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Commutative Ideals in BE-algebras

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Abstract. In this paper we study properties of commutative BE–algebras and we give the construction of quotient $(X/I; *, I)$ of a commutative BE–algebra X via an obstinate ideal I of X. We construct upper semilattice and prove that is a nearlattice. Finally we define and study commutative ideals in BE–algebras.

1. Introduction

Y. Imai and K. Iseki [5] introduced two classes of abstract algebras: BCK–algebras and BCI–algebras. BCI–algebras as a class of logical algebras are the algebraic formulations of the set difference together with its properties in set theory and the implicational functor in logical systems. They are closely related to partially ordered commutative monoids as well as various logical algebras. Their names are originated form the combinators B, C, K and I in combinatory logic. It is known that the class of BCK–algebras is a proper subclass of the class of BCI–algebras [4].

Recently, H. S. Kim and Y. H. Kim [6] defined a BE–algebra. S. S. Ahn and K. S. So [1, 2] defined the notion of ideals in BE–algebras, and then stated and proved several characterizations of such ideals. A. Walendziak [12] introduced the notion of commutative BE–algebra.

In this paper, we investigate several properties of commutative BE–algebras, and construct quotient algebra X/I of a transitive BE–algebra X via an obstinate ideal I. Finally we define commutative ideals on BE–algebras and state the relationship between commutative BE–algebras, and prove some theorems.

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2. Preliminaries

Definition 2.1([6]). An algebra $(X;*,1)$ of type $(2,0)$ is called a BE–algebra if
(BE1) $x*x=1$ for all $x\in X$;
(BE2) $x*1=1$ for all $x\in X$;
(BE3) $1*x=x$ for all $x\in X$;
(BE4) $x*(y*z)=y*(x*z)$ for all $x,y,z\in X$ (exchange).

We can define relation "$\leq$" on X by $x\leq y$ if and only if $x*y=1$.

In a BE–algebra X, we have the following identities:
(p1) $x*(y*x)=1$.
(p2) $x*((x*y)*y))=1$.

Example 2.2([6]). Let $X:=\{1,a,b,c,d,0\}$ be a set with the following table.

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1

Then $(X;*,1)$ is a BE–algebra.

Example 2.3([8]). Let $X=\{1,2,\ldots\}$ and the operation $*$ is defined as follows:

$$x*y=\begin{cases}1 & \text{if } y\leq x\\ y & \text{otherwise}\end{cases}$$

Then $(X;*,1)$ is a BE–algebra.

Note. For simplicity of notation we write X instead of BE–algebra $(X;*,1)$.

Definition 2.4([12]). An algebra $(X;*,1)$ of type $(2,0)$ is called a dual BCK–algebra if
(BE1) $x*x=1$ for all $x\in X$;
(BE2) $x*1=1$ for all $x\in X$;
(dBCK1) $x*y=y*x=1\Longrightarrow x=y$;
(dBCK2) $(x*y)*((y*z)*(x*z))=1$;
(dBCK3) $x*((x*y)*y)=1$.

Lemma 2.5([12]). *Let $(X;*,1)$ be a dual BCK–algebra and $x,y,z\in X$. Then:*
(a) $x*(y*z)=y*(x*z)$,
(b) $1*x=x$.

Proposition 2.6([12]). *Any dual BCK–algebra is a BE–algebra.*

Example 2.7. Example 2.2, is a BE–algebra, but it is not a dual BCK–algebra.

Definition 2.8([2]). A non-empty subset I of X is called an ideal of X if it satisfies:
(I1) $(\forall x \in X)$ $(\forall a \in I)$ $x * a \in I$, i.e, $X * I \subseteq I$;
(I2) $(\forall x \in X)$ $(\forall a, b \in I)$ $(a * (b * x)) * x \in I$.

Example 2.9. In Example 2.2, $\{1, a, b\}$ is an ideal of X, but $\{1, a\}$ is not an ideal of X, because $(a * (a * b)) * b = (a * a) * b = 1 * b = b \notin \{1, a\}$.

Proposition 2.10([2]). *Let I be an ideal of X. If $a \in I$ and $a \leq x$, then $x \in I$.*

Lemma 2.11([11]). *A nonempty subset I of X is an ideal of X if and only if it satisfies*

(1) $1 \in I$;

(2) $(\forall x, z \in X)$ $(\forall y \in I)$ $(x * (y * z) \in I \Rightarrow x * z \in I)$.

Definition 2.12([7]). A subset F of X is said to be a filter when it satisfies the conditions:
(F1) $1 \in F$;
(F2) $x,\ x * y \in F \Rightarrow y \in F$.
Obviously any filter F of a BE–algebra X is a subalgebra, i.e., $1 \in F$ and if x, $y \in F$, then $x * y \in F$.

Example 2.13. In Example 2.2, $F_1 = \{1, a, b\}$ is a filter of X, but $F_2 = \{1, a\}$ is not a filter of X, since $a * b = a \in F_2$ and $a \in F_2$ but, $b \notin F_2$.

Definition 2.14([6]). A BE–algebra X is said to be self distributive if $x*(y*z) = (x * y) * (x * z)$, for all $x, y, z \in X$.

Example 2.15([6]). Let $X := \{1, a, b, c, d\}$ be a set with the following table.

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1

It is easy to see that X is a BE–algebra satisfying self distributive.

Note that the BE-algebra in Example 2.2, is not self distributive, since $d * (a * 0) = d * d = 1$ while $(d * a) * (d * 0) = 1 * a = a$.

Proposition 2.16([10]). *Let X be a self distributive BE–algebra. Then for all $x, y, z \in X$ the following statements hold:*
(1) *if $x \leq y$, then $z * x \leq z * y$;*

(2) $y * z \leq (x * y) * (x * z)$.

Definition 2.17([2]). A BE–algebra X is said to be transitive if for any $x, y, z \in X$,

$$y * z \leq (x * y) * (x * z).$$

Proposition 2.18([7]). *Let X be a transitive BE–algebra. Then for all $x, y, z \in X$ the following statements hold:*
(1) $y \leq z$ *implies* $x * y \leq x * z$;
(2) $y \leq z$ *implies* $z * x \leq y * x$;
(3) $1 \leq x$ *implies* $x = 1$.

Example 2.19([2]). Let $X := \{1, a, b, c\}$ be a set with the following table.

$*$	1	a	b	c
1	1	a	b	c
a	1	1	a	a
b	1	1	1	a
c	1	1	a	1

Then X is a transitive BE–algebra.

Corollary 2.20([2]). *If X is a self distributive BE–algebra, then it is transitive.*

Note. The converse of Corollary 2.20, need not be true in general. In Example 2.19, X is a transitive BE–algebra, but $a*(a*b) = a*a = 1$, while $(a*a)*(a*b) = 1*a = a$, showing that X is not self distributive.

Definition 2.21([9]). A mapping $f : X \to Y$ of BE–algebras is called a BE–homomorphism if $f(x * y) = f(x) * f(y)$, for all $x, y \in X$.

Definition 2.22([10]). Let I be a proper ideal of BE–algebra X. Then I said to be obstinate if, for any $x, y \in X$, $x, y \notin I$ implies $x * y \in I$ and $y * x \in I$.

Example 2.23. In Example 2.2, $I = \{1, a, b\}$ is an obstinate ideal.

Theorem 2.24([10]). *Let I be an obstinate ideal of a self distributive BE–algebra X. Then $(X/I; *, I)$ is also a BE-algebra, which is called to be the quotient algebra via I.*

Definition 2.25([12]). Let X be a BE–algebra. We say that X is commutative if satisfies $(x * y) * y = (y * x) * x$ for all $x, y \in X$.

Example 2.26. In Example 2.15, X is not a commutative BE–algebra because,

$$(c * a) * a = 1 * a = a \neq 1 = c * c = (a * c) * c.$$

Example 2.27([12]). Let $N_0 = N \cup \{0\}$ and let $*$ be the binary operation of N_0 defined by

$$x * y = \begin{cases} 1 & \text{if } y \leq x \\ y - x & \text{if } x < y \end{cases}$$

then $(N_0; *, 0)$ is a commutative BE–algebra.

Example 2.28. In Example 2.3, X is not commutative because, $(4*5)*5 = 5*5 = 1 \neq 4 = 1*4 = (5*4)*4$.

Example 2.29. In Example 2.19, X is not a $BH/BG/BF/BCK/BCI/B$–algebra because $a * 1 = 1 \neq a$ and is not a d–algebra because $1 * a = a \neq 1$, but it is a commutative BE algebra.

Proposition 2.30([12]). *If X is a commutative BE–algebra, then for all $x, y \in X$, if $x * y = 1$ and $y * x = 1$, then $x = y$.*

Theorem 2.31([12]). *If X is a commutative BE–algebra, then X is a dual BCK–algebra.*

Corollary 2.32([12]). *X is a commutative BE–algebra if and only if it is a commutative dual BCK–algebra.*

3. Some results on commutative BE–algebras

From definition of commutativity we have the following proposition.

Proposition 3.1. *X is a commutative dual BCK–algebra if and only if $(x*y)*y \leq (y*x)*x$ for any $x, y \in X$.*

Proposition 3.2. *Suppose that X is a dual BCK-algebra. Then the following are equivalent, for $x, y \in X$,*
(1) *X is commutative;*
(2) *$x \leq y$ implies $y = (y * x) * x$;*
(3) *$(y * x) * x = (((y * x) * x) * y) * y$.*

Proof. (1) $\Rightarrow$ (2). If $x \leq y$, then $x*y = 1$ and so (1) implies $y = 1*y = (x*y)*y = (y*x)*x$.
(2) $\Rightarrow$ (3). Since $y \leq (y*x)*x$, by (2), we have $(y*x)*x = (((y*x)*x)*y)*y$.
(3) $\Rightarrow$ (1). By (3), the following identity holds:

$$(((y * x) * x) * y) * y = (y * x) * x.$$

Also, since $x \leq (y * x) * x$, then

$$((y * x) * x) * y \leq x * y$$

Hence $(x * y) * y \leq (((y * x) * x) * y) * y = (y * x) * x$ and X is commutative by Proposition 3.1. □

Theorem 3.3. *An algebra X is a commutative BE-algebra if and only if the following identities hold: for $x, y, z \in X$.*
(1) $1 * x = x$;
(2) $x * 1 = 1$;
(3) $(z * x) * (y * x) = (x * z) * (y * z)$;
(4) $x * (y * z) = y * (x * z)$.

Proof. To prove the sufficiency, it suffices to show $(BE1)$ and commutativity. substituting z and y by 1 in (3) and using (2) and (1), we have

$$(1 * x) * (1 * x) = (x * 1) * (1 * 1)$$
$$x * (1 * x) = 1 * 1 = 1$$
$$x * x = 1$$

Hence $(BE1)$ holds. Substituting y by 1 in (3) and using (2) we have

$$(z * x) * (1 * x) = (x * z) * (1 * z)$$
$$(z * x) * x = (x * z) * z.$$

Then X is commutative.
Necessity. It suffices to prove (3). By $(BE4)$ and commutativity $(z * x) * (y * x) = y * ((z * x) * x) = y * ((x * z) * z) = (x * z) * (y * z)$.
Then (3) holds. □

Theorem 3.4. *An algebra X is a commutative BE–algebra if and only if the following identities hold: for $x, y, z \in X$*
(1) $(y * 1) * x = x$;
(2) $(y * x) * (z * x) = (x * y) * (z * y)$;
(3) $x * (y * z) = y * (x * z)$.

Proof. Necessity. It suffices to prove (2). By $(BE4)$ and hypothesis we have

$$(y * x) * (z * x) = z * ((y * x) * x) = z * ((x * y) * y) = (x * y) * (z * y).$$

Sufficiency. By (1) we have $1 * x = ((1 * 1) * 1) * x = x$ $(BE3)$.
From (1), (2) and $(BE3)$ we conclude that $1 = 1 * 1 = ((1 * x) * 1) * (1 * 1) = (1 * (1 * x)) * (1 * (1 * x)) = (1 * x) * (1 * x) = x * x$ $(BE1)$.
By $(BE1)$ and (1) we have $1 = (x * 1) * (x * 1) = x * 1$, hence $(BE2)$ holds. It suffices to prove commutativity. From (1), (2), (3), we have
$(y * x) * x = (y * x) * ((y * 1) * x) = (x * y) * ((y * 1) * y) = (x * y) * y$. Then X is commutative. □

Proposition 3.5. *Let X be a transitive BE–algebra. If the least upper bound $x \vee y$ of x and y exists, then for all $z \in X$, the greatest lower bound $(x * z) \wedge (y * z)$ of $x * z$ and $y * z$ exists and $(x \vee y) * z = (x * z) \wedge (y * z)$.*

Proof. If the least upper bound $x \vee y$ of x and y exists, by Proposition 2.16, $(x \vee y) * z \leq x * z$. Similarly, $(x \vee y) * z \leq y * z$. Hence $(x \vee y) * z$ is a lower bound of $(x * z)$ and $(y * z)$. Also, assume that u is any lower bound of $(x * z)$ and $(y * z)$. Then $u \leq x * z$ and $u \leq y * z$. By $(BE4)$ we have $x \leq u * z$ and $y \leq u * z$ and so $x \vee y \leq u * z$. Using $(BE4)$ $u \leq (x \vee y) * z$. Hence $(x \vee y) * z$ is the greatest lower bound of $(x * z)$ and $(y * z)$. Therefore the greatest lower bound $(x * z)$ and $(y * z)$ exists and $(x * z) \wedge (y * z) = (x \vee y) * z$. □

Theorem 3.6. *A dual BCK–algebra X is commutative if and only if $(X; \leq)$ is an upper semilattice with $x \vee y = (y * x) * x$, for any $x, y \in X$.*

Proof. Since $y \leq (y * x) * x$ and $x \leq (y * x) * x$, we have $(y * x) * x$ is an upper bound of x and y for any $x, y \in X$. Let z be any upper bound of x and y. Since $x \leq z$, by Proposition 3.2, $z = (z * x) * x$. Also, since $y \leq z$, we obtain $(y * x) * x \leq (z * x) * x$. Hence $(y * x) * x \leq z$ and $(y * x) * x$ must be least upper bound of x and y.

Conversely, since X is an upper semilattice, we have $x \vee y = y \vee x$, then $(y * x) * x = (x * y) * y$. Hence X is commutative. □

Proposition 3.7. *Let X be a transitive commutative BE–algebra. If there is a lower bound a of x and y, then the greatest lower bound $x \wedge y$ of x and y exists and $x \wedge y = ((x * a) \vee (y * a)) * a$.*

Proof. Since $(X; \leq)$ is an upper semilattice, then $(y * a) \vee (x * a)$ exist and by Proposition 3.5, we have $((y * a) * a) \wedge ((x * a) * a) = ((x * a) \vee (y * a)) * a$. Since X is commutative, we have $(x \vee a) \wedge (y \vee a) = ((x * a) \vee (y * a)) * a$. Hence $x \wedge y = ((x * a) \vee (y * a)) * a$. □

Proposition 3.8. *Suppose that X is a commutative dual BCK–algebra. Then for any $a, x, y \in X$ the following hold:* (1) *$x * y = y$ if and only if $x \vee y = 1$;*

(2) *$(x \vee y) * x = x \vee (y * x) = y * x$;*

(3) *$x * y = y$ implies $y * x = x$;*

(4) *$a \leq y$ implies $(y * a) * (x * a) = x * y$;*

(5) *$a \leq x$ implies $(x * a) \vee (y * a) = (y * x) * (y * a)$;*

(6) *$y \leq z$ implies $x * z = x * (y \vee z)$.*

Proof. (1) If $x * y = y$, by Proposition 3.2, we have $x \vee y = y \vee x = (x * y) * y = y * y = 1$.

Conversely, if $x \vee y = 1$, then $x * y = ((x * y) * y) * y = (x \vee y) * y = 1 * y = y$.

(2) By Proposition 3.2, we have $((y * x) * x) * x = y * x$, it follows

$$y * x = (y \vee x) * x = (x \vee y) * x$$
$$y * x = (y * x) \vee x = x \vee (y * x)$$

Hence $(x \vee y) * x = x \vee (y * x) = y * x$.

(3) If $x * y = y$, then $x \vee y = 1$ by (1). So, (2) gives $y * x = (x \vee y) * x = 1 * x = x$.

(4) If $a \leq y$, then $(y * a) * (x * a) = x * ((y * a) * a) = x * (y \vee a) = x * y$.

(5) If $a \leq x$, then (4) implies $(x * a) \vee (y * a) = ((x * a) * (y * a)) * (y * a) = (y * x) * (y * a)$.

(6) Since $y * z = 1$ and $y \vee z = (y * z) * z$ it follows that $x * (y \vee z) = x * ((y * z) * z) = (y * z) * (x * z) = 1 * (x * z) = (x * z)$. □

Theorem 3.9. *Let $(X; \leq)$ be an upper semilattice with the greatest element* 1. *Then*

$(X;\leq)$ *has its associated commutative* BE*–algebra* $(X;*,1)$ *if and only if satisfies the following conditions: for any* $x,y,z,\in X$,

(1) *If* $x\leq y$*, then there exists one and only one complement* y'_x *of* y *relative to* x *in the sense that* $x\leq y'_x$ *and* $(y'_x)'_x=y$*;*

(2) $((x\vee y)'_x\vee z)'_{(x\vee y)'_x}=((x\vee z)'_x\vee y)'_{(x\vee z)'_x}$*;*

(3) $1'_x=x$.

Proof. (1) Assume that the associated commutative BE–algebra $(X;*,1)$ of $(X;\leq)$ exists, then $x\vee y=(y*x)*x$, for any $x,y\in X$. If $x\leq y$, we define $y'_x=y*x$, then $x\leq y'_x$ and

$$(y'_x)'_x=(y*x)*x=y.$$

Let z be any complement of y relative to x. Then $x\leq z$ and $z'_x=y$. Since $x\leq z$, we have $(z'_x)'_x=z$, then $z=(z'_x)'_x=y'_x$. So the relative complement y'_x is unique. Hence (1) is true.

Next, by Proposition 3.8, we have $(x\vee y)'_x=(x\vee y)*x=y*x$ and so

$$((x\vee y)'_x\vee z)'_{(x\vee y)'_x}=((y*x)\vee z)'_{y*x}=z*(y*x)$$

and

$$((x\vee z)'_x\vee y)'_{(x\vee z)'_x}=((z*x)\vee y)*(z*x)=y*(z*x).$$

Therefore the $(BE4)$ implies that (2) holds.
(3) $1'_x=1*x=x$.

Conversely, because $x\leq x\vee y$ for any $x,y\in X$, by (1), we can define a binary operation $*$ on X by

$$y*x=(x\vee y)'_x$$

No, by (3), we obtain

$$(y*1)*x=((1\vee y)'_1\vee x)'_x=(1'_1\vee x)'_x=(1\vee x)'_x=1'_x=x.$$

Next, by definition of relative complements, we have $x\leq(x\vee y)'_x$ then

$$(y*x)*x=((x\vee y)'_x\vee x)'_x=((x\vee y)'_x)'_x=x\vee y.$$

Likewise, $(x*y)*y=y\vee x$. Thus commutativity of $\vee$ gives

$$(y*x)*x=(x*y)*y\ (a).$$

Note that $x*(y*z)=((y*z)\vee x)'_{y*z}=((y\vee z)'_z\vee x)'_{(y\vee z)'_z}$ and $y*(x*z)=((x*z)\vee y)'_{x*z}=((z\vee x)'_z\vee y)'_{(y\vee z)'_z}$. It follows from (2) that

$$x*(y*z)=y*(x*z)\ (b)$$

Now, left $*$ multiplying both sides of above equality (a) by z, the following holds:

$$z * ((y * x) * x) = z * ((x * y) * y)$$
$$z * ((y * x) * x) = (y * x) * (z * x) = (x * y) * (z * y)$$

By (b) we conclude that

$$(y * x) * (z * x) = (x * y) * (z * y).$$

By Theorem 3.4, X is a commutative BE–algebra.

Finally, since $(y * x) * x = x \vee y$ we have $(X; *, 1)$ is the associated commutative BE–algebra of $(X; \leq)$. □

By a nearlattice we mean a join-semilattice $(S, \vee)$ such that for every $a \in A$, the principal order filter $[a) = \{x \in A : x \geq a\}$ is a lattice with respect to the order by $\vee$[3].

Proposition 3.10. *Let X be a commutative BE–algebra. Then*

(1) *for each $a \in X$, the mapping $f_a : x \to x * a$ is an antitone involution on the section $[a, 1]$.*

(2) *$(A, \leq)$ is a nearlattice with section antitone involution, where*

$$x \vee y = (x * y) * y,$$

and for every $a \in X$, the antitone involution f_a on $[a, 1]$ is given by $f_a(x) = x * a$.

Proof. (1) Since X is a commutative BE–algebra and $a \leq x$, then

$$f_a(f_a(x)) = (x * a) * a = x.$$

If $x \leq y$, then $y * a \leq x * a$ and so $f_a(y) \leq f_a(x)$.

(2) Let X be a commutative BE–algebra. Then we have

1. $x * 1 = 1$ and $1 * x = x$;
2. $(x * y) * y = (y * x) * x$;
3. Since $x \leq (x * y) * y$, $((x * y) * y) * z \leq x * z$. Hence $(((x * y) * y) * z) * (x * z) = 1$.

By Theorem 6.4.4, [3], $(A, \leq)$ is a nearlattice with section antitone involution, where $x \vee y = (x * y) * y$, and for every $a \in X$, the antitone involution f_a on $[a, 1]$ is given by $f_a(x) = x * a$. □

Theorem 3.11. *Let I be an obstinate ideal of a commutative BE–algebra X. Then the quotient algebra $(X/I; *, I)$ is a commutative BE–algebra.*

Proof. Suppose that $C_x, C_y \in X/I$. Then
$(C_x * C_y) * C_y = C_{x*y} * C_y = C_{(x*y)*y} = C_{(y*x)*x} = C_{y*x} * C_x = (C_y * C_x) * C_x$.
This shows that $(X/I; *, I)$ is commutative. □

4. Commutative ideals in BE–algebras

Definition 4.1. *A subset I of X is called a commutative ideal of X if*

(1) $1 \in I$;

(2) $x * (y * z) \in I$ *and* $x \in I$ *implies* $((z * y) * y) * z \in I$,

for all $x, y, z \in X$.

Example 4.2. In Example 2.2, $I = \{1, a, b\}$ is a commutative ideal.

Example 4.3. In Example 2.19, $I = \{1, c\}$ is not an obstinate ideal but it is a commutative ideal.

Proposition 4.4. *If* I *is a commutative ideal of* X*, then it is a filter of* X*.*

Proof. Let $x, y \in X$. If $x * y \in I$ and $x \in I$, since $x * (1 * y) \in I$, we have $((y * 1) * 1) * y \in I$, that is , $y \in I$. Hence I is a filter of X. □

Proposition 4.5. *An ideal* I *of a* BE*–algebra* X *is commutative if and only if* $x * y \in I$ *implies* $((y * x) * x) * y \in I$*, for any* $x, y \in X$*.*

Proof. Necessity. For any $x, y \in X$, if $x*y \in I$, then by $(BE3)$ $1*(x*y) = x*y \in I$ and $1 \in I$. Since I is commutative, it follows that $((y * x) * x) * y \in I$.

Sufficiency. Obviously, $1 \in I$. For any $x, y, z \in X$, if $x * (y * z) \in I$ and $x \in I$, since I is an ideal of X, we have $y * z \in I$, then our hypothesis implies $((z * y) * y) * z \in I$ therefore I is a commutative ideal of X. □

Proposition 4.6. *An ideal* I *of* X *is commutative if and only if* $x * y \in I$ *implies* $((((y * x) * 1) * 1) * ((y * x) * x) * y) \in I$ *for any* $x, y \in X$*.*

Proposition 4.7. *If* $\{I_\lambda : \lambda \in \Lambda\}$ *is a family of commutative ideals of* X*, then so is* $\cap_{\lambda\in\Lambda} I_\lambda$*.*

Proof. Clearly $1 \in \cap_{\lambda\in\Lambda} I_\lambda$. Let $x * y \in \cap_{\lambda\in\Lambda} I_\lambda$. Then $x * y \in I_\lambda$, for all $\lambda \in \Lambda$. By Proposition 4.5, we have $((y * x) * x) * y \in I_\lambda$, for all $\lambda \in \Lambda$ and so $((y * x) * x) * y \in \cap_{\lambda\in\Lambda} I_\lambda$. This completes the proof. □

Note. In Example 2.15, $I_1 = \{1, a, b\}$ and $I_2 = \{1, a, c\}$ are commutative ideals of X, but $I = I_1 \cup I_2 = \{1, a, b.c\}$ is not a commutative ideal, because $c * (a * d) \in I$ and $c \in I$ but $((d * a) * a) * d = (1 * a) * d = a * d = d \notin I$.

Theorem 4.8. *Let* $f : X \to Y$ *be a homomorphism of* BE*–algebras. If* J *is a commutative ideal of* Y*, then* $f^{-1}(J)$ *is a commutative ideal of* X*.*

Proof. Clearly $1 \in f^{-1}(J)$. Let $x, y \in X$ be such that $x * y \in f^{-1}(J)$.Then $f(x) * f(y) = f(x * y) \in J$. Since J is a commutative ideal of Y, then we have $((f(y) * f(x)) * f(x)) * f(y) \in J$ by Proposition 4.5, it follows that
$((f(y) * f(x)) * f(x)) * f(y) = f((y * x) * x) * f(y) = f(((y * x) * x) * y) \in J$, so that $((y * x) * x) * y \in f^{-1}(J)$. This shows that $f^{-1}(J)$ is a commutative ideal of X. □

Theorem 4.9. *(Extension property) Let* I *and* J *be ideals of a transitive* BE*–algebras* X*, where* $I \subseteq J$*. If* I *is a commutative ideal, then* J *is a commutative ideal.*

Proof. Assume that $x * y \in J$ and $u = x * y$. Then $x * (u * y) \in I$. Since I is commutative, by Proposition 4.5, $(((u*y)*x)*x)*(u*y) \in I$. Not that $I \subseteq J$, we have $(((u*y)*x)*x)*(u*y) \in J$. Therefore $u*(((u*y)*x)*x)*y) \in J$. Since $u \in J$, we have $(((u*y)*x)*x)*y) \in J$. Also, since $y \leq u*y$ and X is transitive, then $(((u*y)*x)*x)*y \leq ((y*x)*x)*y$. Hence $((y*x)*x)*y \in J$. Therefore J is a commutative ideal of X. □

Corollary 4.10. *Trivial ideal* $\{1\}$ *is commutative if and only if all ideals of* X *are commutative.*

Theorem 4.11. *A dual* BCK*–algebra* X *is commutative if and only if* $\{1\}$ *is a commutative ideal.*

Proof. Necessity. If $x * y \in \{1\}$, then $x * y = 1$. Since X is commutative, then $y = (y*x)*x$, thus $((y*x)*x)*y = 1 \in \{1\}$. Hence $\{1\}$ is a commutative ideal. Sufficiency, if $x*y = 1$, then $x*y \in \{1\}$. Since $\{1\}$ is a commutative, by Proposition 4.5, $((y*x)*x)*y \in \{1\}$, that is $((y*x)*x)*y = 1$. On the other hand $y*((y*x)*x) = 1$, thus we get that $y = (y*x)*x$. Hence X is commutative. □

Therefore we have the following relation between commutative ideal and commutative BE–algebra.

Theorem 4.12. *Let* X *be a dual* BCK*–algebra. Then the following statement are equivalent:*

(1) X *is a commutative* BE*–algebra;*

(2) $\{1\}$ *is a commutative ideal of* X*;*

(3) *every ideal of* X *is a commutative ideal.*

Theorem 4.13. *An obstinate ideal* A *of a dual* BCK*–algebra* X *is commutative if and only if the quotient algebra* $(X/A; *, A)$ *is commutative.*

Proof. Assume that A is commutative. If $A_x * A_y = A_1$, then $A_{x*y} = A_1$, and so $x*y \in A$. Hence the commutativity of A implies $((y*x)*x)*y \in A$. Therefore $A_{((y*x)*x)*y} = ((A_y * A_x) * A_x) * A_y = A_1$, proving the ideal A_1 of X/A is commutative. Therefore X/A is a commutative BE–algebra by Theorem 4.12.
Conversely, assume that X/A is a commutative BE–algebra, then A_1 is a commutative ideal of X/A by Theorem 4.12. If $x*y \in A$, we have $A_x * A_y = A_1$, then commutativity of A_1 implies $((A_y * A_x) * A_x) * A_y = A_1$. So, $((y*x)*x)*y \in A$. Hence A is a commutative ideal of X. □

5. Conclusion

In this paper, we discuss on concept of commutative BE–algebras and conclude a dual BCK–algebra X is commutative if and only if $(X; \leq)$ is an upper semilattice. Also we conclude quotient algebra of a commutative BE–algebra X via an obstinate ideal of X is commutative. We prove that an obstinate ideal I of dual BCK–algebra X is commutative if and only if the quotient algebra $(X/I; *, I)$ is commutative. Also, notion of commutative ideals in BE–algebras are introduced.

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