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The Existence of an Alternating Sign on a Spanning Tree of Graphs

Dongseok Kim*
Department of Mathematics, Kyonggi University, Suwon, 443-760, Korea
e-mail : dongseok@kgu.ac.kr

Young Soo Kwon and Jaeun Lee
Department of Mathematics, Yeungnam University, Kyongsan, 712-749, Korea
e-mail : ysookwon@ynu.ac.kr *and* julee@yu.ac.kr

Abstract. For a spanning tree T of a connected graph Γ and for a labelling $\phi : E(T) \to \{+, -\}$, ϕ is called an *alternating sign* on a spanning tree T of a graph Γ if for any cotree edge $e \in E(\Gamma) - E(T)$, the unique path in T joining both end vertices of e has alternating signs. In the present article, we prove that any graph has a spanning tree T and an alternating sign on T.

1. Introduction

A *graph* Γ is an ordered pair $\Gamma = (V(\Gamma), E(\Gamma))$ comprising a set $V(\Gamma)$ of vertices together with a set $E(\Gamma)$ of edges. A graph is *signed* if there is a function $\mu : E(\Gamma) \to \{+, -\}$. A *labelling* on Γ means to be a 2-edge coloring which is a function $\phi : E(\Gamma) \to \{+, -\}$ unless stated differently. A graph Γ is *bipartite* if vertices can be divided into two disjoint sets S and T such that every edge connects a vertex in S to one in T. A *tree* is a connected acyclic simple graph. A *spanning tree* is a spanning subgraph that is a tree. A *walk* is an alternating sequence of vertices and edges, beginning and ending with a vertex, where each vertex is incident to both the edge that precedes it and the edge that follows it in the sequence, and where the vertices that precede and follow an edge are the end vertices of that edge. A walk is *closed* if its first and last vertices are the same, and *open* if they are different. A *path* is an open walk which is *simple*, meaning that no vertices (and thus no edges) are repeated. We will often omit edges in path if there is no ambiguity.

For a spanning tree T of a connected graph Γ, a labeling $\phi : E(T) \to \{+, -\}$ is called an *alternating sign* on a spanning tree T of a graph Γ if for any cotree edge $e \in E(\Gamma) \setminus E(T)$, the unique path $v_0, v_1, v_2, \ldots, v_\ell$ in T joining both end vertices of

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* Corresponding Author.

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e satisfies that $\phi(v_i v_{i+1}) \neq \phi(v_{i+1} v_{i+2})$ for any $i = 0, 1, \ldots, \ell - 2$. The motivation of this research is the following: Bipartite graphs naturally rise in knot theory as induced graphs of Seifert surfaces of links as discussed in section 2. Plumbed Seifert surfaces play a key role in the research of the geometry of knot complements [9]. The existence of an alternating sign on a spanning tree of bipartite graphs is a key ingredient to show the existence of such a surface [2, 4, 5].

In the present article, we prove that not only a bipartite graph but also a graph has such a spanning tree T and an alternating sign on T. Note that if any graph without multiple edges and loops has such a tree and an alternating sign, then one can easily show that any graph with multiple edges or loops also has them. Hence in this note, we assume that any graph has no multiple edges and no loops. For any edge $e = \{u, v\}$ in Γ, we simply denote the edge e by uv.

The outline of this paper is as follows. In section 2, we first review some preliminary definitions in graph and knot theory to introduce the origin of the problem, then we prove the existence of a spanning tree and an alternating sign in Theorem 3.2 in section 3. In section 4, we find some applications for knot theory. In section 5, we discuss some further problems.

2. Induced graphs and Seifert surfaces

In this section, we review the origin of the problem which came from the knot theory.

Let L be a link in $\mathbb{S}^3$. A compact orientable surface $\mathcal{F}$ is a *Seifert surface* of L if the boundary of $\mathcal{F}$ is isotopic to L. The existence of such a surface was first proven by Seifert using an algorithm on a diagram of L, named after him as *Seifert's algorithm* [8]. A Seifert surface $\mathcal{F}_L$ of an oriented link L by applying Seifert's algorithm to a link diagram $D(L)$ as shown in Figure 1 (i) is called a *canonical Seifert surface*. From such a canonical Seifert surface, we construct an induced graph $\Gamma(\mathcal{F}_L)$ by collapsing discs to vertices and half twist bands to signed edges as illustrated in Figure 1 (i). From arbitrary Seifert surfaces, these process can be done too. Since a link L is tame and its Seifert surface $\mathcal{F}_L$ is compact, the induced graph $\Gamma(\mathcal{F}_L)$ is a finite graph. Since the Seifert surface $\mathcal{F}_L$ is orientable, it is two sided namely $+, -$. By considering a bipartition $(\{P = \{v \in E(\Gamma(\mathcal{F}_L))|$ the local orientation on disc is $+\}, N = \{v \in E(\Gamma(\mathcal{F}_L))|$ the local orientation on disc is $-\})$ as indicated on each vertices in Figure 1 (ii), $\Gamma(\mathcal{F}_L)$ is a bipartite graph. For bipartite graphs, it is easy to see that the length of a closed path is always even. It is fairly easy to see that the number of Seifert circles (half twisted bands), denoted by $s(\mathcal{F}_L)(c(\mathcal{F}_L))$, is the cardinality of the vertex set, $V(\Gamma(\mathcal{F}_L))$ (edge set $E(\Gamma(\mathcal{F}_L))$, respectively). A spanning tree T of $\Gamma(\mathcal{F}_L)$ is depicted in Figure 1 (iii). The number of edges of a spanning tree of a connected graph with n vertices is $n - 1$. One can see that the length of the path joining both end vertices of $e \in \Gamma(\mathcal{F}_L)$ is odd.

By [7], one can define the top plumbing as follows. Let α be a proper arc on a Seifert surface S. Let D_α be a 3-cell on top of S along a tubular neighborhood C_α of α on S. Let $A_n \subset D_\alpha$ be an annulus of n-full twists such that $A_n \cap \partial D_\alpha = C_\alpha$.

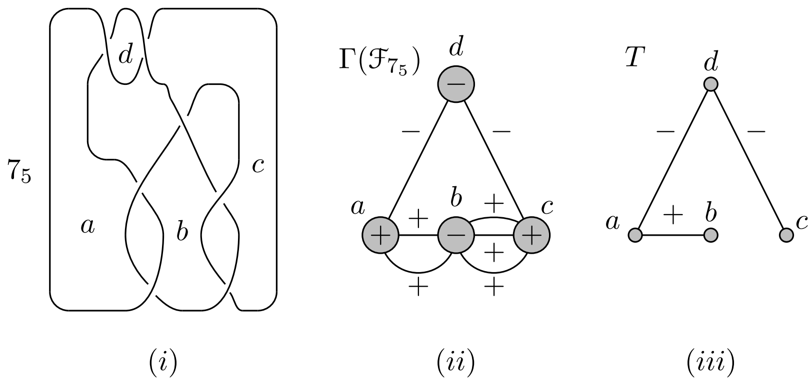

Figure 1: (i) A knot diagram of 7_5 and its Seifert surface $\mathcal{F}_{7_5}$ whose discs are named a, b, c, d, (ii) its corresponding signed induced graph $\Gamma(\mathcal{F}_{7_5})$ and (iii) a spanning tree T of $\Gamma(\mathcal{F}_{7_5})$.

A Seifert surface $\mathcal{F}$ is a *flat plumbing basket surface* if $\mathcal{F} = D_2$ or if $\mathcal{F} = \mathcal{F}_0 *_\alpha A_0$ which can be constructed by plumbing A_0 to a flat plumbing basket surface $\mathcal{F}_0$ along a proper arc $\alpha \subset D_2 \subset \mathcal{F}_0$. We say that a link L admits a *flat plumbing basket presentation* if there exists a flat plumbing basket surface $\mathcal{F}$ such that $\partial\mathcal{F}$ is equivalent to L. In [2], it was shown that every link admits a flat plumbing basket presentation from the closed braid diagrams of L. However, not all diagrams of links are the closed braid diagrams and the modification require several Reidemeister moves as shown in the Alexander theorem. To deal with the canonical Seifert surface $\mathcal{F}_L$, we require Theorem 3.2 to prove the existence of such a flat plumbing basket surface of a link L from its diagram as given in Theorem 4.1.

3. The existence of alternating signs

For a connected graph Γ, if Γ has a Hamilton path, namely, a spanning tree T isomorphic to a path, then the assignment of signs to the edges of T alternately is an alternating sign on T. In this section, we show that even though a graph Γ does not contain a Hamilton path, Γ has such a tree and an alternating sign.

Lemma 3.1. *Let Γ be a connected graph and v_* be a fixed vertex of Γ. Then there is a spanning tree T of Γ such that for any cotree edge $e \in E(\Gamma) \setminus E(T)$, the unique path $v_0, v_1, v_2, \ldots, v_\ell$ in T joining both end vertices of e satisfies that either*

$$d_T(v_*, v_0) < d_T(v_*, v_1) < d_T(v_*, v_2) < \ldots < d_T(v_*, v_\ell)$$

or

$$d_T(v_*, v_0) > d_T(v_*, v_1) > d_T(v_*, v_2) > \ldots > d_T(v_*, v_\ell),$$

where $d_T(v_*, v)$ is the length of the unique path joining v_* and v in T.

Proof. Let $\mathcal{T}_\Gamma$ be the set of all spanning trees of Γ. We define a function $\Psi : \mathcal{T}_\Gamma \to R$ by $\Psi(T) = \sum_{v \in V(\Gamma)} d_T(v_*, v)$, where $d_T(v_*, v)$ is the length of the unique path joining v_* and v in T. Let T_M be a spanning tree of Γ such that $\Psi(T_M) = \max\{\Psi(T) : T \in \mathcal{T}_\Gamma\}$. We will show that T_M is a spanning tree that completes the proof. Assume that T_M does not satisfy the property. Then there is a cotree edge $e = \{u, v\} \in E(\Gamma)\backslash E(T_M)$ which does not satisfy the condition. Let $v_0, v_1, v_2, \ldots, v_\ell$ be the unique path P in T_M from $u = v_0$ to $v = v_\ell$ and let Q be the unique path in T_M from v_* to $v_\ell = v$. Let v_i be the first vertex in P that contained in Q. Then it comes from the choice of e that v_i is neither $v_0 = u$ nor $v_\ell = v$. Moreover, $d_{T_M}(v_*, v_{i-1}) > d_{T_M}(v_*, v_i)$ and $d_{T_M}(v_*, v_i) < d_{T_M}(v_*, v_{i+1})$. For a convenience, let $d_{T_M}(v_*, v_0) \le d_{T_M}(v_*, v_\ell)$. Let T' be the spanning tree of Γ having edge set $E(T) \cup \{e\} \setminus \{v_{i-1}, v_i\}$. Then $d_{T_M}(v_*, w) \le d_{T'}(v_*, w)$ for all $w \in V(\Gamma) \setminus \{v_{i-1}\}$. Since

$$d_{T'}(v_*, v_{i-1}) \ge d_{T'}(v_*, v_0) > d_{T'}(v_*, v_\ell) = d_{T_M}(v_*, v_\ell) \ge d_{T_M}(v_*, v_0) \ge d_{T_M}(v_*, v_{i-1}),$$

we have $\Psi(T_M) = \sum_{v \in V(\Gamma)} d_{T_M}(v_*, v) < \sum_{v \in V(\Gamma)} d_{T'}(v_*, v) = \Psi(T')$. This contradicts the maximality of $\Psi(T_M)$. It completes the proof. □

Theorem 3.2. *Any connected graph Γ has a spanning tree T and an alternating sign on T.*

Proof. Let v_* be a fixed vertex of Γ. By Lemma 3.1, there is a spanning tree T of Γ such that for any cotree edge $e \in E(\Gamma) \setminus E(T)$, the unique path $v_0, v_1, v_2, \ldots, v_\ell$ in T joining both end vertices of e satisfies that either

$$d_T(v_*, v_0) < d_T(v_*, v_1) < d_T(v_*, v_2) < \ldots < d_T(v_*, v_\ell)$$

or

$$d_T(v_*, v_0) > d_T(v_*, v_1) > d_T(v_*, v_2) > \ldots > d_T(v_*, v_\ell).$$

Define $\phi : E(T) \to \{+, -\}$ as follows: For any edge $e = uv \in T$, $\phi(uv) = +$ if $m(u, v)$ is even; and $\phi(uv) = -$ if $m(u, v)$ is odd, where $m(u, v) = \max\{d_T(v_*, u), d_T(v_*, v)\}$. Then one can easily show that ϕ is an alternating sign on T. □

Remark 3.3. The proof of Lemma 3.1 gives an algorithm to find a spanning tree of Γ having an alternating sign and the proof of Theorem 3.2 gives a way to assign alternating signs.

4. An application in knot theory

Let L be a link in $\mathbb{S}^3$. Let Γ be the induced graph of a canonical Seifert surface $\mathcal{F}$ of the link L with the vertex set $V(\Gamma)$ and the edge set $E(\Gamma)$ where $V(\Gamma) = n$ and $E(|\Gamma|) = c(\mathcal{F}) = m$.

Using Theorem 3.2, there is a spanning tree T and an alternating sign ϕ on T such that for any $e \in E(\Gamma) - E(T)$, the unique path in T joining both ends of e has

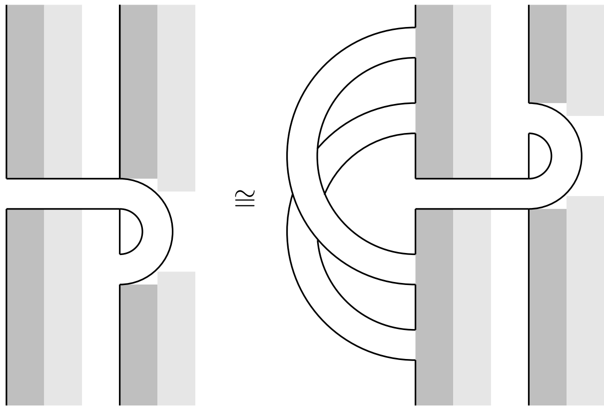

Figure 2: How to change the sign of a twisted band by adding two flat annuli.

an alternating signs. Let $\kappa : E(\Gamma) \to \{+, -\}$ be a labeling representing the original sign of edges in Γ. Let

$$\mathcal{A} = \{e \in E(T) \mid \kappa(e) \neq \phi(e)\}.$$

First if an edge e in $E(T)$ belongs to $\mathcal{A}$, then we have to isotop the link by a type II Reidemeister move. Since we can completely reverse the sign of all edges in the spanning tree T, we may further assume the total number of type II Reidemeister moves in the process is less than or equal to $\left\lceil \frac{n-1}{2} \right\rceil$. Let α be the minimum of the cardinality of the set $\mathcal{A}$ and $n - |\mathcal{A}| - 1$. Now we set D which is starting disc in a flat basket surface to be the disc corresponding to the spanning tree T. Let

$$\mathcal{B} = \{e \in E(\Gamma) - E(T) \mid \kappa(e) \neq \sum_{f \in P_e} \phi(f)\}.$$

If an edge e in $E(\Gamma) - E(T)$ belongs to $\mathcal{B}$, then we can plumb a flat annulus along a curve α corresponding to the path P_e in the spanning tree T. Otherwise, we need to add three flat annuli to make the half twisted band presented by the edge e as shown in Figure 2 [2]. By plumbing all edges in $E(\Gamma) - T$ as described, we have a flat plumbing basket surface of L. Then by summarizing these process of making a flat plumbing surface of L, we obtain the following theorem.

Theorem 4.1([5]). *Let Γ be an induced graph of canonical Seifert surface $\mathcal{F}$ of a link L with the vertex set $V(\Gamma)$ and the edge set $E(\Gamma)$ where $V(\Gamma) = s(\mathcal{F}) = n$ and $E(|\Gamma|) = c(\mathcal{F}) = m$. Let T be a spanning tree of Γ and ϕ an alternating sign on T chosen in Theorem 3.2. Let α be the minimum of $|\mathcal{A}|$ and $n - |\mathcal{A}| - 1$ and let β be*

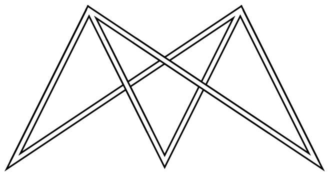

Figure 3: A diagram of the complete bipartite graph $K_{2,3}$ whose boundary is the torus knot $T(2,3)$.

$|\mathcal{B}|$. *Then the flat plumbing basket surface of L can be obtained from D by plumbing $3(m-n)+2(\alpha-\beta)+3$ flat annuli.*

The efficiency of this process and Theorem 4.1 can be found in Example 3.10. [5].

5. Conclusions

As we have discussed in section 2, bipartite graphs naturally raise in knot theory as an induced graphs. For such an induced graph, all edges represents a half twisted band. There is a very similar concept in graph theory but in completely different setup, this is a voltage assignment to determine a two cell embedding of a graph into a surface which use only $0, 1$ where 0 represent a flat band and 1 represent a half twisted band. A detailed definition of voltage assignment and two cell embedding and their relation can be found in [3]. Because the direction of half twist band does change the surface up to homeomorphism, this voltage assignment completely determine the problem of the two cell embeddings of a graph into surfaces. As we have seen in section 2, the induced diagram of a link is always a bipartite graph but not a complete bipartite graph. A very recent article [1] finds an interesting result that every torus knot is a boundary of a complete bipartite graph Γ with a special graph diagram as illustrated in Figure 3 where all voltage assignments on the edge of Γ are 0. We can naturally raise the following questions.

Question 5.1. *For a given link L, is there a graph diagram $D(\Gamma)$ of a complete bipartite graph Γ such that the link L is a boundary of $D(\Gamma)$ where all voltage assignments on the edges of Γ are 0?*

A weaker version of Question 5.1 is given as follows.

Question 5.2. *For a given link L, is there a graph diagram $D(\Gamma)$ of a complete bipartite graph Γ such that the link L is a boundary of $D(\Gamma)$ where all voltage assignments on the edges of Γ are either $0, 1$ and -1?*

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