KYUNGPOOK Math. J. 52(2012), 459-472
http://dx.doi.org/10.5666/KMJ.2012.52.4.459

Dangerous Border-collision Bifurcation for a Piecewise Smooth Nonlinear System

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Abstract. A piecewise smooth system is characterized by non-differentiability on a curve in the phase space. In this paper, we discuss particular bifurcation phenomena in the dynamics of a piecewise smooth system. We consider a two-dimensional piecewise smooth system which is composed of a linear map and a nonlinear map, and analyze the stability of the system to determine the existence of dangerous border-collision bifurcation. We finally present some numerical examples of the bifurcation phenomena in the system.

1. Introduction

Piecewise smooth systems have drawn attention to researchers in natural science and engineering fields. In general, a piecewise smooth system is considered as a system of maps that is piecewise smooth and depends smoothly on a parameter. The systems have been considerably studied in science and engineering fields, and they have provided adequate interpretations for various dynamical processes in mathematical and computational models. In this paper, we take account of a two-dimensional piecewise smooth system and probe its dynamical properties which are involved in the so-called dangerous border-collision bifurcation(abbreviated by DBCB in this paper).

It is well known that piecewise smooth systems exhibit a wide variety of bifurcation phenomena. One of the typical bifurcation phenomena in the system is a border-collision bifurcation. See, for example, [8]. Border-collision bifurcation is a dynamical behavior in which the periodicity of stable period orbits changes when a parameter passes through a critical value. The bifurcation was introduced in [5] through some particular two-dimensional piecewise smooth systems for the first time. See [6] and [7] for further studies of border-collision bifurcation.

In the case that a border-collision bifurcation occurs, for instance, when attract-

Received August 14, 2012; accepted September 4, 2012.
2010 Mathematics Subject Classification: 37E05, 37G15, 37N20.
Key words and phrases: piecewise smooth system, (dangerous) border-collision bifurcation.
This work was partially supported by the Soongsil University Research Fund of 2012, and partially supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology (2012R1A1A1014924).

ing fixed points occur on both sides of the border, it was believed that a stability of the fixed point placed on the border would always remain as an attracting one. Unlike such expectation, however, it was found in [4] that there exist some cases in which the fixed point becomes unstable only at a critical bifurcation value, although attracting fixed points exist for all parameter values except the critical value. In other words, stable, unstable and stable bifurcations appear in order, as a bifurcation parameter in the system passes through the value of the border. Such bifurcation phenomena lead us to a new class of bifurcation, *a dangerous border-collision bifurcation*. It reveals a matter of serious concern for practical systems modeled by piecewise smooth systems, because there is no way of representing any signal of the impending collapse or unboundedness.

Since the DBCB being discovered in [4], one of the remarkable results on the bifurcation is mathematical studies in [1, 2]. Using a piecewise linear system, which is given in [5], the mechanism of occurrence of the bifurcation is fully unraveled [1], and furthermore, the existence of the periodic orbits intriguing the DBCB is mathematically verified [2]. In this paper, another type of two-dimensional piecewise smooth system which is given in [5] is considered. The piecewise system is nonlinear, and we shall show that it displays different dynamical behavior from the piecewise linear system by applying dynamical systems tools similar to those used in the works [1, 2]. In addition, we present a numerical example of the DBCB in the system and give a mathematical illustration on the mechanism of its occurrence.

2. Two-dimensional piecewise smooth systems

In this section, we describe a piecewise nonlinear system to be studied in this paper, and briefly summarize the existing works on the DBCB in piecewise linear systems, especially focused on results in the works [4, 1, 2]. Then we search for the conditions of parameters for the stability and the occurrence of the bifurcation in the piecewise nonlinear system.

2.1. System descriptions

We consider a one-parameter family of maps $F_\mu : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$F_\mu(x,y) = \begin{cases} (a - x^2 + by, x) & \text{if } x \leq \mu, \\ (a + cx + by - (\mu + c)\mu,\ dx + (1-d)\mu) & \text{if } x > \mu, \end{cases} \tag{2.1}$$

where a, b, c and d are parameters, μ is a leading parameter to be used for bifurcation parameter. Applying the map F_μ, we define a piecewise nonlinear system by

$$(x_{n+1}, y_{n+1}) = F_\mu\left(x_n, y_n\right). \tag{2.2}$$

One of the maps in (2.1) is linear and the other is nonlinear. Moreover, the map (2.1) possesses the following properties:

1. two distinct smooth maps are separately defined in two smooth regions;

2. one of the maps are glued to the other continuously on the border $x = \mu$;
3. the two maps have the same value at $x = \mu$ but distinct one-sided derivatives.

In determining the existence of the DBCB, it is crucial to have the information of the stability on the border $x = \mu$. The map F_μ (2.1) is continuous everywhere, but it is not differentiable on the border. This means that it is impossible to analyze the stability of the system (2.2) at the fixed point when $x = \mu$ by using typical Jacobian information. In Sec. 3, we propose an alternative way to obtain some information enough to determine the stability.

2.2. DBCB on piecewise linear systems

In [4], the DBCB is disclosed in a piecewise linear system for the first time. The piecewise linear system used in [4] is given as follows:

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{cases} \begin{pmatrix} p & 1 \\ q & 0 \end{pmatrix}\begin{pmatrix} x_n \\ y_n \end{pmatrix} + \begin{pmatrix} \mu \\ 0 \end{pmatrix}, & \text{if } x_n \leq 0, \\ \begin{pmatrix} r & 1 \\ s & 0 \end{pmatrix}\begin{pmatrix} x_n \\ y_n \end{pmatrix} + \begin{pmatrix} \mu \\ 0 \end{pmatrix}, & \text{if } x_n > 0, \end{cases} \tag{2.3}$$

where p, q, r and s are parameters.

The piecewise smooth systems which are given in (2.2) and (2.3) are originally introduced in the work [5] to discuss the border-collision bifurcation. When the following inequalities

$$|p| < 1 + q \quad \text{and} \quad |r| < 1 + s \tag{2.4}$$

hold, the system (2.3) has an attracting fixed point. Indeed, the Jacobian matrices of the linear system (2.3), where it defined, at the fixed point have the complex eigenvalues. In [3], using some periodic orbits of period n ($n \geq 3$), a set of sufficient conditions in which the DBCB in the system (2.3) occurs is built. Through the periodic orbits of period three, the sufficient conditions are given as follows:

$$\begin{aligned} &1 - pr^2 - qs^2 - ps - qr - rs > 0, \\ &1 + pr^2 - qs^2 + ps + qr + rs > 0, \\ &1 - p^2r - q^2s - qr - ps - rs > 0, \\ &1 + p^2r - q^2s + qr + ps + rs > 0. \end{aligned} \tag{2.5}$$

Using the inequalities in (2.4) and (2.5), one can build a region of parameters which guarantees the existence of the DBCB. Following this way, for each period $n > 3$, one can establish such a region in the parameter space, and finally, have the complete set of regions in the parameter space where the DBCB occurs. See Fig. 1.

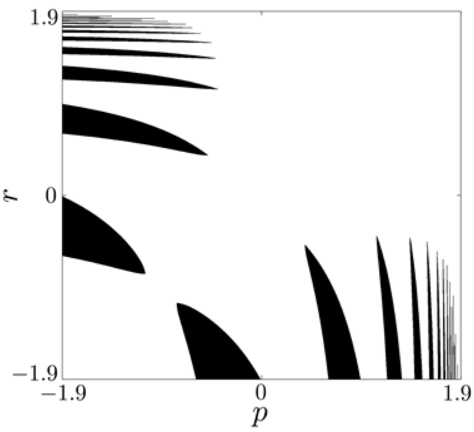


Figure 1: The region (represented in black color) of parameters which guarantee the existence of the DBCB for the piecewise linear system (2.3) obtained using the conditions of existence of complementary orbits at $q = -0.9$ and $s = -0.9$.

2.3. Linear analysis

We rewrite the piecewise nonlinear system (2.2) by plugging the map (2.1) into the system. Then we have

$$(2.6)\quad \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{cases} (a - x_n^2 + by_n, x_n) & \text{if } x_n \le \mu, \\ (a + cx_n + by_n - (\mu + c)\mu,\ dx_n + (1-d)\mu) & \text{if } x_n > \mu. \end{cases}$$

For convenience, we let

$$(2.7)\quad \begin{aligned} H(x, y) &= (a - x^2 + by,\ x), \\ L_\mu(x, y) &= (a + cx + by - (\mu + c)\mu,\ dx + (1-d)\mu). \end{aligned}$$

The map F_μ is smooth in half planes $\{(x, y) \in \mathbb{R}^2 | x < \mu\}$ and $\{(x, y) \in \mathbb{R}^2 | x > \mu\}$, and the line $x = \mu$ is the border of the two half planes, which is a smooth curve. The border line changes as the bifurcation parameter μ varies.

The map F_μ possesses at most three fixed points depending on the value of μ. One can obtain two fixed points from the quadratic equation $H(x, y) = (x, y)$ which are denoted by $(x_H^{\pm}, y_H^{\pm})$, and also, one fixed point from the linear equation

$L_\mu(x,y) = (x,y)$ which is denoted by (x_L, y_L):

$$(x_H^\pm, y_H^\pm) = \left(\frac{b-1 \pm \sqrt{(b-1)^2+4a}}{2}, \frac{b-1 \pm \sqrt{(b-1)^2+4a}}{2} \right),$$
$$(x_L, y_L) = \left(\frac{\mu^2 - (b - bd - c)\mu - a}{bd + c - 1}, \frac{d\mu^2 - (1 - c - d)\mu - ad}{bd + c - 1} \right).$$

By (2.6), the fixed points $(x_H^\pm, y_H^\pm)$ exist if $x_H^\pm \le \mu$, and (x_L, y_L) exists if $x_L \ge \mu$. According to the value of μ, it is determined whether the fixed points are located in smooth regions or the border line. Specifically, when the equalities $x_H^\pm = \mu = x_L$ hold, all the fixed points $(x_H^\pm, y_H^\pm)$ and (x_L, y_L) appear on the border line, and indeed, they are all the same point. This gives us critical bifurcation values $\mu_{\text{crit}}^\pm$;

$$\mu_{\text{crit}}^\pm = \frac{b-1 \pm \sqrt{(b-1)^2+4a}}{2}. \tag{2.8}$$

The system (2.6) is designated to be continuous everywhere. In other words, the two maps H and L_μ given in (2.7) have the same values for all the points on the border. Moreover, when $\mu = \mu_{\text{crit}}^+$, the fixed points (x_H^+, y_H^+) and (x_L, y_L) are located on the border line, and they become the same point. Similarly, when $\mu = \mu_{\text{crit}}^-$, the fixed points (x_H^-, y_H^-) and (x_L, y_L) are located on the border line, and they become the same point. See Fig. 2.

If $\mu \ne \mu_{\text{crit}}^\pm$, the fixed points are located in the smooth regions, if exist. Their stability type can be easily determined by the Jacobian information:

$$J_H(x_H^\pm, y_H^\pm) = \begin{pmatrix} -2x_H^\pm & b \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad J_L(x_L, y_L) = \begin{pmatrix} c & b \\ d & 0 \end{pmatrix}. \tag{2.9}$$

Using the Jacobian matrices in (2.9) one can determine the stability of the system at the fixed points when $\mu \ne \mu_{\text{crit}}^\pm$.

Theorem 2.1. *The following statements hold.*
(a) *If the inequality $-1 < bd < 1 - |c|$ holds, then the fixed point (x_L, y_L) exists and it is asymptotically stable.*
(b) *If the inequalities*

$$|b| < 1 \quad \textit{and} \quad \left| \frac{4a}{(b-1)^2} - 1 \right| < 2 \tag{2.10}$$

hold, then the fixed point (x_H^+, y_H^+) is stable (attracting) whereas the fixed point (x_H^-, y_H^-) is a saddle.

Proof. Clearly, if the inequality $-1 < bd < 1 - |c|$ holds, so do the inequalities $|bd| < 1$ and $|c| < 2$. The eigenvalues λ_i $(i = 1, 2)$ of $J_L(x_L, y_L)$ are

$$\lambda_i = \frac{c \pm \sqrt{c^2 + 4bd}}{2}.$$

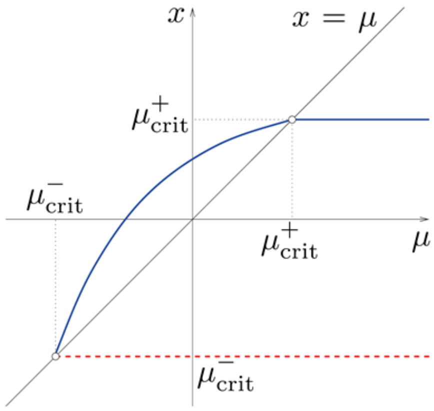


Figure 2: (Color online) Existence, stability and location of fixed points under the assumptions in Theorem 2.1. Below the border line $x = \mu$, the fixed point $(x_H^{\pm}, y_H^{\pm})$ exist, and (x_H^+, y_H^+) is attracting (blue solid line) while (x_H^-, y_H^-) stands for repelling (red dashed line). The fixed point (x_L, y_L) is located in upper the border line and it is attracting (blue solid curve).

In order to show that $|\lambda_i| < 1$, we consider two cases where $c^2 + 4bd < 0$ and $c^2 + 4bd \geq 0$. In the case $c^2 + 4bd < 0$, the eigenvalues λ_i are complex numbers, and the magnitudes of λ_i are both $|bd|$, which is less than one. Meanwhile, in the case $c^2 + 4bd \geq 0$, the eigenvalues λ_i are real numbers, and the following estimation

$$|\lambda_i| \leq \frac{|c| + \sqrt{c^2 + 4bd}}{2} < \frac{|c| + \sqrt{c^2 - 4|c| + 4}}{2} = \frac{|c| + |2 - |c||}{2} = 1,$$

holds because $|c| < 2$. Thus, the fixed point (x_L, y_L) is attracting.

In order to determine the stability at the fixed point (x_H^+, y_H^+), we consider the eigenvalues ν_i $(i = 1, 2)$ of $J_H(x_H^+, y_H^+)$ which are

$$\nu_i = -x_H^+ \pm \sqrt{(x_H^+)^2 + b}.$$

If $(x_H^+)^2 + b < 0$, then for $i = 1, 2$ we have $|\nu_i| = |b| < 1$ by the assumption in (2.10). On the other hand, if $(x_H^+)^2 + b \geq 0$ then, under the assumptions in (2.10), we have

$$-\frac{1-b}{2} < x_H^+ < \frac{1-b}{2},$$

and hence,

$$|\nu_i| \leq |x_H^+| + \sqrt{(x_H^+)^2 + b} < \frac{1-b}{2} + \sqrt{\frac{(1-b)^2}{4} + b} = \frac{1-b}{2} + \frac{|1+b|}{2} = 1.$$

Therefore, (x_H^+, y_H^+) is an attracting fixed point.

We finally determine the stability of the fixed point (x_H^-, y_H^-) by using the eigenvalues of $J_H(x_H^-, y_H^-)$. The eigenvalues are

$$\rho_i = -x_H^- \pm \sqrt{(x_H^-)^2 + b}.$$

for $i = 1, 2$. Under the assumptions in (2.10), we have

$$-\frac{3(1-b)}{2} < x_H^- < -\frac{1-b}{2},$$

and hence,

$$(x_H^-)^2 + b > \frac{(1-b)^2}{4} + b = \frac{(1+b)^2}{4} \geq 0,$$

This implies that the eigenvalues ρ_i $(i = 1, 2)$ are real values, and furthermore,

$$|\rho_1| = \left| -x_H^- - \sqrt{(x_H^-)^2 + b} \right| > \frac{1-b}{2} + \frac{|1+b|}{2} = 1,$$

and

$$|\rho_2| = \left| -x_H^- + \sqrt{(x_H^-)^2 + b} \right| < \sqrt{|b|} < 1.$$

Therefore, (x_H^-, y_H^-) is a saddle fixed point which is unstable. □

Under the assumptions of the parameters in Theorem 2.1 (a) and (b), the inequality $bd + c - 1 < 0$ always holds, and thus, we have the following properties:

1. two fixed points $(x_H^\pm, y_H^\pm)$ exist when $\mu > \mu_{\text{crit}}^+$;
2. two fixed points (x_H^-, y_H^-) and (x_L, y_L) exist when $\mu_{\text{crit}}^- \leq \mu \leq \mu_{\text{crit}}^+$;
3. no fixed point exists when $\mu < \mu_{\text{crit}}^-$.

Theorem 2.1 is applicable to the case $\mu \neq \mu_{\text{crit}}^\pm$ only. Since the system (2.6) is piecewise differentiable, *i.e.*, it is not of C^1 at $\mu = \mu_{\text{crit}}^\pm$, it is impossible to obtain the Jacobian information enough to determine the stability of the system at the fixed points on the border. In this respect, in order to examine the existence of the DBCB on the border, we shall directly investigate trajectories around the fixed points when $\mu = \mu_{\text{crit}}^\pm$ to figure out the stability.

From now on, among the three fixed points, we only consider two of them, that are (x_H^+, y_H^+) and (x_L, y_L). For notational conveniences, we set

$$(x_H, y_H) = (x_H^+, y_H^+), \quad J_H = J_H(x_H^+, y_H^+), \quad \text{and} \quad J_L = J_L(x_L, y_L).$$

The value of the border line on which the fixed points (x_H^+, y_H^+) and (x_L, y_L) meet is $\mu = \mu_{\text{crit}}^+$. For convenience, we denote it by

$$\mu_{\text{crit}} := \mu_{\text{crit}}^+ = \frac{b - 1 + \sqrt{(b-1)^2 + 4a}}{2}. \tag{2.11}$$

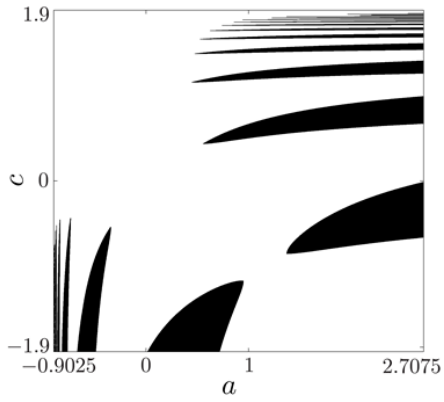


Figure 3: The regions (represented in black color) of dangerous border collision bifurcation for the nonlinear system (2.6) obtained using the conditions of existence of complementary orbits for $b = -0.9$ and $d = 1$.

Figure 2 illustrates the existence and locations of the fixed points. As μ varies before and after the critical value $\mu = \mu_{\text{crit}}$, two distinct fixed points (x_H^+, y_H^+) and (x_L, y_L) exist, and the fixed points are asymptotically stable. However, at the critical value $\mu = \mu_{\text{crit}}$, the two fixed points indicate the same point, which is located on the border line, but their stability types cannot be determined by means of the Jacobian information, because the system is not differentiable at $x = \mu_{\text{crit}}$.

2.4. Parameter regions for DBCB

According to the stability of the fixed point on the border, the DBCB may or may not occur. The parameters c and d are not contained in Eq. (2.11) for critical value μ_{crit}, so that c and d are unable to modify μ_{crit}. The parameters c and d are mainly related to the stability of the map L_μ. We use the Jacobian matrices in (2.9) of the maps H and L_μ to choose the values of parameters a, b, c and d for attracting fixed points at $\mu \neq \mu_{\text{crit}}$. On the other hand, when $\mu = \mu_{\text{crit}}$ one can find a condition of parameters in which a periodic orbit in the system possesses unstable behavior around the fixed points on the border.

We establish the region of the parameter space in which the system (2.6) displays the DBCB. In order to find a condition of the parameters to guarantee the existence of the DBCB, we search for some periodic orbits which are unstable and intersects with the fixed point on the border at $\mu = \mu_{\text{crit}}$. Here, we consider an invariant subspace forming the boundary between the basins of attraction of the two attractors. It is known that the stable manifold of a saddle fixed point can form such a basin boundary. Thus, we study which fixed point can serve this purpose. Under the assumptions in Theorem 2.1, an unstable fixed point orbit does not occur in the region of parameter space.

We now set the values of the parameters b and d as follows: $b = -0.9$ and $d = 1$. Then applying Theorem 2.1, one can obtain a region of the parameter space for a and c as follows:

$$-\frac{1.9^2}{4} < a < \frac{3 \cdot 1.9^2}{4} \quad \text{and} \quad |c| < 1.9. \tag{2.12}$$

Then the fixed points (x_H^+, y_H^+) and (x_L, y_L) are attracting when $\mu \neq \mu_{\text{crit}}$. Meanwhile, at $\mu = \mu_{\text{crit}}$ one can find proper conditions of parameters in which periodic orbits are unstable and intersects with the fixed point on the border. It is a typical mechanism of the DBCB that an unstable periodic orbit crushes the stability at the fixed point on the border. To obtain unstable periodic orbits corresponding to this situation, we compute the equations

$$H^n \circ L_\mu(x, y) = (x, y) \quad \text{and} \quad H \circ L_\mu^n(x, y) = (x, y) \tag{2.13}$$

for $n \geq 1$. When $n = 1$, the solution to Eq. (2.13) does not satisfy the conditions of the system (2.6). This implies that no periodic orbits of period two exist. However, for each $n \geq 2$, there exist the values of the parameters a and c for which the solutions to Eq. (2.13) satisfy the conditions of the system (2.6) so that unstable periodic orbits exist. Figure 3 indicates the region of the parameter spaces for a and c in which the system (2.6) displays the DBCB.

3. An illustrating example

In this section, we illustrate the mechanism for the DBCB in piecewise smooth nonlinear systems through a specific example. Plugging the following values of parameters

$$a = 1.5, \quad b = -0.9, \quad c = -0.7, \quad \text{and} \quad d = 1.0, \tag{3.1}$$

into the system (2.6), we have a piecewise nonlinear system

$$(x_{n+1}, y_{n+1}) = \begin{cases} (1.5 - x_n^2 - 0.9y_n, x_n) & \text{if } x_n \leq \mu, \\ (1.5 - 0.7x_n - 0.9y_n - (\mu - 0.7)\mu,\ x_n) & \text{if } x_n > \mu. \end{cases} \tag{3.2}$$

The values of the parameters in (3.1) satisfy the conditions in Theorem 2.1, and also, plugging the values into Eq. (2.11), we obtain the critical bifurcation value $\mu_{\text{crit}} = 0.6$. When $\mu \neq 0.6$, the fixed points (x_H^+, y_H^+) and (x_L, y_L) exist and their stabilities are both of sink. To prove the existence of the DBCB, we examine the existence of periodic saddle orbits of the piecewise system (3.2) in a neighborhood of the critical value $\mu = 0.6$ and then determine the stability type at the fixed point.

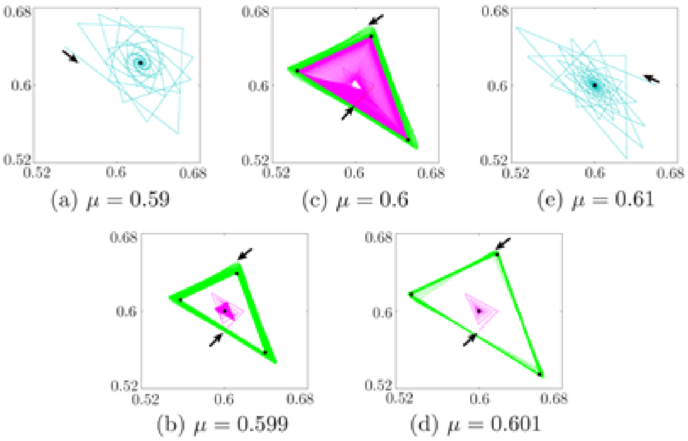


Figure 4: (Color online) The phase plots for the orbits of the system (3.2). The dots indicate attracting fixed points and periodic attracting orbits. Let $\mathbf{z}_0 = (0.6, 0.58)$, $\tilde{\mathbf{z}}_0 = (0.61, 0.66)$, $\mathcal{O}(\mu) = \{\mathbf{z}_n | \mathbf{z}_{i+1} = F_\mu(\mathbf{z}_i), i \in \mathbb{N}\}$, and $\tilde{\mathcal{O}}(\mu) = \{\tilde{\mathbf{z}}_n | \tilde{\mathbf{z}}_{i+1} = F_\mu(\tilde{\mathbf{z}}_i), i \in \mathbb{N}\}$. In (a) and (e), both $\tilde{\mathcal{O}}(\mu)$ and $\mathcal{O}(\mu)$ converge to the fixed point. In (b) and (d), $\mathcal{O}(\mu)$ converges to the fixed point while $\tilde{\mathcal{O}}(\mu)$ converges to the periodic orbit. In (c), both $\mathcal{O}(\mu)$ and $\tilde{\mathcal{O}}(\mu)$ converge to the periodic orbit. No convergent orbits to the fixed point exist.

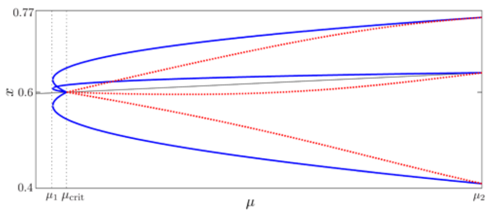


Figure 5: (Color online) The existence of periodic saddle orbits of period three in the system (3.2). The thick solid curves(in blue) represent the location of the x-coordinates of the periodic orbits $P^{\pm}_{(\mu,3)}$ given in (3.3) and (3.4), while the thick dotted curves(in red) represent that of the periodic orbit $Q_{(\mu,3)}$ given in (3.6). The locations of $P^{\pm}_{(\mu,3)}$ and $Q_{(\mu,3)}$ depend on the value of μ. The thin solid line(in black) is the border of the system, $x = \mu$.

3.1. Periodic orbits on the basin boundary

In order to describe a mechanism inducing the occurrence of the DBCB, we observe how the stability type of the fixed point changes when μ passes though μ_{crit}. The main idea of the mechanism is that in the left hand side and the right hand side of the border at $\mu = \mu_{\text{crit}}$ there exist a periodic saddle orbit as well as an attracting fixed point orbit. It is mentioned in Sec. 2.4 that there is no periodic orbit of period two under the assumption of Theorem 2.1. Thus, we consider the existence of periodic orbits of period greater than two in order for μ in a neighborhood of the critical bifurcation values $\mu = \mu_{\text{crit}}$.

For the values of the parameters in (3.1), one can easily obtain periodic orbits of period three from

$$L_\mu^k \circ H \circ L_\mu^{2-k}(x,y) = (x,y) \quad \text{and} \quad H^k \circ L_\mu \circ H^{2-k}(x,y) = (x,y)$$

for $k = 0, 1, 2$. Note that it is impossible to get any solution for periodic orbits from $H^3(x,y) = (x,y)$ and $L_\mu^3(x,y) = (x,y)$. From the equation $L_\mu^2 \circ H(x,y) = (x,y)$, two solutions for x are given as follows:

$$\alpha_0^+ = \alpha_0^+(\mu) = \frac{469 + \sqrt{787200\mu^2 - 551040\mu + 47761}}{820}$$
$$\alpha_0^- = \alpha_0^-(\mu) = \frac{469 - \sqrt{787200\mu^2 - 551040\mu + 47761}}{820},$$

Also, two solutions for y, denoted by β_0^+ and β_0^-, are also obtained corresponding to α_0^+ and α_0^-, respectively. We consider two sets $P_{(\mu,3)}^+$ and $P_{(\mu,3)}^-$ given by

$$P_{(\mu,3)}^+ := \left\{ \left(\alpha_i^+, \beta_i^+\right) = F_\mu^i\left(\alpha_0^+, \beta_0^+\right) \middle|\ i = 0,1,2 \right\}; \tag{3.3}$$

$$P_{(\mu,3)}^- := \left\{ \left(\alpha_i^-, \beta_i^-\right) = F_\mu^i\left(\alpha_0^-, \beta_0^-\right) \middle|\ i = 0,1,2 \right\}. \tag{3.4}$$

We denote two constants μ_1 and μ_2 by

$$\mu_1 = \frac{7}{20} + \frac{\sqrt{5986533}}{9840} \quad \text{and} \quad \mu_2 = -\frac{2}{5} + \frac{\sqrt{433}}{20}. \tag{3.5}$$

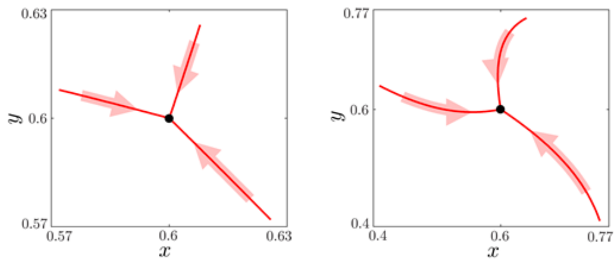
For $\mu_1 < \mu < \mu_{\text{crit}}$, one can show that α_i^+ $(i = 0,1,2)$ are real numbers, and the following inequalities hold:

$$\alpha_0^+ < \mu, \quad \alpha_1^+ > \mu, \quad \text{and} \quad \alpha_2^+ > \mu.$$

Thus, the set $P_{(\mu,3)}^+$ in (3.3) exists as a periodic orbit of period three. Similarly, for $\mu_1 < \mu < \mu_2$, α_i^- $(i = 0,1,2)$ are real numbers, and the inequalities

$$\alpha_0^- < \mu, \quad \alpha_1^- > \mu, \quad \text{and} \quad \alpha_2^- > \mu,$$

hold, and hence, $P_{(\mu,3)}^-$ in (3.4) exists as another periodic orbit of period three.



(a) $\mu_1 \leq \mu < \mu_{\text{crit}}$ (b) $\mu_{\text{crit}} \leq \mu < \mu_2$

Figure 6: Evolution of the periodic saddle orbits $P^+_{(\mu,3)}$ and $Q_{(\mu,3)}$ to the same fixed point. (a) $P^+_{(\mu,3)}$ approaches (x_H, y_H) as $\mu \to \mu_{\text{crit}}$ from μ_1. (b) $Q_{(\mu,3)}$ approaches (x_L, y_L) as $\mu \to \mu_{\text{crit}}$ from μ_2.

On the other hand, another periodic orbit of period three can be obtained from $L_\mu \circ H^2(x,y) = (x,y)$, which provides at most four solutions for x. For $\mu_1 < \mu < \mu_2$, two of them are real and the other two are complex. By checking the order of the rotation of the orbit, one can show that there exists the only one valid solution $\tilde{\alpha}_i$ of the equation above, only if $\mu_{\text{crit}} \leq \mu \leq \mu_2$. For $i = 0, 1, 2$, let $\tilde{\beta}_i$ be the solution for y corresponding to $\tilde{\alpha}_i$. Then the following inequalities hold

$$\tilde{\alpha}_0 < \mu, \quad \tilde{\alpha}_1 < \mu, \quad \text{and} \quad \tilde{\alpha}_2 > \mu.$$

Thus, we obtain another periodic orbit of period three denoted by

$$Q_{(\mu,3)} := \left\{ \left(\tilde{\alpha}_i, \tilde{\beta}_i\right) = F^i_\mu \left(\tilde{\alpha}_0, \tilde{\beta}_0\right) \Big| \ i = 0, 1, 2 \right\}. \tag{3.6}$$

Figure 4 describes how the periodic orbits $P^\pm_{(\mu,3)}$ and $Q_{(\mu,3)}$ change according to the parameter μ.

3.2. Stability of the fixed point at the critical bifurcation value

The stabilities of the periodic orbits $P^\pm_{(\mu,3)}$ and $Q_{(\mu,3)}$ are determined as follows. By computing the Jacobian information, we have

1. $P^-_{(\mu,3)}$ is a periodic attracting orbit for $\mu_1 \leq \mu \leq \mu_2$;

2. $P^+_{(\mu,3)}$ is a periodic saddle orbit for $\mu_1 \leq \mu < \mu_{\rm crit}$;

3. $Q_{(\mu,3)}$ is a periodic saddle orbit for $\mu_{\rm crit} < \mu \leq \mu_2$.

We use the stability information above to determine the stability of the fixed point on the border at $\mu = \mu_{\rm crit}$, and moreover, to understand the behavior of all the orbits which appears at $\mu = \mu_{\rm crit}$.

For Fig. 6 (a) $\mu_1 \leq \mu < \mu_{\rm crit}$, the periodic saddle orbit $P^+_{(\mu,3)}$ exists but the periodic saddle orbit $Q_{(\mu,3)}$ does not. In contrast, for Fig. 6 (b) $\mu_{\rm crit} \leq \mu < \mu_2$, $Q_{(\mu,3)}$ exists but $P^+_{(\mu,3)}$ does not. As μ approaches $\mu_{\rm crit}$, the periodic saddle orbits $P^+_{(\mu,3)}$ approaches the fixed point from the left hand side of the border whereas $Q_{(\mu,3)}$ also approaches the fixed point from the right hand side, and eventually, the two periodic saddle orbits $P^+_{(\mu,3)}$ and $Q_{(\mu,3)}$ are united at $\mu = \mu_{\rm crit}$ and they intersect at the same fixed point. Their local invariant manifolds also collide, and a new local invariant manifold is generated. It is natural that one of the local invariant manifolds becomes dominant, and hence, the stability type at the fixed point is determined by the local invariant manifold, which is unstable.

Only at the critical bifurcation value, a dynamical property at the fixed point is shifted from asymptotically stable to saddle right before and after the stable fixed point, and also, the unstable orbits are united together. This means that the qualitative feature of the periodic saddle orbit transmits to the fixed point. Thus, orbits starting from all points other than the nonsmooth stable manifold become deviated, and finally, they approach another attractor, which is $P^-_{(\mu,3)}$.

It is well-known that the invariant manifolds of a given system is locally influenced by the eigenvectors. The stability type of the fixed point at the critical bifurcation value will be determined by the surviving invariant manifolds of the periodic orbit. Figure 7 (b) illustrates that the surviving invariant manifolds are the ones obtained from the periodic saddle orbit $P^+_{(\mu,3)}$.

4. Conclusions

We have studied the DBCBs found in piecewise nonlinear systems. The occurrence of DBCB depends on the existence of a periodic saddle orbit on the boundary of the basin of the attractor at infinity. In case that no Jacobian information exists at the fixed point, we have suggested a mathematical methodology of how to decide a stability type of the fixed point. We analyzed a qualitative type of the fixed point at the critical bifurcation value and its stability type is of a saddle, which is originated from the qualitative feature of such periodic saddle orbits on the boundary of the basin of the attractor at infinity. In addition, we described a dangerous situation of the numerical simulation without having concrete mathematical theories.

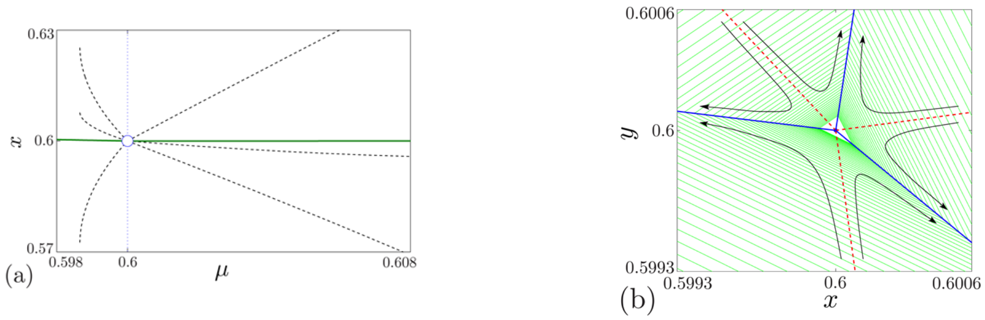


Figure 7: (a) Periodic saddle orbit(dashed curves) and attracting fixed point(solid line). Two periodic saddle orbits coalesce at the fixed point and they modify the stability of the fixed point from stable to unstable. The circle at $\mu = \mu_{\text{crit}}$ indicates the location of the saddle point. (b) Stable manifold(dotted curves) and unstable manifold(solid curves). Any orbit around fixed point does not converge to the fixed point, and instead, eventually escapes from the fixed point along to the unstable manifold.

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