $P^* \in \mathcal{M}$ against the simple alternative $\{R\}$.

4.2. Explicit solution in a complete market. In a complete market which the equivalent martingale measure is unique, i.e., $P^* \in \mathcal{M}$ is a singleton, the static optimal problem can be solved explicitly. Let $\tilde{\alpha} = \frac{\alpha}{E^{P^*}[H]}$. Then $0 < \tilde{\alpha} < 1$ with the assumption, $\alpha < \sup_{P^* \in \mathcal{M}} E^{P^*}[H]$.

Let c be an $(1 - \tilde{\alpha})$ -quantile of $\tilde{\varphi} := \frac{dR}{dR^*} = \frac{dQ}{dP^*}$ under R^* . Then c is written as

$$(4.23) \quad c = \inf\{a \mid R^*[\tilde{\varphi} > a] \leq \tilde{\alpha}\}.$$

Theorem 2.3 implies that if there exists $\psi^* \in \mathcal{R}_0$ satisfying $\int \psi^* dR^* = \tilde{\alpha}$, then for all $\psi \in \mathcal{R}$, ψ^* can be written as

$$(4.24) \quad \psi^* = I_{\{\tilde{\varphi} > c\}} + \kappa I_{\{\tilde{\varphi} = c\}},$$

where κ is defined as

$$(4.25) \quad \kappa = \begin{cases} 0 & \text{if } R^*[\tilde{\varphi} = c] = 0, \\ \frac{\tilde{\alpha} - R^*[\tilde{\varphi} > c]}{R^*[\tilde{\varphi} = c]} & \text{otherwise} \end{cases}.$$

The equation (4.24) implies that ψ^* is a solution of the static problem, in other words if ψ^* satisfies

$$E^{R^*}[\psi] \leq E^{R^*}[\psi^*] = \tilde{\alpha} \quad \forall \psi \in \mathcal{R},$$

then we have

$$E^R[\psi] \leq E^R[\psi^*] \quad \forall \psi \in \mathcal{R}.$$

Now we have only to find the solution $\psi^* \in \mathcal{R}_0$ satisfying

$$(4.26) \quad E^{R^*}[\psi^*] = \int \psi^* dR^* = \tilde{\alpha} \iff E^{P^*}[\psi^* H] = \int \psi^* H dP^* = \alpha.$$

Let us change the expressions in terms of R and R^* back into ones in terms of Q and P^* .

Using the relation

$$\tilde{\alpha} \geq R^*[\tilde{\varphi} > a] = R^* \left[\frac{dQ}{dP^*} > a \right] = \int_{\{\frac{dQ}{dP^*} > a\}} dR^* = \int_{\{\frac{dQ}{dP^*} > a\}} \frac{H}{E^{P^*}[H]} dP^*,$$

the equation (4.23) becomes

$$c := \inf \left\{ a \mid \int_{\{\frac{dQ}{dP^*} > a\}} H dP^* \leq \alpha \right\}.$$

The equations (4.24) and (4.25) becomes

$$\psi^* = I_{\{\frac{dQ}{dP^*} > c\}} + \kappa I_{\{\frac{dQ}{dP^*} = c\}}$$

where κ is

$$\begin{aligned} \kappa &= \frac{\frac{\alpha}{E^{P^*}[H]} - \int_{\{\frac{dQ}{dP^*} > c\}} \frac{H}{E^{P^*}[H]} dP^*}{\int_{\{\frac{dQ}{dP^*} = c\}} \frac{H}{E^{P^*}[H]} dP^*} \\ &= \frac{\alpha - \int_{\{\frac{dQ}{dP^*} > c\}} H dP^*}{\int_{\{\frac{dQ}{dP^*} = c\}} H dP^*} \quad \text{if } P^*[\{dQ/dP^* = c\} \cap \{H > 0\}] > 0, \end{aligned}$$

where c is defined as

$$c = \inf \left\{ a \mid \int_{\{\frac{dQ}{dP^*} > a\}} H dP^* \leq \alpha \right\}.$$

4.3. General solutions in an incomplete market In this subsection, we see the static problem (4.17) becomes the max-min optimal problem. It is used the coherent risk measure defined on the Orlicz space. See the papers [19, 20, 4, 15] for details.

Definition 4.10. Let \mathcal{X} be a linear subspace of L^0 that contains all constants. The acceptance set of a monetary risk measure $\rho : \mathcal{X} \rightarrow \mathbb{R} \cup \{\infty\}$ is given by

$$\mathcal{A}_\rho := \{X \in \mathcal{X} : \rho(X) \leq 0\}.$$

A subset U of \mathcal{X} is an algebraic neighborhood of $x \in \mathcal{X}$ if for every $y \in \mathcal{X}$, there exists an $\epsilon > 0$ such that

$$x + ty \in U \text{ for all } 0 \leq t \leq \epsilon.$$

The algebraic interior of a subset A of \mathcal{X} , denoted by $\text{core}(A)$, consists of all $x \in A$ that have an algebraic neighborhood in A .

We call a function $\Phi : [0, \infty) \rightarrow [0, \infty)$ a Young function if it is left-continuous, convex, $\lim_{x \downarrow 0} \Phi(x) = \Phi(0) = 0$, and $\lim_{x \rightarrow \infty} \Phi(x) = \infty$. It follows from these properties that Φ is increasing and continuous except possibly at a single point, where it jumps to ∞ . So the condition of left-continuity is needed at that one point. The conjugate(or polar) function Ψ of Φ is defined as

$$\Psi(y) := \sup_{x \geq 0} \{xy - \Phi(x)\}, \quad y \geq 0.$$

The function Ψ is a Young function and its conjugate function is Φ . The Orlicz hearts corresponding to Φ defined as

$$M^\Phi := \{X \in L^0 : E^P[\Phi(c|X|)] < \infty \text{ for all } c > 0\}.$$

The Orlicz space for Φ is defined as

$$L^\Phi := \{X \in L^0 : E^P[\Phi(c|X|)] < \infty \text{ for some } c > 0\}.$$

We identify a probability measure Q on (Ω, \mathcal{F}) that is absolutely continuous with respect to P with the Radon-Nikodym derivative $Z_Q = dQ/dP \in L^1$. The set

$$\mathcal{D} := \{Z_Q \in L^1 : Z_Q \geq 0, E^P[Z_Q] = 1\}$$

represents all probability measures on (Ω, \mathcal{F}) that is absolutely continuous with respect to P . Let \mathcal{D}^Ψ be denoted by the intersection

$$\mathcal{D}^\Psi = \mathcal{D} \cap L^\Psi.$$

Theorem 4.11 ([4]). *Let $\rho : M^\Phi \rightarrow \mathbb{R} \cup \{+\infty\}$ be a coherent risk measure with acceptance set*

$$\mathcal{A}_\rho := \{X \in M^\Phi : \rho(X) \leq 0\}.$$

If $\text{core}(\text{dom } \rho) \neq \emptyset$, then ρ is real-valued and can be represented as

$$(4.27) \quad \rho(X) = \max_{Q \in \mathcal{Q}_\Psi} E^Q[-X], \quad X \in M^\Phi,$$

for the $\|\cdot\|_\phi^$ -bounded, convex set*

$$\mathcal{Q}_\Psi := \{Q \in \mathcal{D}^\Psi : E^Q[X] \geq 0 \text{ for all } X \in \mathcal{A}_\rho\}.$$

Assume that the contingent claim H belongs to M^Φ . Let the static problem (4.17) be the primal problem with value

$$(4.28) \quad p := \min_{\psi \in \mathcal{R}_0} \rho((1 - \psi)H)$$

$$(4.29) \quad = \min_{\psi \in L^\infty} \{\rho((1 - \psi)H) + \chi_{\mathcal{R}_0}(\psi)\},$$

where $\chi_{\mathcal{R}_0}(\psi)$ is the indicator function.

Let $\rho : M^\Phi \rightarrow (-\infty, \infty]$ be a coherent risk measure with $\text{core}(\text{dom } \rho) \neq \emptyset$. Then the dual problem of the primal problem (4.28) is given by

$$(4.30) \quad d = \sup_{Q \in \mathcal{D}^\Psi} \inf_{\psi \in \mathcal{R}_0} \{E^Q[(1 - \psi)H]\}.$$

Also the strong duality holds, i.e., $p = d$. If $\tilde{\psi} \in \mathcal{R}_0$ is the solution of (4.28), and $\tilde{Z}_Q = d\tilde{Q}/dP$ is the solution of (4.30), then $(\tilde{Z}_Q, \tilde{\psi})$ is a saddle point of the function $E^Q[(1 - \psi)H]$. Consequently, it holds

$$(4.31) \quad \min_{\psi \in \mathcal{R}_0} \max_{Q \in \mathcal{Q}_\Psi} E^Q[(1 - \psi)H] = \max_{Q \in \mathcal{Q}_\Psi} \min_{\psi \in \mathcal{R}_0} E^Q[(1 - \psi)H].$$

For $\tilde{Z}_Q = \arg \min_{Q \in \mathcal{Q}_\Psi} E[Z_Q(1 - \tilde{\psi})H]$, it can be shown that

$$\max_{\psi \in \mathcal{R}_0} E[Z_Q\psi H] = E[\tilde{Z}_Q\tilde{\psi}H].$$

For each $Q \in \mathcal{Q}_\Psi$ define $p(Q)$ as

$$(4.32) \quad p(Q) := \max_{\psi \in \mathcal{R}_0} E^Q[\psi H].$$

It is shown [20, 19] that Fenchel duality $d(Q)$ of $p(Q)$ is given by

$$(4.33) \quad d(Q) := \inf_{\lambda \in \Lambda_+} \left\{ \int_{\Omega} [HZ_Q - H \int_{\mathcal{M}} Z_Q^* d\lambda] dP + \alpha \lambda(\mathcal{M}) \right\}.$$

Here Λ is the space of all σ -additive signed measures on $(\mathcal{M}, \mathcal{S})$ with bounded variation, where \mathcal{S} is a σ -algebra generated by all subsets of \mathcal{M} .

Also strong duality holds, i.e.

$$(4.34) \quad d(Q) = p(Q) \quad \forall Q \in \mathcal{Q}_\Psi.$$

Moreover, for each $Q \in \mathcal{Q}_\Psi$ there exists a solution $\tilde{\lambda}_Q$ to (4.33). The optimal randomized test $\tilde{\psi}_Q$ of (4.32) has the following structure.

$$\tilde{\psi}_Q(\omega) := \begin{cases} 1, & HZ_Q > H \int_{\mathcal{M}} Z_{Q^*} d\tilde{\lambda}_Q(Q^*) \\ 0, & HZ_Q < H \int_{\mathcal{M}} Z_{Q^*} d\tilde{\lambda}_Q(Q^*), \end{cases} \quad P - a.s.$$

with

$$E^{Q^*}[\tilde{\psi}_Q H] = \alpha \quad \tilde{\lambda}_Q - a.s.$$

It can be shown that

$$(4.35) \quad \max_{Q \in \mathcal{D}^\Psi} \min_{\psi \in R_0} \{E^Q[(1 - \psi)H]\}$$

$$(4.36) \quad = \max_{Q \in \mathcal{D}^\Psi, \lambda \in \Lambda_+} \left\{ E^P \left[HZ_Q \wedge H \int_{\mathcal{M}} Z_{Q^*} d\lambda \right] - \alpha \lambda(\mathcal{M}) \right\}.$$

There exists \tilde{Q} which maximizes the equation (4.30) with respect to $Q \in \mathcal{D}^\Psi$. Because of strong duality (4.34), there exists a solution $\tilde{\lambda} = \tilde{\lambda}_{\tilde{Q}}$ to (4.33). Thus there exists a solution $(\tilde{Q}, \tilde{\lambda})$ of the equation (4.36).

If $(\tilde{Q}, \tilde{\lambda})$ is the solution pair of (4.36), then the solution of the static optimization problem (4.17) is given by

$$\tilde{\psi} := \begin{cases} 1, & HZ_{\tilde{Q}} > H \int_{\mathcal{M}} Z_{Q^*} d\tilde{\lambda}(Q^*) \\ 0, & HZ_{\tilde{Q}} < H \int_{\mathcal{M}} Z_{Q^*} d\tilde{\lambda}(Q^*), \end{cases} \quad P - a.s.$$

with

$$E^{\tilde{Q}}[\tilde{\psi} H] = \alpha \quad \tilde{\lambda} - a.s.$$

5. OPTIMAL PARTIAL HEDGING IN A COMPLETE MARKET

In this section, we will see how to do optimal partial hedging by using the risk measure $\rho(X) = E[X]$ in a complete market [14, 18]. Let $W_t, 0 \leq t \leq T$, be a Brownian motion on a probability space (Ω, \mathcal{F}, P) . Let P^* be a unique equivalent martingale measure on (Ω, \mathcal{F}, P) . Consider a generalized geometric Brownian motion of stock price process whose differential is given by

$$(5.37) \quad dS_t = \mu_t S_t dt + \sigma_t S_t dW_t, \quad 0 \leq t \leq T.$$

This equation can be equivalently written as

$$(5.38) \quad S_t = S_0 \exp \left\{ \int_0^t \sigma_s dW_s + \int_0^t \left(\mu_s - \frac{1}{2} \sigma_s^2 \right) ds \right\}.$$

that the interest rate is zero, $\mu_t = \mu (> 0)$, $\sigma_t = \sigma (> 0)$ are constants for simplicity. If we define $Y := -W_T/\sqrt{T}$, then Y is a standard normal random variable. The geometric Brownian motion of stock price process (5.38) becomes

$$\begin{aligned} S_T &= S_0 \exp \left\{ \sigma W_T + \left(\mu - \frac{1}{2} \sigma^2 \right) T \right\} \\ (5.39) \quad &= S_0 \exp \left\{ -\sigma \sqrt{T} Y + \left(\mu - \frac{1}{2} \sigma^2 \right) T \right\}. \end{aligned}$$

The Girsanov's Theorem implies that the equivalent martingale measure P^* is given by

$$\begin{aligned} \frac{dP^*}{dP} &= \exp \left\{ -\int_0^T \Theta_t dW_t - \frac{1}{2} \int_0^T \|\Theta_t\|^2 dt \right\} = \exp \left(-\frac{\mu}{\sigma} W_T - \frac{1}{2} \left(\frac{\mu}{\sigma} \right)^2 T \right) \\ (5.40) \quad &= \exp \left(\frac{\mu}{\sigma} \sqrt{T} Y - \frac{1}{2} \left(\frac{\mu}{\sigma} \right)^2 T \right) = \text{const} \cdot S_T^{-\mu/\sigma^2}, \end{aligned}$$

where $\Theta_t := \mu_t/\sigma_t$ is the market price of risk [13, 11]. The process W^* defined as

$$W_t^* = W_t + \int_0^t \Theta_u du = W_t + \frac{\mu}{\sigma} t$$

is a Brownian motion under P^* . Consider a European call option $H = (S_T - K)^+$ with a strike price K . Then the claim H can be replicated completely with the initial capital

$$H_0 = E^{P^*}[H] = S_0 N(d_+) - KN(d_-),$$

where N denotes the standard normal distribution function, and

$$d_{\pm} = \frac{\ln(S_0/K)}{\sigma\sqrt{T}} \pm \frac{1}{2}\sigma\sqrt{T}.$$

For the optimal partial hedging, let $\alpha < E^{P^*}[H]$, i.e., α be smaller than the Black-Scholes price $H_0 = E^{P^*}[H]$. Note that from the equation (5.40)

$$\varphi = \frac{dP}{dP^*} > a \iff S_T > a' \quad \text{for some constant } a'.$$

As we have shown in the previous section,

$$\psi^* = I_{\{S_T > c\}} + \kappa I_{\{S_T = c\}}$$

is the solution to the problem (4.18) and (4.19), and c is determined by

$$\alpha = E^{P^*}[H(I_{\{S_T > c\}} + \kappa I_{\{S_T = c\}})].$$

For simplicity, we assume that $P[S_T = c] = 0$. Then ψ^* becomes $\psi^* = I_{\{S_T > c\}}$. Let us calculate α . From (5.39) and (5.40) we have

$$\begin{aligned}
 \alpha &= E^{P^*} [H(I_{\{S_T > c\}})] = \int_{\{S_T > c\}} (S_T - K)^+ dP^* \\
 &= e^{-\frac{1}{2}(\frac{\mu}{\sigma})^2 T} \int_{\{S_T > c\}} (S_T - K)^+ e^{\frac{\mu}{\sigma}\sqrt{TY}} dP \\
 &= e^{-\frac{1}{2}(\frac{\mu}{\sigma})^2 T} \int_{\{S_T > c\} \cap \{S_T > K\}} (S_T - K) e^{\frac{\mu}{\sigma}\sqrt{TY}} dP \\
 &= e^{-\frac{1}{2}(\frac{\mu}{\sigma})^2 T} \left(\int_{\{S_T > c\} \cap \{S_T > K\}} S_T e^{\frac{\mu}{\sigma}\sqrt{TY}} dP \right. \\
 (5.41) \quad &\quad \left. + K \int_{\{S_T > c\} \cap \{S_T > K\}} e^{\frac{\mu}{\sigma}\sqrt{TY}} dP \right)
 \end{aligned}$$

If $c \leq K$, then $\{S_T > K\} \subset \{S_T > c\}$ and

$$\alpha = E^{P^*} [H(I_{\{S_T > c\}})] = E^{P^*} [H],$$

which is a contradiction to the assumption $E^{P^*} [H] < \alpha$. So $c > K$ and $\{S_T > c\} \subset \{S_T > K\}$. First calculate $S_T > c$. From (5.39), we have

$$\begin{aligned}
 S_T > c &\iff Y < \frac{\ln \frac{S_0}{c} + (\mu - \sigma^2/2)T}{\sigma\sqrt{T}} = \frac{\ln(S_0/c)}{\sigma\sqrt{T}} - \frac{\sigma\sqrt{T}}{2} + \frac{\mu}{\sigma}\sqrt{T} \\
 &= d_c + \frac{\mu}{\sigma}\sqrt{T},
 \end{aligned}$$

where d_c is defined as

$$d_c = \frac{\ln(S_0/c)}{\sigma\sqrt{T}} - \frac{1}{2}\sigma\sqrt{T}.$$

We can calculate

$$\begin{aligned}
 \int_{\{S_T > c\}} S_T e^{\frac{\mu}{\sigma}\sqrt{TY}} dP &= S_0 e^{(\mu - \frac{\sigma^2}{2})T} \frac{1}{\sqrt{2\pi}} \int_{\{d_c + \frac{\mu}{\sigma}\sqrt{T} > y\}} e^{(\frac{\mu}{\sigma} - \sigma)\sqrt{T}y} e^{-\frac{1}{2}y^2} dy \\
 &= S_0 e^{\frac{1}{2}(\frac{\mu}{\sigma}\sqrt{T})^2} \frac{1}{\sqrt{2\pi}} \int_{\{d_c + \sigma\sqrt{T} > z\}} e^{-\frac{1}{2}z^2} dz \\
 &= S_0 e^{\frac{1}{2}(\frac{\mu}{\sigma}\sqrt{T})^2} N(d_c + \sigma\sqrt{T}),
 \end{aligned}$$

by changing a variable with $z = y - (\mu/\sigma - \sigma)\sqrt{T}$. We can also calculate

$$\begin{aligned} \int_{\{S_T > c\}} e^{\frac{\mu}{\sigma}\sqrt{T}Y} dP &= \frac{1}{\sqrt{2\pi}} \int_{\{d_c + \frac{\mu}{\sigma}\sqrt{T} > y\}} e^{(\frac{\mu}{\sigma})\sqrt{T}y} e^{-\frac{1}{2}y^2} dy \\ &= e^{\frac{1}{2}(\frac{\mu}{\sigma}\sqrt{T})^2} \frac{1}{\sqrt{2\pi}} \int_{\{d_c > z\}} e^{-\frac{1}{2}z^2} dz \\ &= e^{\frac{1}{2}(\frac{\mu}{\sigma}\sqrt{T})^2} N(d_c), \end{aligned}$$

by changing a variable with $z = y - (\mu/\sigma)\sqrt{T}$. Thus α in (5.41) becomes

$$\begin{aligned} \alpha &= e^{-\frac{1}{2}(\frac{\mu}{\sigma}\sqrt{T})^2} \left\{ S_0 e^{\frac{1}{2}(\frac{\mu}{\sigma}\sqrt{T})^2} N(d_c + \sigma\sqrt{T}) - K e^{\frac{1}{2}(\frac{\mu}{\sigma}\sqrt{T})^2} N(d_c) \right\} \\ &= S_0 N(d_c + \sigma\sqrt{T}) - KN(d_c). \end{aligned}$$

Thus the modified claim $\tilde{\psi}H = HI_{\{S_T > c\}} = (S_T - c)^+ + (c - K)I_{\{S_T > c\}}$ should be hedged, and the price of the modified claim at time t is given by

$$E^{P^*}[\tilde{\psi}H | \mathcal{F}_t] = S_t N\left(\frac{\ln(S_t/c)}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t}\right) - KN\left(\frac{\ln(S_t/c)}{\sigma\sqrt{T-t}} - \frac{1}{2}\sigma\sqrt{T-t}\right).$$

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