

Strong Convergence of Modified Iteration Processes for Relatively Weak Nonexpansive Mappings

DARUNI BOONCHARI

Department of Mathematics, Mahasarakham University, Maha Sarakham 44150, Thailand; and Centre of Excellence in Mathematics, CHE, Sriayudthaya Rd., Bangkok 10400, Thailand
e-mail : daruni.b@msu.ac.th

SATIT SAEJUNG*

Department of Mathematics, Khon Kaen University, Khon Kaen 40002, Thailand; and Centre of Excellence in Mathematics, CHE, Sriayudthaya Rd., Bangkok 10400, Thailand
e-mail : saejung@kku.ac.th

ABSTRACT. We adapt the concept of shrinking projection method of Takahashi et al. [J. Math. Anal. Appl. 341(2008), 276–286] to the iteration scheme studied by Kim and Lee [Kyungpook Math. J. 48(2008), 685–703] for two relatively weak nonexpansive mappings. By letting one of the two mappings be the identity mapping, we also obtain strong convergence theorems for a single mapping with two types of computational errors. Finally, we improve Kim and Lee’s convergence theorem in the sense that the same conclusion still holds without the uniform continuity of mappings as was the case in their result.

1. Introduction and Preliminaries

Many problems in nonlinear analysis, for example, variational problem, convex feasibility problem, equilibrium problem, etc., can be formulated as a problem of finding a fixed point of a certain mapping or a common fixed point of a family of mappings. Iterative method plays an important role in approximation of such a point. This paper deals with a class of nonlinear mappings which is closely related to the resolvent of maximal monotone operators (see [1, 2, 11, 12]).

Let C be a nonempty subset of a real Banach space E , let T be a mapping from C into itself. Throughout the paper, we denote strong convergence of a sequence $\{x_n\}$ in E to $x \in E$ by $x_n \rightarrow x$, weak convergence by $x_n \rightharpoonup x$. A point $p \in C$ is

* Corresponding Author.

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an asymptotic fixed point (a strongly asymptotic fixed point, resp.) of T if there exists a sequence $\{x_n\}$ in C such that $x_n \rightarrow p$ ($x_n \rightarrow p$, resp.) and $x_n - Tx_n \rightarrow 0$. We denote $F(T)$, $\widehat{F}(T)$ and $\widetilde{F}(T)$ by the set of fixed points of T , asymptotic fixed points of T and strongly asymptotic fixed points of T , respectively. A Banach space E is said to be strictly convex if $\frac{\|x+y\|}{2} < 1$ for $x, y \in S(E) = \{z \in E : \|z\| = 1\}$ and $x \neq y$. It is also said to be uniformly convex if for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that $\frac{\|x+y\|}{2} < 1 - \delta$ for $x, y \in S(E)$ and $\|x - y\| \geq \epsilon$. The space E is said to be smooth if the limit

$$\lim_{t \rightarrow 0} \frac{\|x + tx\| - \|x\|}{t}$$

exists for all $x, y \in S(E)$. It is also said to be uniformly smooth if the limit exists uniformly in $x, y \in S(E)$.

Let E be a smooth Banach space. Throughout this paper, we denote by φ the function defined by

$$\varphi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2 \text{ for all } x, y \in E,$$

where J is the normalized duality mapping from E to the dual space E^* given by the following relation

$$\langle x, Jx \rangle = \|x\|^2 = \|Jx\|^2.$$

We know that if E is smooth, strictly convex and reflexive, then the duality mapping J is one-to-one and onto.

Following Matsushita and Takahashi [7], a mapping $T : C \rightarrow E$, where C is a nonempty subset of a smooth Banach space E , is said to be relatively nonexpansive if the following conditions are satisfied:

- (R1) $F(T)$ is nonempty;
- (R2) $\varphi(u, Tx) \leq \varphi(u, x)$ for all $u \in F(T)$, $x \in C$;
- (R3) $\widehat{F}(T) = F(T)$.

If T satisfies (R1), (R2) and

- (R3') $\widetilde{F}(T) = F(T)$,

then T is called a relatively weak nonexpansive mapping [16]. Relative nonexpansiveness implies relatively weak nonexpansiveness but the converse is not true. The following interesting example (see [4, 15, 17]) shows that there exists a relatively weak nonexpansive mapping which is not a relatively nonexpansive mapping.

Example 1.1. Let $E = \ell^2$ where

$$\ell^2 = \left\{ \xi = (\xi_1, \xi_2, \dots, \xi_n, \dots) : \sum_{n=1}^{\infty} |\xi_n|^2 < \infty \right\},$$

$$\|\xi\| = \left(\sum_{n=1}^{\infty} |\xi_n|^2 \right)^{1/2}, \text{ for all } \xi = (\xi_1, \xi_2, \dots, \xi_n, \dots) \in \ell^2.$$

Let $\{x_n\} \subset E$ be a sequence defined by

$$\begin{aligned} x_0 &= (1, 0, 0, 0, \dots) \\ x_1 &= (1, 1, 0, 0, \dots) \\ x_2 &= (1, 0, 1, 0, 0, \dots) \\ &\vdots \\ x_n &= (1, 0, 0, 0, \dots, 0, \overset{(n+1)\text{-th}}{1}, 0, 0, \dots) \end{aligned}$$

Define a mapping $T : E \rightarrow E$ by

$$T(x) = \begin{cases} \frac{n}{n+1}x_n & \text{if } x = x_n \ (\exists n \geq 1), \\ -x & \text{if } x \neq x_n \ (\forall n \geq 1). \end{cases}$$

Then the mapping T is relatively weak nonexpansive but *not* relatively nonexpansive.

For a closed convex subset C of a smooth, strictly convex and reflexive Banach space E , Alber [1] introduced the generalized projection Π_C from E onto C as follows:

$$\Pi_C(x) = \operatorname{argmin}_{y \in C} \varphi(y, x) \text{ for all } x \in E.$$

If E is a Hilbert space, then $\varphi(y, x) = \|y - x\|^2$ and Π_C becomes the metric projection of E onto C . It is clear that Alber's generalized projection is a relatively nonexpansive mapping. For more example, see [6, 9].

In 2004, Masushita and Takahashi [6, 7] proved convergence theorems for finding a fixed point of a single relatively nonexpansive mapping. The purpose of this paper is to prove strong convergence of the sequence generated by the shrinking projection method introduced by Takahashi et al. [13] and the hybrid method introduced by Nakajo and Takahashi [8]. We strengthen the result of Kim and Lee [5] in the sense that the same conclusion still holds without the uniform continuity of mappings as was the case in [5].

Lemma 1.2([3]). *Let E be a smooth and uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous, and convex function $h : [0, 2r] \rightarrow \mathbb{R}$ such that $h(0) = 0$ and*

$$h(\|x - y\|) \leq \varphi(x, y)$$

for all $x, y \in B_r := \{z \in E : \|z\| \leq r\}$.

Lemma 1.3([3, Proposition 2]). *Let E be a smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{z_n\}$ be two sequences of E . If either $\{x_n\}$ or $\{z_n\}$ is bounded and $\lim_{n \rightarrow \infty} \varphi(x_n, z_n) = 0$, then $\lim_{n \rightarrow \infty} \|x_n - z_n\| = 0$.*

Lemma 1.4([3, Proposition 5]). *Let C be a nonempty closed convex subset of a*

smooth, strictly convex and reflexive Banach space E . Then

$$\varphi(x, \Pi_C y) + \varphi(\Pi_C y, y) \leq \varphi(x, y)$$

for all $x \in C$ and $y \in E$.

Lemma 1.5([14]). *Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous, and convex function $g : [0, 2r] \rightarrow \mathbb{R}$ such that $g(0) = 0$ and*

$$\|tx + (1-t)y\|^2 \leq t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)g(\|x-y\|)$$

for all $x, y \in B_r$ and $t \in [0, 1]$.

Lemma 1.6([5, Lemma 2.3]). *Let C be a nonempty closed convex subset of a smooth Banach space E , $x, y, z \in E$ and $\lambda \in [0, 1]$. If $a \in \mathbb{R}$, then the set*

$$D := \{v \in C : \varphi(v, z) \leq \lambda\varphi(v, x) + (1-\lambda)\varphi(v, y) + a\}$$

is closed and convex.

The following result is an analogue of Xu's inequality [14] with respect to φ .

Theorem 1.7. *Let E be a smooth Banach space. Then, for any $x, y, u \in E$ and $\alpha \in [0, 1]$,*

$$\varphi(u, J^{-1}(\alpha Jx + (1-\alpha)Jy)) \leq \alpha\varphi(u, x) + (1-\alpha)\varphi(u, y).$$

In addition, if E is a uniformly smooth Banach space and $r > 0$, then there exists a strictly increasing, continuous, and convex function $g : [0, 2r] \rightarrow \mathbb{R}$ such that $g(0) = 0$ and

$$\varphi(u, J^{-1}(\alpha Jx + (1-\alpha)Jy)) \leq \alpha\varphi(u, x) + (1-\alpha)\varphi(u, y) - \alpha(1-\alpha)g(\|Jx - Jy\|)$$

for all $\alpha \in [0, 1]$, $x, y \in B_r$ and $u \in E$.

Proof. Since the first inequality can be extracting from the proof of latter inequality, we omit the proof. For the latter inequality, let $x, y \in B_r$. Note that E is uniformly smooth if and only if E^* is uniformly convex. By Lemma 1.5, there exists a strictly increasing, continuous, and convex function $g : [0, 2r] \rightarrow \mathbb{R}$ such that $g(0) = 0$ and

$$\|\alpha Jx + (1-\alpha)Jy\|^2 \leq \alpha\|Jx\|^2 + (1-\alpha)\|Jy\|^2 - \alpha(1-\alpha)g(\|Jx - Jy\|)$$

for all $\alpha \in [0, 1]$. Then, for any $u \in E$,

$$\begin{aligned} & \varphi(u, J^{-1}(\alpha Jx + (1-\alpha)Jy)) \\ &= \|u\|^2 - 2\langle u, \alpha Jx + (1-\alpha)Jy \rangle + \|\alpha Jx + (1-\alpha)Jy\|^2 \\ &\leq \|u\|^2 - 2\alpha\langle u, Jx \rangle - 2(1-\alpha)\langle u, Jy \rangle \\ &\quad + \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)g(\|Jx - Jy\|) \end{aligned}$$

$$= \alpha\varphi(u, x) + (1 - \alpha)\varphi(u, y) - \alpha(1 - \alpha)g(\|Jx - Jy\|). \quad \square$$

2. Result

Motivated by the iterative sequence studied by Kim and Lee in [5] and the shrinking projection method introduced by Takahashi et al. in [13], we prove the following theorem.

Theorem 2.1. *Let C be a nonempty closed convex subset of a uniformly smooth and uniformly convex Banach space E . Let $\{T_1, T_2 : C \rightarrow C\}$ be a pair of relatively weak nonexpansive mappings with $F := F(T_1) \cap F(T_2) \neq \emptyset$. Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ such that $\limsup_{n \rightarrow \infty} \alpha_n(1 - \alpha_n) > 0$ and $\lim_{n \rightarrow \infty} \beta_n = 1$. Define a sequence $\{x_n\}$ in C by the algorithm:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ C_1 = C, \\ y_n = J^{-1}(\alpha_n J T_2 z_n + (1 - \alpha_n) J T_1 x_n), \\ z_n = \beta_n x_n + (1 - \beta_n) e_n, \\ C_{n+1} = \{v \in C_n : \varphi(v, y_n) \leq \alpha_n \varphi(v, z_n) + (1 - \alpha_n) \varphi(v, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \text{ for all } n \geq 1, \end{cases}$$

where $\{e_n\}$ is a bounded sequence in C . Then $x_n \rightarrow \Pi_F x_0$.

Proof. First, we prove that $\{x_n\}$ is well defined. From Lemma 1.6, we can prove by induction that C_n is closed and convex for all $n \geq 1$. Next we claim that $F \subset C_n$ for $n \geq 1$. Clearly, $F \subset C_1 = C$. Assume that $F \subset C_k$ for some k . Let $p \in F$. Then, it follows from Lemma 1.7 that

$$\begin{aligned} \varphi(p, y_{k+1}) &\leq \alpha_{k+1} \varphi(p, T_2 z_{k+1}) + (1 - \alpha_{k+1}) \varphi(p, T_1 x_{k+1}) \\ &\leq \alpha_{k+1} \varphi(p, z_{k+1}) + (1 - \alpha_{k+1}) \varphi(p, x_{k+1}). \end{aligned}$$

So, $p \in C_{k+1}$. By induction, we get the claim. Hence $\{x_n\}$ is well defined.

Second, we show that $\{x_n\}$ is a Cauchy sequence. By $x_{n+1} = \Pi_{C_{n+1}} x_0$ and $x_{n+1} \in C_n$, we have

$$\varphi(x_n, x_0) \leq \varphi(x_{n+1}, x_n) + \varphi(x_n, x_0) \leq \varphi(x_{n+1}, x_0) \leq \varphi(\Pi_F x_0, x_0)$$

for all $n \in \mathbb{N}$. Thus $\{\varphi(x_n, x_0)\}$ is non-decreasing and bounded. This implies that $\lim_{n \rightarrow \infty} \varphi(x_n, x_0)$ exists and hence $\{x_n\}$ is bounded. Since $x_{n+m} \in C_{n+m} \subset C_n$ for all n, m and $x_n = \Pi_{C_n} x_0$, we have

$$\begin{aligned} \varphi(x_{n+m}, x_n) &= \varphi(x_{n+m}, \Pi_{C_n} x_0) \\ &\leq \varphi(x_{n+m}, x_0) - \varphi(\Pi_{C_n} x_0, x_0) \\ &= \varphi(x_{n+m}, x_0) - \varphi(x_n, x_0). \end{aligned}$$

Since $\{x_n\}$ is bounded, it follows from Lemma 1.2 that there exists a strictly increasing, continuous, and convex function h such that $h(0) = 0$ and

$$h(\|x_{n+m} - x_n\|) \leq \varphi(x_{n+m}, x_n) \leq \varphi(x_{n+m}, x_0) - \varphi(x_n, x_0) \text{ for all } n, m \in \mathbb{N}.$$

Since $\lim_{n \rightarrow \infty} \varphi(x_n, x_0)$ exists, we have $\{x_n\}$ is a Cauchy sequence.

Suppose that $x_n \rightarrow z \in E$. We finally prove that $z = \Pi_F x_0$. Since $\beta_n \rightarrow 1$ and $\{e_n\}$ is bounded, we have

$$z_n = \beta_n x_n + (1 - \beta_n) e_n \rightarrow z.$$

Consequently, we get that

$$\lim_{n \rightarrow \infty} \varphi(x_{n+1}, z_n) = \lim_{n \rightarrow \infty} \varphi(x_{n+1}, x_n) = \varphi(z, z) = 0.$$

From $x_{n+1} \in C_{n+1}$, we get that

$$\varphi(x_{n+1}, y_n) \leq \alpha_n \varphi(x_{n+1}, z_n) + (1 - \alpha_n) \varphi(x_{n+1}, x_n) \rightarrow 0.$$

Using Lemma 1.3 and $\{x_n\}$ is bounded, we have

$$\lim_{n \rightarrow \infty} \|x_{n+1} - y_n\| = 0.$$

Since J is uniformly norm-to-norm continuous on bounded sets, we get that

$$\lim_{n \rightarrow \infty} Jx_n = \lim_{n \rightarrow \infty} Jy_n = \lim_{n \rightarrow \infty} Jz_n = Jz.$$

Since $\{T_2 z_n\}$ and $\{T_1 x_n\}$ are bounded, let $r = \sup_{n \in \mathbb{N}} \{\|T_2 z_n\|, \|T_1 x_n\|\}$. By Lemma 1.5, there exists a strictly increasing, continuous, and convex function $g : [0, 2r] \rightarrow \mathbb{R}$ such that $g(0) = 0$ and for any $u \in F$

$$\begin{aligned} & \alpha_n (1 - \alpha_n) g(\|JT_2 z_n - JT_1 x_n\|) \\ & \leq \alpha_n \varphi(u, T_2 z_n) + (1 - \alpha_n) \varphi(u, T_1 x_n) - \varphi(u, y_n) \\ & \leq \alpha_n \varphi(u, z_n) + (1 - \alpha_n) \varphi(u, x_n) - \varphi(u, y_n) \\ & = \alpha_n (\varphi(u, z_n) - \varphi(u, x_n)) + (\varphi(u, x_n) - \varphi(u, y_n)) \rightarrow 0. \end{aligned}$$

It follows from $\limsup_{n \rightarrow \infty} \alpha_n (1 - \alpha_n) > 0$ that there exists $\{\alpha_{n_k}\} \subset \{\alpha_n\}$ such that $\lim_{k \rightarrow \infty} \alpha_{n_k} (1 - \alpha_{n_k}) > 0$ and hence

$$\alpha_{n_k} (1 - \alpha_{n_k}) g(\|JT_2 z_{n_k} - JT_1 x_{n_k}\|) \rightarrow 0.$$

By the properties of g , we have

$$\lim_{n \rightarrow \infty} \|JT_2 z_{n_k} - JT_1 x_{n_k}\| = 0.$$

Notice that $Jy_{n_k} = \alpha_{n_k} JT_2 z_{n_k} + (1 - \alpha_{n_k}) JT_1 x_{n_k}$, we have

$$\|Jy_{n_k} - JT_1 x_{n_k}\| = \alpha_{n_k} \|JT_2 z_{n_k} - JT_1 x_{n_k}\| \rightarrow 0.$$

Since J^{-1} is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{k \rightarrow \infty} \|y_{n_k} - T_1 x_{n_k}\| = \lim_{k \rightarrow \infty} \|T_2 z_{n_k} - T_1 x_{n_k}\| = 0$$

and hence

$$\lim_{k \rightarrow \infty} T_1 x_{n_k} = \lim_{k \rightarrow \infty} T_2 z_{n_k} = z.$$

Since both T_1 and T_2 are relatively weak nonexpansive, we get that $z \in F(T_1) \cap F(T_2)$. From $x_n = \Pi_{C_n} x_0$ and $F \subset C_n$, we get that

$$\varphi(x_n, x_0) \leq \varphi(p, x_0)$$

for all $p \in F$. Since $x_n \rightarrow z \in F$, we have $\varphi(z, x_0) \leq \varphi(p, x_0)$. By the property of Π_C , we obtain $z = \Pi_F x_0$. \square

A careful reading of the proof of Theorem 2.1 gives the following result for the case $T_1 = \text{identity}$ with a weaker assumption $\limsup_{n \rightarrow \infty} \alpha_n > 0$.

Theorem 2.2. *Let C be a nonempty closed convex subset of a uniform smooth and uniformly convex Banach space E . Let $T : C \rightarrow C$ be a relatively weak nonexpansive mappings. Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ such that $\limsup_{n \rightarrow \infty} \alpha_n > 0$ and $\lim_{n \rightarrow \infty} \beta_n = 1$. Define a sequence $\{x_n\}$ in C by the algorithm:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ C_1 = C, \\ y_n = J^{-1}(\alpha_n J T(\beta_n x_n + (1 - \beta_n)e_n) + (1 - \alpha_n) J x_n), \\ C_{n+1} = \{v \in C_n : \varphi(v, y_n) \leq \alpha_n \varphi(v, \beta_n x_n + (1 - \beta_n)e_n) + (1 - \alpha_n) \varphi(v, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \text{ for all } n \geq 1, \end{cases}$$

where $\{e_n\}$ is a bounded sequence in C . Then $x_n \rightarrow \Pi_F x_0$.

From the proof of Theorem 2.1 we also have the following result for the case $T_2 = \text{identity}$ with a weaker assumption $\limsup_{n \rightarrow \infty} (1 - \alpha_n) > 0$.

Theorem 2.3. *Let C be a nonempty closed convex subset of a uniformly smooth and uniformly convex Banach space E . Let $T : C \rightarrow C$ be a relatively weak nonexpansive mappings. Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ such that $\limsup_{n \rightarrow \infty} (1 - \alpha_n) > 0$ and $\lim_{n \rightarrow \infty} \beta_n = 1$. Define a sequence $\{x_n\}$ in C by the algorithm:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ C_1 = C, \\ y_n = J^{-1}(\alpha_n J(\beta_n x_n + (1 - \beta_n)e_n) + (1 - \alpha_n) J T x_n), \\ C_{n+1} = \{v \in C_n : \varphi(v, y_n) \leq \alpha_n \varphi(v, \beta_n x_n + (1 - \beta_n)e_n) + (1 - \alpha_n) \varphi(v, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \text{ for all } n \geq 1, \end{cases}$$

where $\{e_n\}$ is a bounded sequence in C . Then $x_n \rightarrow \Pi_F x_0$.

The following result is an improvement of [5, Theorem 3.1] in the sense that the uniform continuity of T_2 is not assumed. Recall that the set of all weak subsequential limits of a sequence $\{x_n\}$ is denoted by $\omega_w\{x_n\}$, that is, $\omega_w\{x_n\} = \{p : \exists\{x_{n_k}\} \subset \{x_n\} \text{ such that } x_{n_k} \rightharpoonup p\}$.

Theorem 2.4. *Let C be a nonempty closed convex subset of a uniformly smooth and uniformly convex Banach space E . Let $\{T_1, T_2 : C \rightarrow C\}$ be a pair of relatively nonexpansive mappings with $F := F(T_1) \cap F(T_2) \neq \emptyset$. Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ such that $\liminf_{n \rightarrow \infty} \alpha_n(1 - \alpha_n) > 0$ and $\lim_{n \rightarrow \infty} \beta_n = 1$. Define a sequence $\{x_n\}$ in C by the algorithm:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ y_n = J^{-1}(\alpha_n J T_2 z_n + (1 - \alpha_n) J T_1 x_n), \\ z_n = \beta_n x_n + (1 - \beta_n) e_n, \\ C_n = \{v \in C : \varphi(v, y_n) \leq \alpha_n \varphi(v, z_n) + (1 - \alpha_n) \varphi(v, x_n)\}, \\ Q_n = \{v \in C : \langle x_n - v, J x_n - J x_0 \rangle\} \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \text{ for all } n \geq 1, \end{cases}$$

where $\{e_n\}$ is a bounded sequence in C . Then $x_n \rightarrow \Pi_F x_0$.

Proof. We employ the proof of [5] and obtain that

$$\lim_{n \rightarrow \infty} \|x_n - T_1 x_n\| = \lim_{n \rightarrow \infty} \|x_n - T_2 z_n\| = \lim_{n \rightarrow \infty} \|z_n - x_n\| = 0.$$

From the last two limits, we have

$$\lim_{n \rightarrow \infty} \|z_n - T_2 z_n\| = 0 \text{ and } \omega_w\{x_n\} = \omega_w\{z_n\}.$$

Since both T_1 and T_2 are relatively nonexpansive,

$$\omega_w\{x_n\} \subset \hat{F}(T_1) \cap \hat{F}(T_2) = F(T_1) \cap F(T_2).$$

The conclusion follows immediately from [5, Lemma 2.4]. \square

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References

- [1] Y. I. Alber, Metric and generalized projection operators in Banach spaces: properties and applications. Theory and applications of nonlinear operators of accretive and monotone type, Lecture Notes in Pure and Appl. Math., Dekker, New York, **178**(1996), 15-50.

- [2] K. Aoyama, F. Kohsaka and W. Takahashi, Strong convergence theorems by shrinking and hybrid projection methods for relatively nonexpansive mappings in Banach spaces, Proceeding of the 5th international conference on nonlinear analysis and convex analysis, Taiwan, (2007), 7-26.
- [3] S. Kamimura and W. Takahashi, *Strong convergence of a proximal-type algorithm in a Banach space*, SIAM J. Optim., **13**(2002), 938-945.
- [4] J. Kang, Y. Su and X. Zhang, *Hybrid algorithm for fixed points of weak relatively nonexpansive mappings and applications*, Nonlinear Anal., **4**(4)(2010), 755-765.
- [5] T. H. Kim and H. J. Lee, *Strong convergence of modified iteration processes for relatively nonexpansive mappings*, Kyungpook Math. J., **48**(4)(2008), 685-703.
- [6] S. Matsushita and W. Takahashi, *Weak and strong convergence theorems for relatively nonexpansive mappings in Banach spaces*, Fixed Point Theory Appl., (1)(2004), 37-47.
- [7] S. Matsushita and W. Takahashi, *A strong convergence theorem for relatively nonexpansive mappings in a Banach space*, J. Approx. Theory, **134**(2)(2005), 257-266.
- [8] K. Nakajo and W. Takahashi, *Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups*, J. Math. Anal. Appl., **279**(2)(2003), 372-379.
- [9] S. Reich, *A weak convergence theorem for the alternating method with Bregman distances*, Theory and applications of nonlinear operators of accretive and monotone type, Lecture Notes in Pure and Appl. Math., Dekker, New York, **178**(1996), 313-318.
- [10] Y. Su, D. Wang and M. Shang, *Strong convergence of monotone hybrid algorithm for hemi-relatively nonexpansive mappings*, Fixed Point Theory Appl., Art. ID 284613, (2008), 8 pp.
- [11] W. Takahashi, *Convex Analysis and Approximation Fixed points*, Yokohama Publishers, Yokohama, (2000), (Japanese).
- [12] W. Takahashi, *Nonlinear Functional Analysis, Fixed Point Theory and Its Applications*, Yokohama Publishers, Yokohama, (2000).
- [13] W. Takahashi, Y. Takeuchi and R. Kubota, *Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces* J. Math. Anal. Appl., **341**(1)(2008), 276-286.
- [14] H. K. Xu, *Inequalities in Banach spaces with applications*, Nonlinear Anal., **16**(12)(1991), 1127-1138.
- [15] Y. Xu and Y. Su, *On the weak relatively nonexpansive mappings in Banach spaces*, Fixed Point Theory Appl., Art. ID 189751, (2010), 7 pp.
- [16] H. Zegeye and N. Shahzad, *Strong convergence theorems for monotone mappings and relatively weak nonexpansive mappings*, Nonlinear Anal., **70**(7)(2009), 2707-2716.
- [17] X. Zhang and Y. Su, *Convergence theorems for two families of weak relatively nonexpansive mappings and a family of equilibrium problems*, Commun. Korean Math. Soc., **25**(2010), 583-607.