KYUNGPOOK Math. J. 52(2012), 443-457
http://dx.doi.org/10.5666/KMJ.2012.52.4.443

Screen Slant Lightlike Submanifolds of Indefinite Sasakian Manifolds

S. M. Khursheed Haider*, Advin and Mamta Thakur
Department of Mathematics, Faculty of Natural Sciences, Jamia Millia Islamia, New Delhi-110025, India
e-mail : smkhaider@rediffmail.com, advin.maseih@gmail.com
and mthakur09@gmail.com

Abstract. In this paper, we introduce screen slant lightlike submanifold of an indefinite Sasakian manifold and give examples. We prove a characterization theorem for the existence of screen slant lightlike submanifolds. We also obtain integrability conditions of both screen and radical distributions, prove characterization theorems on the existence of minimal screen slant lightlike submanifolds and give an example of proper minimal screen slant lightlike submanifolds of R_2^9.

1. Introduction

The theory of submanifolds of a Riemannian or semi-Riemannian manifold is one of the interesting topics of differential geometry. It is well known that semi-Riemannian manifolds [9] have many similarities with their Riemannian case. However the lightlike submanifolds are different since there normal vector bundle intersect with tangent bundle making it more interesting to study these submanifolds. The general theory of a lightlike submanifold has been developed by Kupeli [7] and Duggal-Sahin [4]. In[2], slant immersions were defined by B.Y Chen as a natural generalization of both holomorphic and totally real immersions. A.Lotta [8] extended the notion of slant immersions to the setting of almost contact metric manifolds and proved some properties of such immersions. Recently, B.Sahin[10] initiated the study of slant lightlike submanifolds of indefinite Hermitian manifolds and obtained some interesting results. Continuing to the study of screen slant lightlike submanifolds of Kaehler manifolds, in[11], he has obtained characterization theorems for screen slant and minimal screen slant lightlike submanifolds with examples.

In the present paper, we introduce and study screen slant lightlike submanifolds of an indefinite Sasakian manifold. The paper is arranged as follows. In section 2,

* Corresponding Author.

Received December 8, 2010; accepted October 9, 2012.
2010 Mathematics Subject Classification: 53C40, 53C15, 53C50.
Key words and phrases: Degenerate metric, screen slant lightlike submanifold, indefinite Sasakian manifold.

we give basic formulae and definition for an indefinite Sasakian manifold and its lightlike submanifolds which we shall use later. In section 3, we prove a characterization theorem and give examples of proper screen slant submanifolds. We also obtain integrability conditions for both screen and radical distributions. In the last section we prove characterization theorems on the existence of minimal screen slant lightlike submanifolds and give an example of proper minimal screen slant lightlike submanifolds of R_2^9.

2. Preliminaries

We follow [4] for the notation and fundamental equations for lightlike submanifolds used in this paper. A submanifold M^m immersed in a semi-Riemannian manifold $(\overline{M}^{m+n}, \overline{g})$ is called a lightlike submanifold if it is a lightlike manifold with respect to the metric g induced from $\overline{g}$ and the radical distribution $Rad\,TM$ is of rank r, where $1 \leq r \leq m$. Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $Rad\,TM$ in TM, i.e.,

$$TM = Rad\,TM \oplus_{orth} S(TM).$$

Consider a screen transversal vector bundle $S(TM^{\perp})$, which is a semi-Riemannian complementry vector bundle of $Rad\,TM$ in $TM^{\perp}$. Since for any local basis $\{\xi_i\}$ of $Rad\,TM$, there exist a local null frame $\{N_i\}$ of sections with values in the orthogonal complement of $S(TM^{\perp})$ in $[S(TM)]^{\perp}$ such that $\overline{g}(\xi_i, N_j) = \delta_{ij}$, it follows that there exist a lightlike transversal vector bundle $ltr(TM)$ locally spanned by $\{N_i\}$ [[4] ,pg-193]. Let $tr(TM)$ be complementary (but not orthogonal)vector bundle to TM in $T\overline{M}|_M$.Then

$$tr(TM) = ltr(TM) \oplus_{orth} S(TM^{\perp}),$$

$$T\overline{M}|_M = S(TM) \oplus_{orth} [(Rad\ \ TM) \oplus ltr(TM)] \oplus_{orth} S(TM^{\perp}).$$

Following are four subcases of a lightlike submanifold $(M, g, S(TM), S(TM^{\perp})$.
Case 1: r-lightlike if $r < min\{m, n\}$.
Case 2: Co-isotropic if $r = n < m$; $S(TM^{\perp}) = \{0\}$.
Case 3: Isotropic if $r = m < n$; $S(TM) = \{0\}$.
Csae 4: Totally lightlike if $r = m = n$; $S(TM) = \{0\} = S(TM^{\perp})$.
The Gauss and Weingarten equations are

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \ \ \forall\, X, Y \in \Gamma(TM) \tag{2.1}$$

$$\overline{\nabla}_X U = -A_U X + \nabla^t_X U, \ \ \forall\, X \in \Gamma(TM), U \in \Gamma(tr(TM)),$$

where $\{\nabla_X Y, A_U X\}$ and $\{h(X, Y), \nabla^t_X U\}$ belongs to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively, ∇ and ∇^t are linear connections on M and on the vector bundle $tr(TM)$, respectively. Moreover, we have

$$\overline{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y) \tag{2.2}$$

$$\overline{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X, N) \tag{2.3}$$

$$\overline{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X, W) \tag{2.4}$$

for each $X, Y \in \Gamma(TM)$, $N \in \Gamma(ltr(TM))$ *and* $W \in \Gamma(S(TM^{\perp}))$. Denote the projection of TM on S(TM) by P. Then, by using (2.1), (2.2),(2.3), and (2.4) and a metric connection $\overline{\nabla}$, we obtain

$$\overline{g}(h^s(X, Y), W) + \overline{g}(Y, D^l(X, W)) = g(A_W X, Y) \tag{2.5}$$

$$\overline{g}(D^s(X, N), W) = \overline{g}(N, A_W X). \tag{2.6}$$

From the decomposition of the tangent bundle of a lightlike submanifold, we have

$$\nabla_X PY = \nabla^*_X PY + h^*(X, PY),$$

$$\nabla_X \xi = -A^*_\xi X + \nabla^{*t}_X \xi,$$

for $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(Rad\, TM)$. By using above equations we obtain

$$\overline{g}(h^l(X, PY), \xi) = g(A^*_\xi X, PY) \tag{2.7}$$

$$\overline{g}(h^*(X, PY), N) = g(A_N X, PY)$$

$$\overline{g}(h^l(X, \xi), \xi) = 0, A^*_\xi \xi = 0.$$

In general, the induced connection ∇ on M is not a metric connection. Since $\overline{\nabla}$ is a metric connection, by using (2.2) we get

$$(\nabla_X g)(Y, Z) = \overline{g}(h^l(X, Y), Z) + \overline{g}(h^l(X, Z), Y).$$

An odd dimensional semi-Riemannian manifold $(\overline{M}, \overline{g})$ is called a contact metric manifold [[3],[6]] if there exists a (1,1) tensor field ϕ, a vector field V, called the characteristic vector field, and its 1-form η satisfying

$$\left.\begin{array}{l} \overline{g}(\phi X, \phi Y) = \overline{g}(X, Y) - \epsilon\eta(X)\eta(Y), \overline{g}(V, V) = \epsilon \\ \phi^2 X = -X + \eta(X)V, \overline{g}(X, V) = \epsilon\eta(X), \\ d\eta(X, Y) = \overline{g}(X, \phi Y), \forall X, Y \in \Gamma(TM), \end{array}\right\} \tag{2.8}$$

where $\epsilon = \pm 1$. It follows that $\phi V = 0$, $\eta o \phi = 0$, $\eta(V) = \epsilon$.

Then $(\phi, V, \eta, \overline{g})$ is called a contact metric structure of $\overline{M}$. We say that $\overline{M}$ has a normal contact structure if $N_\phi + d\eta \otimes V = 0$, where N_ϕ is the Nijenhuis tensor field of ϕ [14]. A normal contact metric manifold is called a Sasakian manifold [13] for which we have

$$\overline{\nabla}_X V = \phi X, \tag{2.9}$$

$$(\overline{\nabla}_X \phi)Y = -\overline{g}(X,Y)V + \epsilon\eta(Y)X. \tag{2.10}$$

Without loss of generality, we may assume that $\epsilon = 1$ throughout this paper.

3. Screen slant lightlike submanifolds

In the present section, we introduce and study screen slant lightlike submanifolds of an indefinite Sasakian manifold. We first recall the definition of screen slant lightlike submanifolds given by Sahin and Yildirim[12].

Definition 3.1.([12]) Let $(M, g, S(TM))$ be a 2q-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$ of index $2q < dim(M)$ with structure vector field V tangent to M. Then we say that M is a screen slant lightlike submanifold of $\overline{M}$ if the following conditions are satisfied:
i) $Rad\,TM$ is invariant with respect to ϕ,i.e $\phi(Rad\,TM) = Rad\,TM$
ii) For any $x \in U \subset M$ and for any non-zero vector field X tangent to $S(TM) = D\perp\{V\}$, linearly independent on V, the angle $\theta(X)$ between ϕX and $S(TM)$ is constant, where D is the complementary distribution to V in screen distribution $S(TM)$.

The constant angle $\theta(X)$ is called the slant angle of $S(TM)$. From now on, we suppose that $(M, g, S(TM))$ is a $2q(< dim(M))$ lightlike submanifold of an indefinite Sasakian manifold with constant index 2q and denote it by M and V is always tangent to M.

For any vector field $X \in (S(TM))$, we write

$$\phi X = TX + \omega X \tag{3.1}$$

where $TX \in \Gamma(TM)$ and $\omega X \in \Gamma(tr(TM))$.

For the existence of proper screen slant lightlike submanifolds in indefinite Sasakian manifolds, we give the following examples. We first recall the Sasakian structure defined on R_q^{2m+1}.

Hereafter, $(R_q^{2m+1}, \phi_0, V, \eta, g)$ will denote the manifold R_q^{2m+1} with its usual Sasakian structure given by

$$\eta = \frac{1}{2}(dz - \sum_{i=1}^{m} y^i dx^i), V = 2\partial z$$

$$\overline{g} = \eta \otimes \eta + \frac{1}{4}(-\sum_{i=1}^{\frac{q}{2}} dx^i \otimes dx^i + dy^i \otimes dy^i + \sum_{i=q+1}^{m} dx^i \otimes dx^i + dy^i \otimes dy^i)$$

$$\phi_0(\sum_{i=1}^{m}(X_i\partial x^i + Y_i\partial y^i) + Z\partial z) = \sum_{i=1}^{m}(Y_i\partial x^i - X_i\partial y^i) + \sum_{i=1}^{m} Y_i\, y^i\partial z$$

where (x^i, y^i, z) are the cartesian co-ordinates.

Example 3.1. For any $\alpha > 0$, let M be a submanifold of $\overline{M} = (R_2^{11}, \overline{g})$ defined by

$$x^1 = u\cosh\alpha,\;\; x^2 = u,\;\; x^3 = s\cos\alpha_1,\;\; x^4 = t\sin\alpha_1,\;\; x^5 = u\sinh\alpha$$

$$y^1 = v\cosh\alpha,\;\; y^2 = v,\;\; y^3 = t\cos\alpha_1,\;\; y^4 = s\sin\alpha_1,\;\; y^5 = v\sinh\alpha$$

Then, a local frame of TM is given by

$$\xi_1 = 2(\cosh\alpha\partial x_1 + \partial x_2 + \sinh\alpha\partial x_5 + (y^1\cosh\alpha + y^2 + y^5\sinh\alpha)\partial z)$$

$$\xi_2 = 2(\cosh\alpha\partial y_1 + \partial y_2 + \sinh\alpha\partial y_5)$$
$$X_1 = 2(\cos\alpha_1\partial x_3 + \sin\alpha_1\partial y_4 + y^3\cos\alpha_1\partial z)$$
$$X_2 = 2(\sin\alpha_1\partial x_4 + \cos\alpha_1\partial y_3 + y^4\sin\alpha_1\partial z)$$
$$V = 2\partial z.$$

Thus M is a 2-lightlike submanifold with $Rad\,TM = span\{\xi_1, \xi_2\}$, which is invariant with respect to ϕ_0. Let $S(TM) = D\perp\{V\} = span\{X_1, X_2\}\perp\{V\}$. Then, $S(TM)$ is Riemannian vector subbundle and it can be easily proved that $S(TM)$ is a slant distribution with slant angle $2\alpha_1$. Moreover, the screen transversal bundle $S(TM^\perp)$ is spanned by

$$W_1 = 2(-\sin\alpha_1\partial x_3 + \cos\alpha_1\partial y_4 - y^3\sin\alpha_1\partial z)$$

$$W_2 = 2(\cos\alpha_1\partial x_4 - \sin\alpha_1\partial y_3 + y^4\cos\alpha_1\partial z)$$

and $ltr(TM)$ is spanned by

$$N_1 = -\cosh\alpha\partial x_1 + \partial x_2 - \sinh\alpha\partial x_5 + (-y^1\cosh\alpha + y^2 - y^5\sinh\alpha)\partial z$$

$$N_2 = -\cosh\alpha\partial y_1 + \partial y_2 - \sinh\alpha\partial y_5.$$

Example 3.2. Let M be a submanifold of $\overline{M} = (R_2^9, \overline{g})$ defined by

$$x^1 = u,\;\; x^2 = s\sin t,\;\; x^3 = \sin s,\;\; x^4 = v$$

$$y^1 = -v,\;\; y^2 = s\cos t,\;\; y^3 = \cos s,\;\; y^4 = u,$$

where $t, s \in (0, \frac{\pi}{2})$. It is easy to see that a local frame of TM is given by

$$\xi_1 = 2(\partial x_1 + \partial y_4 + y^1\partial z)$$

$$\xi_2 = 2(-\partial y_1 + \partial x_4 + y^4\partial z)$$

$$X_1 = 2(s\cos t\partial x_2 - s\sin t\partial y_2 + y^2 s\cos t\partial z)$$
$$X_2 = 2(\sin t\partial x_2 + \cos s\partial x_3 + \cos t\partial y_2 - \sin s\partial y_3 + (y^2\sin t + y^3\cos s)\partial z)$$
$$V = 2\partial z.$$

We see that M is a 2-lightlike submanifold with $Rad\ TM = span\{\xi_1, \xi_2\}$ and $\phi_0(Rad\ TM) = Rad\ TM$. Thus $Rad\ TM$ is invariant with respect to ϕ_0. We choose $S(TM) = D\perp\{V\} = span\{X_1, X_2\}\perp\{V\}$ which is Riemannian and can be easily proved that $S(TM)$ is a slant distribution with slant angle $\theta = \frac{\pi}{4}$. Hence M is a screen slant lightlike submanifold of R^9_2. Moreover, the screen transversal bundle $S(TM^\perp)$ is spanned by

$$W_1 = 2(\sin s\partial x_3 + \cos s\partial y_3 + y^3\sin s\partial z)$$
$$W_2 = 2(\sin t\partial x_2 - \cos s\partial x_3 + \cos t\partial y_2 + \sin s\partial y_3 + (y^2\sin t - y^3\cos s)\partial z)$$

and the lightlike transversal bundle $ltr(TM)$ is spanned by

$$N_1 = -\partial x_1 + \partial y_4 - y^1\partial z$$
$$N_2 = \partial y_1 + \partial x_4 + y^4\partial z.$$

Example 3.3. For any $k, \theta > 0$, let M be a submanifold of $\overline{M} = (R^{13}_2, \overline{g})$ defined by

$$x^1 = u,\ \ x^2 = u\sin\theta,\ \ x^3 = t,\ \ x^4 = k\cos t,\ \ x^5 = k\cos s,\ \ x^6 = u\cos\theta$$
$$y^1 = v,\ \ y^2 = v\sin\theta,\ \ y^3 = s,\ \ y^4 = k\sin t,\ \ y^5 = k\sin s, y^6 = v\cos\theta$$

Then TM is spanned by

$$\xi_1 = 2(\partial x_1 + \sin\theta\partial x_2 + \cos\theta\partial x_6 + (y^1 + y^2\sin\theta + y^6\cos\theta)\partial z)$$
$$\xi_2 = 2(\partial y_1 + \sin\theta\partial y_2 + \cos\theta\partial y_6)$$
$$X_1 = 2(\partial x_3 - ksint\partial x_4 + k\cos t\partial y_4 + (y^3 - y^4 k\sin t)\partial z)$$
$$X_2 = 2(-k\sin s\partial x_5 + \partial y_3 + k\cos s\partial y_5 - y^5 k\sin s\partial z)$$
$$V = 2\partial z.$$

Hence M is a 2-lightlike submanifold with invariant $Rad\ TM = span\{\xi_1, \xi_2\}$. By direct calculation, we obtain $S(TM) = span\{X_1, X_2\}\perp\{V\}$ is slant distribution with slant angle $\theta = \cos^{-1}(\frac{1}{1+k^2})$. Thus M is a screen slant lightlike submanifold of R^{13}_2. Moreover, the screen transversal vector bundle $S(TM^\perp)$ is spanned by

$$W_1 = 2(k\cos t\partial x_4 + k\sin t\partial y_4 + y^4 k\cos t\partial z)$$
$$W_2 = 2(k\cos s\partial x_5 + k\sin s\partial y_5 + y^5 k\cos s\partial z)$$

and the lightlike transversal bundle $ltr(TM)$ is spanned by

$$N_1 = -\partial x_1 + \sin\theta\partial x_2 + \cos\theta\partial x_6 + (-y^1 + y^2\sin\theta + y^6\cos\theta)\partial z$$

$$N_2 = -\partial y_1 + \sin\theta\partial y_2 + \cos\theta\partial y_6.$$

Let M be a screen slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. We denote the projection morphism on the distribution $Rad\,TM$ and $S(TM)$ by Q and P, respectively. Then we have

$$X = QX + PX \tag{3.2}$$

for any $X \in \Gamma(TM)$, where QX denotes the component of X in $Rad\,TM$ and PX denotes the component of X in $S(TM)$. Applying ϕ on (3.2), we obtain

$$\phi X = \phi QX + \phi PX$$

$$= TQX + TPX + \omega PX, \tag{3.3}$$

from which we have

$$\phi QX = TQX, \quad \omega QX = 0 \;\; and \;\; TPX \in \Gamma(S(TM))$$

On the other hand, the screen transversal bundle $S(TM^\perp)$ has the following decomposition

$$S(TM^\perp) = \omega P(S(TM)) \perp \nu$$

Then, for any $W \in \Gamma(S(TM^\perp))$, we write

$$\phi W = BW + CW, \tag{3.4}$$

where $BW \in \Gamma(S(TM))$ and $CW \in \Gamma(\nu)$.

Now, we prove a characterization theorem for the existence of a screen slant lightlike submanifold in indefinite Sasakian manifolds.

Theorem 3.1. *Let M be a $2q$-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$ with constant index $2q < dim(M)$. Then, M is a screen slant lightlike submanifold if and only if there exists a constant $\lambda \in [-1, 0]$ such that*

$$(PoT)^2 X = \lambda(X - \eta(X)V) \tag{3.5}$$

for any $X \in \Gamma(S(TM))$. Moreover, in this case $\lambda = -cos^2\theta|_{S(TM)}$.

Proof. Let M be a screen slant lightlike submanifold of an indefinite Sasakian manifold with constant angle θ which is independent of $X \in \Gamma(S(TM) - \{V\})$ and $x \in U \subset M$. Then

$$cos\theta(X) = -\frac{\overline{g}(X, (PoT)^2 X)}{|\phi X||TPX|}. \tag{3.6}$$

On the other hand, we have

$$cos\theta(X) = \frac{|TPX|}{|\phi X|}. \tag{3.7}$$

From (3.6) and (3.7), we conclude that

$$cos^2\theta = -\frac{\overline{g}(X, (PoT)^2 X)}{|\phi X|^2}.$$

Since $\theta(X)$ is a constant, we observe that $(PoT)^2 X = \lambda(X - \eta(X)V), \lambda \in [-1, 0]$ which proves our assertion.

Conversely, making use of (2.8) and (3.4), for $\xi \in \Gamma(RadTM), N \in \Gamma(ltr(TM))$, we obtain

$$\overline{g}(\phi\xi, N) = -g(\xi, BN + C_1 N + C_2 N + C_3 N),$$

where $BN \in \Gamma(TM), C_1 N \in \Gamma(ltr(TM)), C_2 N \in \Gamma(\omega(S(TM)))$ and $C_3 N \in \Gamma(\mu)$. Thus

$$\overline{g}(\phi\xi, N) = -g(\xi, C_1 N) \neq 0,$$

which shows that $\phi\xi \in \Gamma(RadTM)$, i.e, $RadTM$ is invariant with respect to ϕ. To prove (ii) of definition 3.2, taking inner product of (3.5) with $X \in \Gamma(S(TM))$, we have

$$g((PoT)^2 X, X) = -cos^2\theta\{g(X, X) - \eta(X)\eta(X)\}.$$

Using (2.8) in the above equation, we get

$$g(\phi TPX, X) = -cos^2\theta\{g(\phi X, \phi X)\}. \tag{3.8}$$

Now, from (2.8),(3.7) and (3.8), we have

$$-g(\phi X, TPX) = -cos\theta(\frac{|TPX|}{|\phi X|})g(\phi X, \phi X).$$

The above equation can be written as

$$cos\theta = \frac{g(\phi X, TPX)}{|TPX||\phi X|},$$

which proves (ii). □

From the above Theorem, we have

Corollary 3.2. *Let M be screen slant lightlike submanifold of $\overline{M}$. Then*

$$g(TPX, TPY) = \cos^2\theta|_{S(TM)}[g(X, Y) - \eta(X)\eta(Y)] \tag{3.9}$$

and

$$\overline{g}(\omega PX, \omega PY) = sin^2\theta|_{S(TM)}[g(X, Y) - \eta(X)\eta(Y)] \tag{3.10}$$

for any $X, Y \in \Gamma(TM)$.

Proof. Using (2.8) and (3.3), we have

$$g(TPX, TPY) = -g(X, (PoT)^2Y)$$

for any $X, Y \in \Gamma(S(TM))$. From the above equation and considering Theorem 3.1(ii), we have

$$g(TPX, TPY) = \cos^2\theta(g(X, Y) - \eta(X)\eta(Y)),$$

which proves (3.9) while (3.10) follows from (3.3),(2.8) and (3.9).

Differentiating (3.3) and comparing the tangential and transversal parts we have

$$(\nabla_X T)Y = A_{\omega PY}X + Bh^s(X, Y) - g(X, Y)V + \eta(Y)X \tag{3.11}$$

$$\phi h^l(X, Y) = h^l(X, \phi QY) + h^l(X, TPY) + D^l(X, \omega PY)$$

$$(\nabla_X \omega)Y = -h^s(X, \phi QY) - h^s(X, TPY) + Ch^s(X, Y) \tag{3.12}$$

for any $X, Y \in \Gamma(TM)$, where $(\nabla_X T)Y = \nabla_X \phi QY + \nabla_X TPY - \phi Q\nabla_X Y - TP\nabla_X Y$ and $(\nabla_X \omega)Y = \nabla^s_X \omega PY - \omega P\nabla_X Y$ □

Now, we are in a position to discuss the integrability of the radical and screen distributions involved in the definition of a screen slant lightlike submanifold of indefinite Sasakian manifolds.

Theorem 3.2. *Let M be a screen slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then*
i) The radical distribution $Rad\,TM$ is integrable if and only if the screen transversal second fundamental form of M satisfies

$$h^s(X, \phi Y) = h^s(\phi X, Y), \quad \forall\, X, Y \in \Gamma(Rad\,TM)$$

ii) The screen distribution $S(TM)$ is integrable if and only if

$$Q(\nabla_X TPY - \nabla_Y TPX) = Q(A_{\omega PY}X - A_{\omega PX}Y), \ \forall\, X, Y \in \Gamma(S(TM)).$$

Proof. Using (3.12), for any $X, Y \in \Gamma(Rad\,TM)$, we get

$$\omega P\nabla_X Y = h^s(X, \phi Y) - Ch^s(X, Y),$$

from which we can easily have

$$\omega P[X, Y] = h^s(X, \phi Y) - h^s(\phi X, Y).$$

Thus assertion(i) follows from the above equation. To prove assertion (ii), we use (3.11) to get

$$\nabla_X TPY - A_{\omega PY}X = \phi Q\nabla_X Y + TP\nabla_X Y + Bh^s(X, Y) - g(X, Y)V + \eta(Y)X$$

for any $X, Y \in \Gamma(S(TM))$. From the above equation, we can easily see that

$$\nabla_X TPY - \nabla_Y TPX - A_{\omega PY}X + A_{\omega PX}Y = \phi Q[X, Y] + TP[X, Y] + \eta(Y)X - \eta(X)Y. \tag{3.13}$$

Considering the components of $Rad\,TM$ of (3.13), we get

$$\phi Q[X, Y] = Q(\nabla_X TPY - \nabla_Y TPX) + Q(A_{\omega PX}Y - A_{\omega PY}X),$$

which proves (ii). □

Theorem 3.3. *Let M be a screen slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then the screen distribution defines a totally geodesic foliation if and only if $\phi A_{\omega PY}X + A_{\omega PTPY}X$ has no component in Rad TM for $X, Y \in \Gamma(S(TM))$.*

Proof. From (2.10) and (2.2), a direct calculation shows that

$$\overline{g}(\nabla_X Y, N) = \overline{g}(\overline{\nabla}_X TPY, \phi N) - \overline{g}(A_{\omega PY}X, \phi N) \tag{3.14}$$

for any $N \in \Gamma(ltr(TM))$. Using (2.10),(2.2),(3.3) and (2.4) in the first expression of (3.14), we get

$$\overline{g}(\nabla_X Y, N) = -\overline{g}(\nabla_X (PoT)^2 Y, N) + \overline{g}(A_{\omega PTPY}X, N) - \overline{g}(A_{\omega PY}X, \phi N) \tag{3.15}$$

From (3.15) and considering Theorem 3.1, we have

$$\overline{g}(\nabla_X Y, N) = \cos^2\theta\,\overline{g}(\nabla_X Y, N) + \overline{g}(A_{\omega PTPY}X, N) - \overline{g}(A_{\omega PY}X, \phi N)$$

which implies that

$$\sin^2\theta\,\overline{g}(\nabla_X Y, N) = \overline{g}(A_{\omega PTPY}X + \phi A_{\omega PY}X, N). \tag{3.16}$$

Thus, proof follows from (3.16). □

Theorem 3.4. *Let M be a screen slant lightlike submanifold of an indefinite*

Sasakian manifold $\overline{M}$. Then T can not be parallel even if M satisfies only the condition $\phi(RadTM) \subset RadTM$.

Proof. From (2.9), we have

$$\overline{\nabla}_X V = -\phi X$$

for any $X \in \Gamma(T\overline{M})$. Using (2.2) and (3.1) in the above equation, we get

$$\nabla_X V + h^l(X,Y) + h^s(X,V) = -TX - \omega X.$$

Comparing tangential components on both sides, we get

$$\nabla_X V = -TX.$$

for every $X \in \Gamma(TM)$. Since $TV = 0$, we observe that

$$(\nabla_X T)V = -T^2 X. \tag{3.17}$$

On the other hand, since $RadTM$ is $\phi-$invariant, we have $T^2X = -X$ for every $X \in \Gamma(RadTM)$. Thus, our assertion follows from the fact that $T^2X = -X$ and (3.17). □

4. Minimal screen slant submanifolds

We recall here the definition of a minimal lightlike submanifold M of a semi-Riemannian manifold $\overline{M}$ which has been introduced by Bejan-Duggal in [1] as follows.

Definition 4.1. We say that a lightlike submanifold $(M, g, S(TM))$ isometrically immersed in a semi-Riemannian manifold $(\overline{M}, \overline{g})$ is minimal if
i) $h^s = 0$ on $Rad\,TM$ and
ii) trace h = 0, where trace is written with respect to g restricted to $S(TM)$.
In the case (ii) the condition (i) is trivial. Moreover it has been shown in [1] that above definition is independent of $S(TM)$ and $S(TM^\perp)$, but it depends on the choice of the transversal bundle $tr(TM)$. It is important to note that, as in the semi-Riemannian case, any lightlike totally geodesic submanifold M is minimal.

Now, we give an example of a minimal proper screen slant lightlike submanifold of R^9_2 which is neither totally geodesic nor totally umbilical.

Example 4.1. Let $\overline{M} = (R^9_2, \overline{g})$, be a semi-Euclidean space of signature $(-,-,+,+,+,+,+,+,+)$ with respect to canonical basis $(\partial x_1, \partial y_1, \partial x_2, \partial y_2, \partial x_3, \partial y_3, \partial x_4, \partial y_4, \partial z)$. Consider a complex structure ϕ defined by

$$\phi(x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4, z) = (-y_1, x_1, -y_2, x_2, -x_4\cos\alpha - y_3\sin\alpha,$$
$$-y_4\cos\alpha + x_3\sin\alpha, x_3\cos\alpha + y_4\sin\alpha,$$
$$y_3\cos\alpha - x_4\sin\alpha, y^3(-x_4\cos\alpha - y_3\sin\alpha)$$

$$+ y^4(x_3 \cos\alpha + y_4 \sin\alpha))$$

for $\alpha \in (0, \frac{\pi}{2})$. Let M be a submanifold of (R^9_2, ϕ) given by

$$x^1 = u_1, \;\; x^2 = u_1 \cos\theta - u_2 \sin\theta, \;\; x^3 = u_3, \;\; x^4 = u_4$$

$$y^1 = u_2, \;\; y^2 = u_1 \sin\theta + u_2 \cos\theta, \;\; y^3 = \sin u_3 \sinh u_4, \;\; y^4 = \cos u_3 \cosh u_4.$$

Then, a local frame of TM is given by

$$Z_1 = 2(\partial x_1 + \cos\theta \partial x_2 + \sin\theta \partial y_2 + (y^1 + y^2 \cos\theta)\partial z)$$

$$Z_2 = 2(\partial y_1 - \sin\theta \partial x_2 + \cos\theta \partial y_2 - y^2 \sin\theta \partial z)$$

$$Z_3 = 2(\partial x_3 + \cos u_3 \sinh u_4 \partial y_3 - \sin u_3 \cosh u_4 \partial y_4 + y^3 \partial z)$$

$$Z_4 = 2(\partial x_4 + \sin u_3 \cosh u_4 \partial y_3 + \cos u_3 \sinh u_4 \partial y_4 + y^4 \partial z)$$

$$V = 2\partial z.$$

Hence M is a 2-lightlike submanifold with invariant $Rad\ TM = span\{Z_1, Z_2\}$. We choose $S(TM) = span\{Z_3, Z_4\} \perp \{V\}$ which is Riemannian and after direct calculation, we obtain that $S(TM)$ is a slant distribution with angle α. It is easy to see that the screen transversal bundle is spanned by

$$W_1 = 2(-\cosh u_4 \sinh u_4 \partial x_3 + \cos u_3 \cosh u_4 \partial y_3 - \sin u_3 \cos u_3 \partial x_4 - \sin u_3 \sinh u_4 \partial y_4$$

$$+(-y^3 \cosh u_4 \sinh u_4 - y^4 \sin u_3 \cos u_3)\partial z)$$

$$W_2 = 2(\sin u_3 \cos u_3 \partial x_3 + \sin u_3 \sinh u_4 \partial y_3 - \cosh u_4 \sinh u_4 \partial x_4 + \cos u_3 \cosh u_4 \partial y_4$$

$$+(y^3 \sin u_3 \cos u_3 - y^4 \cosh u_4 \sinh u_4)\partial z)$$

and $ltr(TM)$ is spanned by

$$N_1 = -\partial x_1 + \cos\theta \partial x_2 + \sin\theta \partial y_2 + (-y^1 + y^2 \cos\theta)\partial z$$

$$N_2 = -\partial y_1 - \sin\theta \partial x_2 + \cos\theta \partial y_2 + (-y^1 - y^2 \sin\theta)\partial z.$$

By direct calculation and using Gauss formula , we get

$$h^l = 0, \;\; h^s(X, Z_1) = 0, \;\; h^s(X, Z_2) = 0 \;\; \forall X \in \Gamma(TM)$$

and

$$h^s(Z_3, Z_3) = \frac{-2(\sinh^2 u_4 + \cos^2 u_3)}{\cosh^4 u_4 - \sin^4 u_3} W_2$$

$$h^s(Z_4, Z_4) = \frac{2(\sinh^2 u_4 + \cos^2 u_3)}{\cosh^4 u_4 - \sin^4 u_3} W_2$$

$$h^s(Z_3, Z_4) = \frac{2(\sinh^2 u_4 + \cos^2 u_3)}{\cosh^4 u_4 - \sin^4 u_3} W_1.$$

Hence M is a minimal proper screen slant lightlike submanifold of R^9_2 and the induced connection on M is a metric connection. Note that M is neither totally geodesic nor totally umbilical.

Next, we prove two characterization theorems for minimal slant lightlike submanifolds. First we give the following lemma which we shall use later.

Lemma 4.2. *Let M be a proper screen slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$ such that $dim(D) = dim(S(TM^\perp))$. If $\{e_1, \cdots\cdots, e_m\}$ is a local orthonormal basis of $\Gamma(D)$, then $\{\csc\theta\omega e_1, \cdots\cdots, \csc\theta\omega e_m\}$ is a orthonormal basis of $S(TM^\perp)$.*

Proof. Since $\{e_1, \cdots\cdots, e_m\}$ is a local orthonormal basis of D and D is Riemannian, from corollary 3.1 , we obtain

$$\overline{g}(\csc\theta\omega e_i, \csc\theta\omega e_j) = \delta_{ij},$$

where $i, j = 1, 2, \cdots, m$. Thus the proof is complete. □

Theorem 4.3. *Let M be a proper screen slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$ with structure vector field tangent to M. Then M is minimal if and only if*

$$traceA^*_{\xi_j}|_{S(TM)} = 0,\ traceA_{W_\alpha}|_{S(TM)} = 0 \ \ and\ \ g(D^l(X, W), Y) = 0$$

for $X, Y \in \Gamma(Rad\,TM)$ and $W \in \Gamma(S(TM^\perp))$, where $\{\xi_j\}_{j=1}^r$ is a basis of $Rad\,TM$ and $\{W_\alpha\}_{\alpha=1}^m$ is a basis of $S(TM^\perp)$.

Proof. Since $\overline{\nabla}_V V = 0$, from (2.2), we have $h^l(V, V) = h^s(V, V) = 0$. Now take an orthonormal frame $\{e_1, \cdots\cdots, e_m\}$ of D. We know that $h^l = 0$ on $Rad\ TM$ ([1],Proposition 4.1). Thus M is minimal if and only if

$$\sum_{k=1}^{m} h(e_k, e_k) = 0$$

and $h^s = 0$ on $Rad\,TM$. Using (2.5) and (2.7) we get

$$\sum_{k=1}^{m} h(e_k, e_k) = \sum_{k=1}^{m} \frac{1}{r} \sum_{j=1}^{r} g(A^*_{\xi_j} e_k, e_k) N_j + \frac{1}{m} \sum_{\alpha=1}^{m} g(A_{W_\alpha} e_k, e_k) W_\alpha. \tag{4.1}$$

On the other hand, from (2.5), we get $h^s = 0$ on Rad TM if

$$\overline{g}(D^l(X, W), Y) = 0 \tag{4.2}$$

for $X, Y \in \Gamma(Rad\ TM)$ and $W \in \Gamma(S(TM^\perp))$. Thus proof follows from (4.1) and (4.2). □

Theorem 4.4. *Let M be a proper slant lightlike submanifold of an indefinite*

Sasakian manifold $\overline{M}$ *with structure vector field tangent to* M *such that* $dim(D) = dim(S(TM^{\perp}))$. *Then* M *is minimal if and only if*

$$traceA^{*}_{\xi_j}|_{S(TM)} = 0, traceA_{\omega e_i}|_{S(TM)} = 0 \ \ and \ \ \overline{g}(D^l(X, \omega e_i), Y) = 0,$$

for $X, Y \in \Gamma(Rad\ TM)$, *where* $\{\xi_k\}_{k=1}^{r}$ *is a basis of* $Rad\ TM$ *and* $\{e_j\}_{j=1}^{m}$ *is a basis of* D.

Proof. Since $\overline{\nabla}_V V = 0$, from (2.2), we have $h^l(V, V) = h^s(V, V) = 0$. We know that $h^l = 0$ on $Rad\ TM$ ([1],Proposition 4.1). By virtue of lemma 4.1, $\{\csc\theta\omega e_1, \cdots\cdots, \csc\theta\omega e_m\}$ is an orthonormal basis of $S(TM^{\perp})$. Thus we can write

$$h^s(X, X) = \sum_{i=1}^{m} A_i \csc\theta\omega e_i \ \ \forall\ X \in \Gamma(TM)$$

for some functions A_i, $i \in (1, 2, \cdots, \cdots, m)$. Hence we obtain

$$h^s(X, X) = \sum_{i=1}^{m} \csc\theta g(A_{\omega e_i} X, X)\omega e_i$$

for $X \in \Gamma(S(TM))$. Thus our assertion follows from Theorem 4.3. □

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