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Conditional Integral Transforms on a Function Space

Dong Hyun Cho
Department of Mathematics, Kyonggi University, Suwon, 443-760, Korea
e-mail : j94385@kyonggi.ac.kr

Abstract. Let $C^r[0,t]$ be the function space of the vector-valued continuous paths $x : [0,t] \to \mathbb{R}^r$ and define $X_t : C^r[0,t] \to \mathbb{R}^{(n+1)r}$ and $Y_t : C^r[0,t] \to \mathbb{R}^{nr}$ by $X_t(x) = (x(t_0), x(t_1), \cdots, x(t_{n-1}), x(t_n))$ and $Y_t(x) = (x(t_0), x(t_1), \cdots, x(t_{n-1}))$, respectively, where $0 = t_0 < t_1 < \cdots < t_n = t$. In the present paper, using two simple formulas for the conditional expectations over $C^r[0,t]$ with the conditioning functions X_t and Y_t, we establish evaluation formulas for the analogue of the conditional analytic Fourier-Feynman transform for the function of the form

$$\exp\left\{\int_0^t \theta(s, x(s))\, d\eta(s)\right\}\psi(x(t)), \quad x \in C^r[0,t]$$

where η is a complex Borel measure on $[0,t]$ and both $\theta(s,\cdot)$ and ψ are the Fourier-Stieltjes transforms of the complex Borel measures on $\mathbb{R}^r$.

1. Introduction and an analogue of the r-dimensional Wiener space

Let $C_0[0,t]$ be the space of real-valued continuous functions x on $[0,t]$ with $x(0) = 0$. It is well known that the space $C_0[0,t]$ is equipped with the Wiener measure which is a probability measure. On the space, Yeh [10, 11, 12] introduced an inversion formula that a conditional expectation can be found by a Fourier-transform. In [2], the author and his co-authors introduced a simple formula for the conditional Wiener integrals over $C_0(\mathbb{B})$, the space of the abstract Wiener space $\mathbb{B}$-valued continuous functions which vanish at 0. Using the formula, they established various evaluation formulas for the conditional analytic Wiener and Feynman integrals of the functionals on $C_0(\mathbb{B})$ in a certain Banach algebra which corresponds to the Cameron and Storvick's Banach algebra $\mathcal{S}''$ [1]. In [3], the author evaluated conditional analytic Fourier-Feynman transform for functional of the form

$$\exp\left\{\int_0^t \theta(s, x(s))\, d\zeta(s)\right\}\phi(x(t)), \quad x \in C_0(\mathbb{B})$$

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where ζ is a complex Borel measure on $[0,t]$ and both $\theta(s,\cdot)$ and ϕ are the Fourier-Stieltjes transforms of the complex Borel measures on the real separable Hilbert space embedded in $\mathbb{B}$. Note that $\theta(s,\cdot)$ and ϕ are defined on $\mathbb{B}$, the abstract Wiener space [8].

On the other hand, let $C[0,t]$ denote the space of real-valued continuous functions on the interval $[0,t]$. Ryu and Im [6, 9] introduced a probability measure w_φ on $(C[0,t], \mathcal{B}(C[0,t]))$, where $\mathcal{B}(C[0,t])$ denotes the Borel σ-algebra on $C[0,t]$ and φ is a probability distribution on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. This measure space is a generalization of the Wiener space $C_0[0,t]$. In the Wiener space, every path x starts at the origin, that is, $x(0)=0$. If the paths x start at any points, that is, if $x \in C[0,t]$, certain properties on $C_0[0,t]$ can not hold or some of them should be modified. Fortunately, in [4, 5], the author could derive two simple formulas for the conditional w_φ-integrals of the functions on $C[0,t]$ with the vector-valued conditioning functions $X : C[0,t] \to \mathbb{R}^{n+1}$ and $Y : C[0,t] \to \mathbb{R}^n$ given by $X(x) = (x(t_0), x(t_1), \cdots, x(t_n))$ and $Y(x) = (x(t_0), x(t_1), \cdots, x(t_{n-1}))$, where $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = t$ is a partition of $[0,t]$. These formulas express the conditional w_φ-integrals directly in terms of the non-conditional w_φ-integrals.

Let $C^r[0,t]$ be the product space of $C[0,t]$ and define $X_t : C^r[0,t] \to \mathbb{R}^{(n+1)r}$ and $Y_t : C^r[0,t] \to \mathbb{R}^{nr}$ by $X_t(x) = (x(t_0), x(t_1), \cdots, x(t_{n-1}),\ x(t_n))$ and $Y_t(x) = (x(t_0), x(t_1), \cdots,\ x(t_{n-1}))$. In the present paper, with the conditioning functions X_t and Y_t, we introduce two simple formulas for the conditional expectations over $C^r[0,t]$, an r-dimensional analogue of Wiener space. We then establish evaluation formulas for an analogue of the conditional analytic Fourier-Feynman transform for the function of the form

$$\exp\left\{\int_0^t \theta(s, x(s))\ d\eta(s)\right\}\psi(x(t)), \quad x \in C^r[0,t]$$

where η is a complex Borel measure on $[0,t]$ and both $\theta(s,\cdot)$ and ψ are the Fourier-Stieltjes transforms of the complex Borel measures on $\mathbb{R}^r$.

Throughout this paper, let $\mathbb{C}$ and $\mathbb{C}_+$ denote the sets of complex numbers and complex numbers with positive real parts, respectively.

Now, we introduce the probability measure w_φ on $(C[0,t], \mathcal{B}(C[0,t]))$.

For a positive real t, let $C = C[0,t]$ be the space of all real-valued continuous functions on the closed interval $[0,t]$ with the supremum norm. For $\vec{t} = (t_0, t_1, \cdots, t_n)$ with $0 = t_0 < t_1 < \cdots < t_n \le t$, let $J_{\vec{t}} : C[0,t] \to \mathbb{R}^{n+1}$ be the function given by

$$J_{\vec{t}}(x) = (x(t_0), x(t_1), \cdots, x(t_n)).$$

For $B_j\ (j = 0, 1, \cdots, n)$ in $\mathcal{B}(\mathbb{R})$, the subset $J_{\vec{t}}^{-1}(\prod_{j=0}^n B_j)$ of $C[0,t]$ is called an interval and let $\mathcal{I}$ be the set of all such intervals. For a probability measure φ on

$(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, let

$$m_\varphi\bigg(J_{\vec{t}}^{-1}\bigg(\prod_{j=0}^{n} B_j\bigg)\bigg) = \bigg[\prod_{j=1}^{n}\frac{1}{2\pi(t_j-t_{j-1})}\bigg]^{\frac{1}{2}}\int_{B_0}\int_{\prod_{j=1}^{n}B_j} \exp\bigg\{-\frac{1}{2}\sum_{j=1}^{n}\frac{(u_j-u_{j-1})^2}{t_j-t_{j-1}}\bigg\}d(u_1,\cdots,u_n)d\varphi(u_0).$$

Then $\mathcal{B}(C[0,t])$ coincides with the smallest σ-algebra generated by $\mathcal{I}$ and there exists a unique probability measure w_φ on $(C[0,t], \mathcal{B}(C[0,t]))$ such that $w_\varphi(I) = m_\varphi(I)$ for all I in $\mathcal{I}$. This measure w_φ is called an analogue of the Wiener measure associated with the probability measure φ [6, 9]. Let r be a positive integer and $C^r = C^r[0,t]$ be the product space of $C[0,t]$ with the product measure w_φ^r. Since $C[0,t]$ is a separable Banach space, we have $\mathcal{B}(C^r[0,t]) = \prod_{j=1}^{r}\mathcal{B}(C[0,t])$. This probability measure space $(C^r[0,t], \mathcal{B}(C^r[0,t]), w_\varphi^r)$ is called an analogue of the r-dimensional Wiener space.

Lemma 1.1([6, Lemma 2.1])**.** *If $f : \mathbb{R}^{n+1} \to \mathbb{C}$ is a Borel measurable function, then we have*

$$\int_C f(x(t_0), x(t_1), \cdots, x(t_n))dw_\varphi(x)$$
$$\stackrel{*}{=} \bigg[\prod_{j=1}^{n}\frac{1}{2\pi(t_j-t_{j-1})}\bigg]^{\frac{1}{2}}\int_{\mathbb{R}}\int_{\mathbb{R}^n} f(u_0,u_1,\cdots,u_n)\exp\bigg\{-\frac{1}{2}\sum_{j=1}^{n}\frac{(u_j-u_{j-1})^2}{t_j-t_{j-1}}\bigg\} d(u_1,\cdots,u_n)d\varphi(u_0)$$

where $\stackrel{*}{=}$ means that if either side exists, then both sides exist and they are equal.

Definition 1.2. Let $F : C^r[0,t] \to \mathbb{C}$ be w_φ^r-integrable and let Z be a random vector on $C^r[0,t]$ assuming that the value space of Z is a normed space with the Borel σ-algebra. Then we have the conditional expectation $E[F|Z]$ of F given Z from a well known probability theory. Further, there exists a P_Z-integrable complex-valued function Ξ on the value space of Z such that $E[F|Z](x) = (\Xi \circ Z)(x)$ for w_φ^r-a.e. $x \in C^r[0,t]$, where P_Z is the probability distribution of Z. The function Ξ is called the conditional w_φ^r-integral of F given Z and it is also denoted by $E[F|Z]$.

2. The simple formulas for conditional $\mathbf{w}_\varphi^{\mathbf{r}}$-integrals

Let $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = t$ be a partition of $[0,t]$. For any x in $C^r[0,t]$, define the polygonal function $[x]$ of x on $[0,t]$ by

$$[x](s) = \sum_{j=1}^{n}\chi_{(t_{j-1},t_j]}(s)\bigg(\frac{t_j-s}{t_j-t_{j-1}}x(t_{j-1}) + \frac{s-t_{j-1}}{t_j-t_{j-1}}x(t_j)\bigg) + \chi_{\{0\}}(s)x(0) \tag{2.1}$$

for $s \in [0,t]$, where $\chi_{(t_{j-1},t_j]}$ and $\chi_{\{0\}}$ denote the indicator functions of $(t_{j-1},t_j]$ and $\{0\}$, respectively. For $\vec{\xi_n} = (\xi_0, \xi_1, \cdots, \xi_{n-1}, \xi_n) \in \mathbb{R}^{(n+1)r}$, we define the polygonal function $[\vec{\xi_n}]$ of $\vec{\xi_n}$ as (2.1) replacing $x(t_j)$ by ξ_j for $j = 0, 1, \cdots, n$.

Now, we introduce a lemma which is useful to prove several theorems. The proof follows immediately from Corollary 2.5 in [4].

Lemma 2.1. *The processes $\{x(s) - [x](s) : t_{j-1} \le s \le t_j\}$ on $C^r[0,t]$, for $j = 1, \cdots, n$, are stochastically independent.*

In the following two theorems, we introduce two simple formulas for the conditional w_φ^r-integrals on $C^r[0,t]$. Their proofs follow immediately from Theorem 2.9 in [4] and Theorem 2.5 in [5].

Theorem 2.2. *Let $F : C^r[0,t] \to \mathbb{C}$ be w_φ^r-integrable and $X_t : C^r[0,t] \to \mathbb{R}^{(n+1)r}$ be given by*

$$X_t(x) = (x(t_0), x(t_1), \cdots, x(t_{n-1}), x(t_n)) \tag{2.2}$$

for $x \in C^r[0,t]$. Then we have for P_{X_t}-a.e. $\vec{\xi_n} \in \mathbb{R}^{(n+1)r}$

$$E[F|X_t](\vec{\xi_n}) = E[F(x - [x] + [\vec{\xi_n}])], \tag{2.3}$$

where P_{X_t} is the probability distribution of X_t on $(\mathbb{R}^{(n+1)r}, \mathcal{B}(\mathbb{R}^{(n+1)r}))$.

Theorem 2.3. *Let $F : C^r[0,t] \to \mathbb{C}$ be w_φ^r-integrable and $Y_t : C^r[0,t] \to \mathbb{R}^{nr}$ be given by*

$$Y_t(x) = (x(t_0), x(t_1), \cdots, x(t_{n-1})) \tag{2.4}$$

for $x \in C^r[0,t]$. Then we have for P_{Y_t}-a.e. $\vec{\xi}_{n-1} = (\xi_0, \xi_1, \cdots, \xi_{n-1}) \in \mathbb{R}^{nr}$

$$\begin{aligned} E[F|Y_t](\vec{\xi}_{n-1}) &= \left[\frac{1}{2\pi(t-t_{n-1})}\right]^{\frac{r}{2}} \int_{\mathbb{R}^r} E[F(x - [x] + [\vec{\xi_n}])] \\ &\quad \times \exp\left\{-\frac{\|\xi_n - \xi_{n-1}\|_{\mathbb{R}^r}^2}{2(t-t_{n-1})}\right\} d\xi_n \end{aligned} \tag{2.5}$$

where P_{Y_t} is the probability distribution of Y_t on $(\mathbb{R}^{nr}, \mathcal{B}(\mathbb{R}^{nr}))$ and $[\vec{\xi_n}]$ denotes the polygonal function of $(\xi_0, \xi_1, \cdots, \xi_{n-1}, \xi_n) \in \mathbb{R}^{(n+1)r}$.

For a function $F : C^r[0,t] \to \mathbb{C}$ and $\lambda > 0$, let $F^\lambda(x) = F(\lambda^{-\frac{1}{2}}x)$, $X_t^\lambda(x) = X_t(\lambda^{-\frac{1}{2}}x)$ and $Y_t^\lambda(x) = Y_t(\lambda^{-\frac{1}{2}}x)$, where X_t and Y_t are given by (2.2) and (2.4), respectively. Suppose that $E[F^\lambda]$ exists for each $\lambda > 0$. By the definition of the conditional w_φ^r-integral and (2.3), we have

$$E[F^\lambda|X_t^\lambda](\vec{\xi_n}) = E[F(\lambda^{-\frac{1}{2}}(x - [x]) + [\vec{\xi_n}])]$$

for $P_{X_t^\lambda}$-a.e. $\vec{\xi}_n \in \mathbb{R}^{(n+1)r}$, where $P_{X_t^\lambda}$ is the probability distribution of X_t^λ on the Borel class of $\mathbb{R}^{(n+1)r}$. Throughout this paper, for $y \in C^r[0,t]$ let

$$I_F^\lambda(y, \vec{\xi}_n) = E[F(y + \lambda^{-\frac{1}{2}}(x - [x]) + [\vec{\xi}_n])]$$

unless otherwise specified, where the expectation is taken over the variable x. Moreover, under the notations used in Theorem 2.3, we have by (2.5)

$$\begin{aligned} & E[F^\lambda | Y_t^\lambda](\vec{\xi}_{n-1}) \\ (2.6) \qquad = & \left[\frac{\lambda}{2\pi(t - t_{n-1})}\right]^{\frac{r}{2}} \int_{\mathbb{R}^r} I_F^\lambda(0, \vec{\xi}_n) \exp\left\{-\frac{\lambda}{2} \frac{\|\xi_n - \xi_{n-1}\|_{\mathbb{R}^r}^2}{t - t_{n-1}}\right\} d\xi_n \end{aligned}$$

for $P_{Y_t^\lambda}$-a.e. $\vec{\xi}_{n-1} \in \mathbb{R}^{nr}$, where $P_{Y_t^\lambda}$ is the probability distribution of Y_t^λ on the Borel class of $\mathbb{R}^{nr}$. From now on, for $y \in C^r[0,t]$ let $K_F^\lambda(y, \vec{\xi}_{n-1})$ be given by (2.6) replacing 0 by y.

If $I_F^\lambda(0, \vec{\xi}_n)$ has the analytic extension $J_\lambda^*(F)(\vec{\xi}_n)$ on $\mathbb{C}_+$ as a function of λ, then it is called the conditional analytic Wiener w_φ^r-integral of F given X_t with the parameter λ and denoted by

$$E^{anw_\lambda}[F|X_t](\vec{\xi}_n) = J_\lambda^*(F)(\vec{\xi}_n)$$

for $\vec{\xi}_n \in \mathbb{R}^{(n+1)r}$. Moreover, if for a nonzero real q, $E^{anw_\lambda}[F|X_t](\vec{\xi}_n)$ has the limit as λ approaches to $-iq$ through $\mathbb{C}_+$, then it is called the conditional analytic Feynman w_φ^r-integral of F given X_t with the parameter q and denoted by

$$E^{anf_q}[F|X_t](\vec{\xi}_n) = \lim_{\lambda \to -iq} E^{anw_\lambda}[F|X_t](\vec{\xi}_n).$$

Similarly, the definitions of $E^{anw_\lambda}[F|Y_t](\vec{\xi}_{n-1})$ and $E^{anf_q}[F|Y_t](\vec{\xi}_{n-1})$ are understood with $K_F^\lambda(0, \vec{\xi}_{n-1})$.

3. Time-dependent conditional Fourier-Feynman transform

For a given extended real number p with $1 < p \le \infty$, suppose that p and p' are related by $\frac{1}{p} + \frac{1}{p'} = 1$(possibly $p' = 1$ if $p = \infty$). Let F_n and F be measurable functions such that for $\rho > 0$

$$\lim_{n \to \infty} \int_{C^r} |F_n(\rho y) - F(\rho y)|^{p'} dw_\varphi^r(y) = 0.$$

Then we write

$$\underset{n \to \infty}{\text{l.i.m.}} (w^{p'})(F_n) \approx F$$

and call F the limit in the mean of order p'. A similar definition is understood when n is replaced by a continuously varying parameter.

Now, we define an analytic conditional Fourier-Feynman transform of the functions on $C^r[0,t]$.

Definition 3.1. Let F be defined on $C^r[0,t]$ and let X_t be given by (2.2). For $\lambda \in \mathbb{C}_+$ and for w_φ^r-a.e. $y \in C^r[0,t]$, let

$$T_\lambda[F|X_t](y, \vec{\xi}_n) = E^{anw_\lambda}[F(y+\cdot)|X_t](\vec{\xi}_n)$$

for P_{X_t}-a.e. $\vec{\xi}_n \in \mathbb{R}^{n+1}$ if it exists. For a non-zero real q and for w_φ^r-a.e. $y \in C^r[0,t]$, we define the L_1 analytic conditional Fourier-Feynman transform $T_q^{(1)}[F|X_t]$ of F by the formula

$$T_q^{(1)}[F|X_t](y, \vec{\xi}_n) = E^{anf_q}[F(y+\cdot)|X_t](\vec{\xi}_n)$$

for P_{X_t}-a.e. $\vec{\xi}_n \in \mathbb{R}^{n+1}$ if it exists. For $1 < p \le \infty$ we define the L_p analytic conditional Fourier-Feynman transform $T_q^{(p)}[F|X_t]$ of F by the formula

$$T_q^{(p)}[F|X_t](\cdot, \vec{\xi}_n) \approx \underset{\lambda \to -iq}{\text{l.i.m.}} (w^{p'})(T_\lambda[F|X_t](\cdot, \vec{\xi}_n))$$

for P_{X_t}-a.e. $\vec{\xi}_n \in \mathbb{R}^{n+1}$, where λ approaches to $-iq$ through $\mathbb{C}_+$.

Similar definitions are understood with $K_F^\lambda(y, \vec{\xi}_{n-1})$ if we replace X_t by Y_t which is given by (2.4).

Let η be a complex valued Borel measure on $[0,t]$. Then $\eta = \mu + \nu$ can be decomposed uniquely into the sum of a continuous measure μ and a discrete measure ν. Further, let $\delta_{p_{l,j}}$ denote the Dirac measure with total mass 1 concentrated at $p_{l,j}$.

Let $\mathcal{M}(\mathbb{R}^r)$ be the class of all complex Borel measures on $\mathbb{R}^r$ and $\mathcal{G}^*$ be the set of all $\mathbb{C}$-valued functions θ on $[0,\infty) \times \mathbb{R}^r$ which have the form

$$\theta(s, \vec{u}) = \int_{\mathbb{R}^r} \exp\{i\langle \vec{u}, \vec{v}\rangle\} d\sigma_s(\vec{v}) \tag{3.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the dot product on $\mathbb{R}^r$ and $\{\sigma_s : s \in [0,\infty)\}$ is the family from $\mathcal{M}(\mathbb{R}^r)$ satisfying the following conditions;

(i) for each Borel subset E of $\mathbb{R}^r$, $\sigma_s(E)$ is a Borel measurable function of s on $[0,t]$,

(ii) $\|\sigma_s\| \in L_1([0,t], \mathcal{B}([0,t]), |\eta|)$.

Now we have the following theorem.

Theorem 3.2. *Let X_t be given by (2.2) and let $\eta = \mu + \sum_{l=1}^n \sum_{j=1}^{r_l} w_{l,j}\delta_{p_{l,j}}$, where $w_{l,j} \in \mathbb{C}$ for all (l,j) and $0 = t_0 < p_{1,1} < p_{1,2} < \cdots < p_{1,r_1} < t_1 < p_{2,1} < \cdots <$*

$p_{2,r_2} < t_2 < \cdots < t_{n-1} < p_{n,1} < \cdots < p_{n,r_n} < t_n = t$. Further, let k be a positive integer and let

$$F_k(x) = \left[\int_0^t \theta(s, x(s)) d\eta(s)\right]^k \text{ for } x \in C^r[0,t], \tag{3.2}$$

where $\theta \in \mathcal{G}^*$ is given by (3.1). Then for $\lambda \in \mathbb{C}_+$, $y \in C^r[0,t]$ and $\vec{\xi}_n = (\xi_0, \xi_1, \cdots, \xi_n) \in \mathbb{R}^{(n+1)r}$, $T_\lambda[F_k|X_t](y, \vec{\xi}_n)$ exists and it is given by

$$T_\lambda[F_k|X_t](y, \vec{\xi}_n) = k! \sum_{q_1+\cdots+q_n=k} \prod_{l=1}^n A(l, \lambda, y, \xi_{l-1}, \xi_l, q_l)$$

where

$$\begin{aligned} &A(l, \lambda, y, \xi_{l-1}, \xi_l, q_l) \\ = &\sum_{m_{l,0}+m_{l,1}+\cdots+m_{l,r_l}=q_l} \left(\prod_{j=1}^{r_l} \frac{w_{l,j}^{m_{l,j}}}{m_{l,j}!}\right) \sum_{j_0+j_1+\cdots+j_{r_l}=m_{l,0}} \int_{\Delta_{m_{l,0};j_0,\cdots,j_{r_l}}} \int_{\mathbb{R}^{q_l r}} D(l, y, \\ &\xi_{l-1}, \xi_l, \vec{v}_l, \vec{h}_l, \vec{s}_l) \exp\bigg\{-\frac{1}{2\lambda} \sum_{u=0}^{r_l} \sum_{v=1}^{j_u+1} (s_{l,u,v} - s_{l,u,v-1}) \bigg\| \sum_{\beta=u+1}^{r_l} \sum_{\gamma=1}^{j_\beta+1} \frac{t_l - s_{l,\beta,\gamma}}{t_l - t_{l-1}} \\ &\times \vec{v}_{l,\beta,\gamma} + \sum_{\gamma=v}^{j_u+1} \frac{t_l - s_{l,u,\gamma}}{t_l - t_{l-1}} \vec{v}_{l,u,\gamma} + \sum_{\gamma=1}^{v-1} \frac{t_{l-1} - s_{l,u,\gamma}}{t_l - t_{l-1}} \vec{v}_{l,u,\gamma} + \sum_{\beta=0}^{u-1} \sum_{\gamma=1}^{j_\beta+1} \frac{t_{l-1} - s_{l,\beta,\gamma}}{t_l - t_{l-1}} \\ &\times \vec{v}_{l,\beta,\gamma} \bigg\|_{\mathbb{R}^r}^2 \bigg\} d\left(\prod_{u=0}^{r_l} \prod_{v=1}^{j_u} \sigma_{s_{l,u,v}} \times \prod_{u=1}^{r_l} \sigma_{p_{l,u}}^{m_{l,u}}\right)(\vec{v}_l, \vec{h}_l) d\mu^{m_{l,0}}(\vec{s}_l) \end{aligned}$$

with the conventions those $s_{l,0,0} = t_{l-1}$, $s_{l,u,0} = p_{l,u} = s_{l,u-1,j_{u-1}+1}$ for $u = 1, \cdots, r_l$, $s_{l,r_l,j_{r_l}+1} = t_l$, $\vec{s}_l = (s_{l,0,1}, \cdots, s_{l,0,j_0}, \cdots, s_{l,r_l,1}, \cdots, s_{l,r_l,j_{r_l}})$, $\Delta_{m_{l,0};j_0,\cdots,j_{r_l}} = \{\vec{s}_l : t_{l-1} < s_{l,0,1} < \cdots < s_{l,0,j_0} < p_{l,1} < s_{l,1,1} < \cdots < s_{l,1,j_1} < p_{l,2} < \cdots < p_{l,r_l} < s_{l,r_l,1} < \cdots < s_{l,r_l,j_{r_l}} < t_l\}$, $\vec{v}_l = (\vec{v}_{l,0,1}, \cdots, \vec{v}_{l,0,j_0}, \vec{v}_{l,1,1}, \cdots, \vec{v}_{l,1,j_1}, \cdots, \vec{v}_{l,r_l,1}, \cdots, \vec{v}_{l,r_l,j_{r_l}})$, $\vec{v}_{l,r_l,j_{r_l}+1} = \vec{0} \in \mathbb{R}^r$, $\vec{h}_l = (\vec{h}_{l,1,1}, \cdots, \vec{h}_{l,1,m_{l,1}}, \vec{h}_{l,2,1}, \cdots, \vec{h}_{l,2,m_{l,2}}, \cdots, \vec{h}_{l,r_l,1}, \cdots, \vec{h}_{l,r_l,m_{l,r_l}})$; $\vec{v}_{l,u-1,j_{u-1}+1} = \sum_{v=1}^{m_{l,u}} \vec{h}_{l,u,v}$ for $u = 1, \cdots, r_l$,

$$\begin{aligned} &D(l, y, \xi_{l-1}, \xi_l, \vec{v}_l, \vec{h}_l, \vec{s}_l) \\ = &\exp\bigg\{ i \sum_{u=0}^{r_l} \sum_{v=1}^{j_u+1} \Big\langle y(s_{l,u,v}) + \frac{t_l - s_{l,u,v}}{t_l - t_{l-1}} \xi_{l-1} + \frac{s_{l,u,v} - t_{l-1}}{t_l - t_{l-1}} \xi_l, \vec{v}_{l,u,v} \Big\rangle \bigg\}, \end{aligned} \tag{3.3}$$

$\sum_{\beta=u+1}^{r_l} \sum_{\gamma=1}^{j_\beta+1} \frac{t_l - s_{l,\beta,\gamma}}{t_l - t_{l-1}} \vec{v}_{l,\beta,\gamma} = \vec{0}$ if $u = r_l$, $\sum_{\gamma=1}^{v-1} \frac{t_{l-1} - s_{l,u,\gamma}}{t_l - t_{l-1}} \vec{v}_{l,u,\gamma} = \vec{0}$ if $v = 1$ and $\sum_{\beta=0}^{u-1} \sum_{\gamma=1}^{j_\beta+1} \frac{t_{l-1} - s_{l,\beta,\gamma}}{t_l - t_{l-1}} \vec{v}_{l,\beta,\gamma} = \vec{0}$ if $u = 0$.

Proof. For $\lambda > 0$, $y \in C^r[0,t]$ and $\vec{\xi}_n = (\xi_0, \xi_1, \cdots, \xi_n) \in \mathbb{R}^{(n+1)r}$, we have by Lemma 2.1 and the binomial expansion

$$\begin{aligned} & I^{\lambda}_{F_k}(y, \vec{\xi}_n) \\ = & \sum_{q_1+\cdots+q_n=k} \frac{k!}{q_1! \cdots q_n!} \prod_{l=1}^{n} \int_{C^r} \bigg[\int_{t_{l-1}}^{t_l} \theta(s, y(s) + \lambda^{-\frac{1}{2}}(x(s) - [x](s)) + [\vec{\xi}_n](s)) \\ & d\mu(s) + \sum_{j=1}^{r_l} w_{l,j} \theta(p_{l,j}, y(p_{l,j}) + \lambda^{-\frac{1}{2}}(x(p_{l,j}) - [x](p_{l,j})) + [\vec{\xi}_n](p_{l,j})) \bigg]^{q_l} dw^r_{\varphi}(x). \end{aligned}$$

Using the simplex method [7] and the Fubini's theorem we have by (3.1)

$$\begin{aligned} & I^{\lambda}_{F_k}(y, \vec{\xi}_n) \\ = & k! \sum_{q_1+\cdots+q_n=k} \prod_{l=1}^{n} \sum_{m_{l,0}+m_{l,1}+\cdots+m_{l,r_l}=q_l} \bigg(\prod_{j=1}^{r_l} \frac{w_{l,j}^{m_{l,j}}}{m_{l,j}!} \bigg) \sum_{j_0+j_1+\cdots+j_{r_l}=m_{l,0}} \\ & \int_{\Delta_{m_{l,0};j_0,\cdots,j_{r_l}}} \int_{C^r} \bigg[\int_{\mathbb{R}^{m_{l,0}r}} \exp\bigg\{ i \sum_{u=0}^{r_l} \sum_{v=1}^{j_u} \langle y(s_{l,u,v}) + \lambda^{-\frac{1}{2}}(x(s_{l,u,v}) - \\ & [x](s_{l,u,v})) + [\vec{\xi}_n](s_{l,u,v}), \vec{v}_{l,u,v} \rangle \bigg\} d\bigg(\prod_{u=0}^{r_l} \prod_{v=1}^{j_u} \sigma_{s_{l,u,v}} \bigg)(\vec{v}_l) \bigg] \bigg[\int_{\mathbb{R}^{(m_{l,1}+\cdots+m_{l,r_l})r}} \\ & \exp\bigg\{ i \sum_{u=1}^{r_l} \sum_{v=1}^{m_{l,u}} \langle y(s_{l,u,0}) + \lambda^{-\frac{1}{2}}(x(s_{l,u,0}) - [x](s_{l,u,0})) + [\vec{\xi}_n](s_{l,u,0}), \vec{h}_{l,u,v} \rangle \bigg\} \\ & d\bigg(\prod_{u=1}^{r_l} \sigma_{p_{l,u}}^{m_{l,u}} \bigg)(\vec{h}_l) \bigg] dw^r_{\varphi}(x) d\mu^{m_{l,0}}(\vec{s}_l), \end{aligned}$$

where $\vec{s}_l$, $\vec{v}_l$, $\vec{h}_l$ and $\Delta_{m_{l,0};j_0,\cdots,j_{r_l}}$ are given by the assumptions, and $s_{l,u,0} = p_{l,u}$ for $u = 1, \cdots, r_l$. For $u = 1, \cdots, r_l$, let $s_{l,u-1,j_{u-1}+1} = s_{l,u,0}$, $\vec{v}_{l,u-1,j_{u-1}+1} = \sum_{v=1}^{m_{l,u}} \vec{h}_{l,u,v}$, $s_{l,r_l,j_{r_l}+1} = t_l$, $s_{l,0,0} = t_{l-1}$ and $\vec{v}_{l,r_l,j_{r_l}+1} = \vec{0} \in \mathbb{R}^r$. Then we have

$$\begin{aligned} & I^{\lambda}_{F_k}(y, \vec{\xi}_n) \\ = & k! \sum_{q_1+\cdots+q_n=k} \prod_{l=1}^{n} \sum_{m_{l,0}+m_{l,1}+\cdots+m_{l,r_l}=q_l} \bigg(\prod_{j=1}^{r_l} \frac{w_{l,j}^{m_{l,j}}}{m_{l,j}!} \bigg) \sum_{j_0+j_1+\cdots+j_{r_l}=m_{l,0}} \\ & \int_{\Delta_{m_{l,0};j_0,\cdots,j_{r_l}}} \int_{\mathbb{R}^{q_l r}} D(l, y, \xi_{l-1}, \xi_l, \vec{v}_l, \vec{h}_l, \vec{s}_l) \int_{C^r} \exp\bigg\{ i\lambda^{-\frac{1}{2}} \sum_{u=0}^{r_l} \sum_{v=1}^{j_u+1} \\ & \bigg\langle \frac{t_l - s_{l,u,v}}{t_l - t_{l-1}}(x(s_{l,u,v}) - x(t_{l-1})) + \frac{t_{l-1} - s_{l,u,v}}{t_l - t_{l-1}}(x(t_l) - x(s_{l,u,v})), \\ & \vec{v}_{l,u,v} \bigg\rangle \bigg\} dw^r_{\varphi}(x) d\bigg(\prod_{u=0}^{r_l} \prod_{v=1}^{j_u} \sigma_{s_{l,u,v}} \times \prod_{u=1}^{r_l} \sigma_{p_{l,u}}^{m_{l,u}} \bigg)(\vec{v}_l, \vec{h}_l) d\mu^{m_{l,0}}(\vec{s}_l) \end{aligned}$$

where $D(l, y, \xi_{l-1}, \xi_l, \vec{v}_l, \vec{h}_l, \vec{s}_l)$ is given by (3.3). Let

$$S(1) = \left[\prod_{u=0}^{r_1}\prod_{v=1}^{j_u+1}\frac{1}{2\pi(s_{1,u,v}-s_{1,u,v-1})}\right]^{\frac{r}{2}}$$

and for $l = 2, \cdots, n$ let

$$S(l) = \left[\frac{1}{2\pi t_{l-1}}\prod_{u=0}^{r_l}\prod_{v=1}^{j_u+1}\frac{1}{2\pi(s_{l,u,v}-s_{l,u,v-1})}\right]^{\frac{r}{2}}.$$

Let $m(1) = m_{1,0} + r_1 + 1$ and $m(l) = m_{l,0} + r_l + 2$ if $l = 2, \cdots, n$. Then we have by an application of Lemma 1.1

$$\begin{aligned}
&I_{F_k}^{\lambda}(y, \vec{\xi}_n)\\
= \;& k! \sum_{q_1+\cdots+q_n=k}\prod_{l=1}^{n}\sum_{m_{l,0}+m_{l,1}+\cdots+m_{l,r_l}=q_l}\left(\prod_{j=1}^{r_l}\frac{w_{l,j}^{m_{l,j}}}{m_{l,j}!}\right)\sum_{j_0+j_1+\cdots+j_{r_l}=m_{l,0}}\\
&\int_{\Delta_{m_{l,0};j_0,\cdots,j_{r_l}}}\int_{\mathbb{R}^{q_l r}} D(l, y, \xi_{l-1}, \xi_l, \vec{v}_l, \vec{h}_l, \vec{s}_l) S(l)\int_{\mathbb{R}^r}\int_{\mathbb{R}^{m(l)r}}\exp\bigg\{ i\lambda^{-\frac{1}{2}}\sum_{u=0}^{r_l}\sum_{v=1}^{j_u+1}\\
&\left\langle \frac{t_l - s_{l,u,v}}{t_l - t_{l-1}}(\vec{\zeta}_{l,u,v} - \vec{\zeta}_{l,0,0}) + \frac{t_{l-1} - s_{l,u,v}}{t_l - t_{l-1}}(\vec{\zeta}_{l,r_l,j_{r_l}+1} - \vec{\zeta}_{l,u,v}), \vec{v}_{l,u,v}\right\rangle - \frac{1}{2}\sum_{u=0}^{r_l}\\
&\sum_{v=1}^{j_u+1}\frac{\|\vec{\zeta}_{l,u,v} - \vec{\zeta}_{l,u,v-1}\|_{\mathbb{R}^r}^2}{s_{l,u,v} - s_{l,u,v-1}} - \frac{\|\vec{\zeta}_{l,0,0} - \vec{\zeta}_{1,0,0}\|_{\mathbb{R}^r}^2}{2t_{l-1} + \delta_{l1}}\bigg\} d\vec{\zeta}_l d\varphi^r(\vec{\zeta}_{1,0,0}) d\Bigg(\prod_{u=0}^{r_l}\prod_{v=1}^{j_u}\sigma_{s_{l,u,v}}\\
&\times\prod_{u=1}^{r_l}\sigma_{p_{l,u}}^{m_{l,u}}\Bigg)(\vec{v}_l, \vec{h}_l) d\mu^{m_{l,0}}(\vec{s}_l)
\end{aligned}$$

where $\vec{\zeta}_{l,u-1,j_{u-1}+1} = \vec{\zeta}_{l,u,0}$ for $u = 1, \cdots, r_l$ and $\vec{\zeta}_1 = (\vec{\zeta}_{1,0,1}, \cdots, \vec{\zeta}_{1,0,j_0+1}, \vec{\zeta}_{1,1,1}, \cdots, \vec{\zeta}_{1,1,j_1+1}, \cdots, \vec{\zeta}_{1,r_1,1}, \cdots, \vec{\zeta}_{1,r_1,j_{r_1}+1})$, $\vec{\zeta}_l = (\vec{\zeta}_{l,0,0}, \vec{\zeta}_{l,0,1}, \cdots, \vec{\zeta}_{l,0,j_0+1}, \vec{\zeta}_{l,1,1}, \cdots, \vec{\zeta}_{l,1,j_1+1}, \cdots, \vec{\zeta}_{l,r_l,1}, \cdots, \vec{\zeta}_{l,r_l,j_{r_l}+1})$ if $l = 2, \cdots, n$. Let $\vec{\eta}_{l,u,v} = \vec{\zeta}_{l,u,v} - \vec{\zeta}_{l,u,v-1}$ for $u = 0, 1, \cdots, r_l$; $v = 1, \cdots, j_u+1$ and $\vec{\eta}_{1,0,0} = \vec{0}$, $\vec{\eta}_{l,0,0} = \vec{\zeta}_{l,0,0} - \vec{\zeta}_{1,0,0}$ if $l = 2, \cdots, n$. Then we have by the change of variable theorem

$$\begin{aligned}
&I_{F_k}^{\lambda}(y, \vec{\xi}_n)\\
= \;& k! \sum_{q_1+\cdots+q_n=k}\prod_{l=1}^{n}\sum_{m_{l,0}+m_{l,1}+\cdots+m_{l,r_l}=q_l}\left(\prod_{j=1}^{r_l}\frac{w_{l,j}^{m_{l,j}}}{m_{l,j}!}\right)\sum_{j_0+j_1+\cdots+j_{r_l}=m_{l,0}}\\
&\int_{\Delta_{m_{l,0};j_0,\cdots,j_{r_l}}}\int_{\mathbb{R}^{q_l r}} D(l, y, \xi_{l-1}, \xi_l, \vec{v}_l, \vec{h}_l, \vec{s}_l) S(l)\int_{\mathbb{R}^{m(l)r}}\exp\bigg\{ i\lambda^{-\frac{1}{2}}\sum_{u=0}^{r_l}\sum_{v=1}^{j_u+1}
\end{aligned}$$

$$\Big\langle \frac{t_l - s_{l,u,v}}{t_l - t_{l-1}} \Big(\sum_{\beta=0}^{u-1} \sum_{\gamma=1}^{j_\beta+1} \vec{\eta}_{l,\beta,\gamma} + \sum_{\gamma=1}^{v} \vec{\eta}_{l,u,\gamma} \Big) + \frac{t_{l-1} - s_{l,u,v}}{t_l - t_{l-1}} \Big(\sum_{\gamma=v+1}^{j_u+1} \vec{\eta}_{l,u,\gamma} + \sum_{\beta=u+1}^{r_l} \sum_{\gamma=1}^{j_\beta+1} \vec{\eta}_{l,\beta,\gamma} \Big), \vec{v}_{l,u,v} \Big\rangle - \frac{1}{2} \sum_{u=0}^{r_l} \sum_{v=1}^{j_u+1} \frac{\|\vec{\eta}_{l,u,v}\|_{\mathbb{R}^r}^2}{s_{l,u,v} - s_{l,u,v-1}} - \frac{\|\vec{\eta}_{l,0,0}\|_{\mathbb{R}^r}^2}{2t_{l-1} + \delta_{l1}} \Big\} d\vec{\eta}_l d\Big(\prod_{u=0}^{r_l} \prod_{v=1}^{j_u} \sigma_{s_{l,u,v}} \times \prod_{u=1}^{r_l} \sigma_{p_{l,u}}^{m_{l,u}} \Big)(\vec{v}_l, \vec{h}_l) d\mu^{m_{l,0}}(\vec{s}_l)$$

where $\vec{\eta}_1 = (\vec{\eta}_{1,0,1}, \cdots, \vec{\eta}_{1,0,j_0+1}, \vec{\eta}_{1,1,1}, \cdots, \vec{\eta}_{1,1,j_1+1}, \cdots, \vec{\eta}_{1,r_1,1}, \cdots, \vec{\eta}_{1,r_1,j_{r_1}+1})$, $\vec{\eta}_l = (\vec{\eta}_{l,0,0}, \vec{\eta}_{l,0,1}, \cdots, \vec{\eta}_{l,0,j_0+1}, \vec{\eta}_{l,1,1}, \cdots, \vec{\eta}_{l,1,j_1+1}, \cdots, \vec{\eta}_{l,r_l,1}, \cdots, \vec{\eta}_{l,r_l,j_{r_l}+1})$ if $l \in \{2, \cdots, n\}$, $\sum_{\beta=0}^{u-1} \sum_{\gamma=1}^{j_\beta+1} \vec{\eta}_{l,\beta,\gamma} = \vec{0}$ if $u = 0$, $\sum_{\gamma=v+1}^{j_u+1} \vec{\eta}_{l,u,\gamma} = \vec{0}$ if $v = j_u + 1$ and $\sum_{\beta=u+1}^{r_l} \sum_{\gamma=1}^{j_\beta+1} \vec{\eta}_{l,\beta,\gamma} = \vec{0}$ if $u = r_l$. Now we have

$$\begin{aligned} & I_{F_k}^{\lambda}(y, \vec{\xi}_n) \\ = {} & k! \sum_{q_1+\cdots+q_n=k} \prod_{l=1}^{n} \sum_{m_{l,0}+m_{l,1}+\cdots+m_{l,r_l}=q_l} \Big(\prod_{j=1}^{r_l} \frac{w_{l,j}^{m_{l,j}}}{m_{l,j}!} \Big) \sum_{j_0+j_1+\cdots+j_{r_l}=m_{l,0}} \\ & \int_{\Delta_{m_{l,0};j_0,\cdots,j_{r_l}}} \int_{\mathbb{R}^{q_l r}} D(l, y, \xi_{l-1}, \xi_l, \vec{v}_l, \vec{h}_l, \vec{s}_l) S(l) \int_{\mathbb{R}^{m(l)r}} \exp\Big\{ i\lambda^{-\frac{1}{2}} \sum_{u=0}^{r_l} \sum_{v=1}^{j_u+1} \Big\langle \\ & \vec{\eta}_{l,u,v}, \sum_{\beta=u+1}^{r_l} \sum_{\gamma=1}^{j_\beta+1} \frac{t_l - s_{l,\beta,\gamma}}{t_l - t_{l-1}} \vec{v}_{l,\beta,\gamma} + \sum_{\gamma=v}^{j_u+1} \frac{t_l - s_{l,u,\gamma}}{t_l - t_{l-1}} \vec{v}_{l,u,\gamma} + \sum_{\gamma=1}^{v-1} \frac{t_{l-1} - s_{l,u,\gamma}}{t_l - t_{l-1}} \\ & \vec{v}_{l,u,\gamma} + \sum_{\beta=0}^{u-1} \sum_{\gamma=1}^{j_\beta+1} \frac{t_{l-1} - s_{l,\beta,\gamma}}{t_l - t_{l-1}} \vec{v}_{l,\beta,\gamma} \Big\rangle - \frac{1}{2} \sum_{u=0}^{r_l} \sum_{v=1}^{j_u+1} \frac{\|\vec{\eta}_{l,u,v}\|_{\mathbb{R}^r}^2}{s_{l,u,v} - s_{l,u,v-1}} - \\ & \frac{\|\vec{\eta}_{l,0,0}\|_{\mathbb{R}^r}^2}{2t_{l-1} + \delta_{l1}} \Big\} d\vec{\eta}_l d\Big(\prod_{u=0}^{r_l} \prod_{v=1}^{j_u} \sigma_{s_{l,u,v}} \times \prod_{u=1}^{r_l} \sigma_{p_{l,u}}^{m_{l,u}} \Big)(\vec{v}_l, \vec{h}_l) d\mu^{m_{l,0}}(\vec{s}_l) \\ = {} & k! \sum_{q_1+\cdots+q_n=k} \prod_{l=1}^{n} A(l, \lambda, y, \xi_{l-1}, \xi_l, q_l) \end{aligned}$$

where the last equality follows from the well known integration formula

$$\int_{\mathbb{R}} \exp\{-au^2 + ibu\} du = \Big(\frac{\pi}{a}\Big)^{\frac{1}{2}} \exp\Big\{-\frac{b^2}{4a}\Big\} \tag{3.4}$$

for $a \in \mathbb{C}_+$ and any real b. By the Morera's theorem, the theorem follows. □

Corollary 3.3. *Under the assumptions given as in Theorem 3.2, with one exception $\eta = \mu$, that is, assuming that η has no discrete part, we have for $\lambda \in \mathbb{C}_+$, $y \in C^r[0,t]$*

and $\vec{\xi}_n \in \mathbb{R}^{(n+1)r}$

$$T_\lambda[F_k|X_t](y,\vec{\xi}_n) = k! \sum_{q_1+\cdots+q_n=k} \prod_{l=1}^{n} A_\mu(l,\lambda,y,\vec{\xi}_n,q_l)$$

where

$$\begin{aligned} & A_\mu(l,\lambda,y,\vec{\xi}_n,q_l) \\ = \; & \int_{\Delta_{q_l}} \int_{\mathbb{R}^{q_l r}} \exp\bigg\{ i \sum_{u=1}^{q_l} \langle y(s_{l,u}) + [\vec{\xi}_n](s_{l,u}), \vec{v}_{l,u} \rangle - \frac{1}{2\lambda} \sum_{u=1}^{q_l+1} (s_{l,u} - s_{l,u-1}) \\ & \times \bigg\| \sum_{\beta=u}^{q_l} \frac{t_l - s_{l,\beta}}{t_l - t_{l-1}} \vec{v}_{l,\beta} + \sum_{\beta=1}^{u-1} \frac{t_{l-1} - s_{l,\beta}}{t_l - t_{l-1}} \vec{v}_{l,\beta} \bigg\|_{\mathbb{R}^r}^2 \bigg\} d\bigg(\prod_{u=1}^{q_l} \sigma_{s_{l,u}} \bigg)(\vec{v}_l) d\mu^{q_l}(\vec{s}_l) \end{aligned}$$

with the conventions those $s_{l,0} = t_{l-1}$, $s_{l,q_l+1} = t_l$, $\vec{s}_l = (s_{l,1}, s_{l,2}, \cdots, s_{l,q_l})$, $\Delta_{q_l} = \{\vec{s}_l : t_{l-1} < s_{l,1} < s_{l,2} < \cdots < s_{l,q_l} < t_l\}$, $\vec{v}_l = (\vec{v}_{l,1}, \vec{v}_{l,2}, \cdots, \vec{v}_{l,q_l})$, $\sum_{\beta=u}^{q_l} \frac{t_l - s_{l,\beta}}{t_l - t_{l-1}} \vec{v}_{l,\beta} = \vec{0}$ *if* $u = q_l + 1$ *and* $\sum_{\beta=1}^{u-1} \frac{t_{l-1} - s_{l,\beta}}{t_l - t_{l-1}} \vec{v}_{l,\beta} = \vec{0}$ *if* $u = 1$.

Corollary 3.4. *Under the assumptions given as in Theorem 3.2 with one exception* $\eta = \sum_{l=1}^{n} \sum_{j=1}^{r_l} w_{l,j} \delta_{p_{l,j}}$, *that is, assuming that* η *has no continuous part, we have for* $\lambda \in \mathbb{C}_+$, $y \in C^r[0,t]$ *and* $\vec{\xi}_n \in \mathbb{R}^{(n+1)r}$

$$T_\lambda[F_k|X_t](y,\vec{\xi}_n) = k! \sum_{q_1+\cdots+q_n=k} \prod_{l=1}^{n} A_\nu(l,\lambda,y,\vec{\xi}_n,q_l)$$

where

$$\begin{aligned} & A_\nu(l,\lambda,y,\vec{\xi}_n,q_l) \\ = \; & \sum_{m_{l,1}+\cdots+m_{l,r_l}=q_l} \bigg(\prod_{j=1}^{r_l} \frac{w_{l,j}^{m_{l,j}}}{m_{l,j}!} \bigg) \int_{\mathbb{R}^{q_l r}} \exp\bigg\{ i \sum_{u=1}^{r_l} \sum_{v=1}^{m_{l,u}} \langle y(p_{l,u}) + [\vec{\xi}_n](p_{l,u}), \vec{h}_{l,u,v} \rangle \\ & - \frac{1}{2\lambda} \sum_{u=1}^{r_l+1} (p_{l,u} - p_{l,u-1}) \bigg\| \sum_{\beta=u}^{r_l} \sum_{\gamma=1}^{m_{l,\beta}} \frac{t_l - p_{l,\beta}}{t_l - t_{l-1}} \vec{h}_{l,\beta,\gamma} + \sum_{\beta=1}^{u-1} \sum_{\gamma=1}^{m_{l,\beta}} \frac{t_{l-1} - p_{l,\beta}}{t_l - t_{l-1}} \vec{h}_{l,\beta,\gamma} \bigg\|_{\mathbb{R}^r}^2 \\ & \bigg\} d\bigg(\prod_{u=1}^{r_l} \sigma_{p_{l,u}}^{m_{l,u}} \bigg)(\vec{h}_l) \end{aligned}$$

with the conventions those $p_{l,0} = t_{l-1}$, $p_{l,r_l+1} = t_l$, $\vec{h}_l = (\vec{h}_{l,1,1}, \cdots, \vec{h}_{l,1,m_{l,1}}, \vec{h}_{l,2,1}, \cdots, \vec{h}_{l,2,m_{l,2}}, \cdots, \vec{h}_{l,r_l,1}, \cdots, \vec{h}_{l,r_l,m_{l,r_l}})$, $\sum_{\beta=u}^{r_l} \sum_{\gamma=1}^{m_{l,\beta}} \frac{t_l - p_{l,\beta}}{t_l - t_{l-1}} \vec{h}_{l,\beta,\gamma} = \vec{0}$ *if* $u = r_l + 1$ *and* $\sum_{\beta=1}^{u-1} \sum_{\gamma=1}^{m_{l,\beta}} \frac{t_{l-1} - p_{l,\beta}}{t_l - t_{l-1}} \vec{h}_{l,\beta,\gamma} = \vec{0}$ *if* $u = 1$.

Theorem 3.5. *Let* $1 \le p \le \infty$ *and* q *be a nonzero real number. Then, under the* *assumptions given as in Theorem 3.2, $T_q^{(p)}[F_k|X_t](y,\vec{\xi}_n)$ exists for $y \in C^r[0,t]$ and $\vec{\xi}_n = (\xi_0, \xi_1, \cdots, \xi_n) \in \mathbb{R}^{(n+1)r}$, and it is given by*

$$T_q^{(p)}[F_k|X_t](y,\vec{\xi}_n) = k! \sum_{q_1+\cdots+q_n=k} \prod_{l=1}^{n} A(l,-iq,y,\xi_{l-1},\xi_l,q_l) \tag{3.5}$$

where A is given as in Theorem 3.2.

Proof. For $1 \le p \le \infty$ let $T_q^{(p)}[F_k|X_t](y,\vec{\xi}_n)$ be given by the right hand side of (3.5). If $\lambda \in \mathbb{C}_+$ or $\lambda = -iq$, we have by the simplex method and the binomial expansion

$$\begin{aligned}
&\Big| k! \sum_{q_1+\cdots+q_n=k} \prod_{l=1}^{n} A(l,\lambda,y,\xi_{l-1},\xi_l,q_l) \Big| \\
\le\ & k! \sum_{q_1+\cdots+q_n=k} \prod_{l=1}^{n} \Big[\sum_{m_{l,0}+m_{l,1}+\cdots+m_{l,r_l}=q_l} \frac{1}{m_{l,0}!} \Big[\int_{t_{l-1}}^{t_l} \|\sigma_{s_l}\| d|\mu|(s_l) \Big]^{m_{l,0}} \\
& \times \prod_{j=1}^{r_l} \frac{(|w_{l,j}| \|\sigma_{p_{l,j}}\|)^{m_{l,j}}}{m_{l,j}!} \Big] \\
=\ & k! \sum_{q_1+\cdots+q_n=k} \prod_{l=1}^{n} \frac{1}{q_l!} \Big[\int_{t_{l-1}}^{t_l} \|\sigma_{s_l}\| d|\eta|(s_l) \Big]^{q_l} = \Big[\int_0^t \|\sigma_s\| d|\eta|(s) \Big]^k
\end{aligned}$$

so that for $\lambda \in \mathbb{C}_+$ we have

$$|T_\lambda[F_k|X_t](y,\vec{\xi}_n) - T_q^{(p)}[F_k|X_t](y,\vec{\xi}_n)| \le 2 \Big[\int_0^t \|\sigma_s\| d|\eta|(s) \Big]^k.$$

Hence when $1 < p \le \infty$, for $\frac{1}{p} + \frac{1}{p'} = 1$ and $\rho > 0$, we have

$$\begin{aligned}
&\int_{C^r} |T_\lambda[F_k|X_t](\rho y,\vec{\xi}_n) - T_q^{(p)}[F_k|X_t](\rho y,\vec{\xi}_n)|^{p'} dw_\varphi^r(y) \\
\le\ & \int_{C^r} \Big[2 \Big[\int_0^t \|\sigma_s\| d|\eta|(s) \Big]^k \Big]^{p'} dw_\varphi^r(y) = 2^{p'} \Big[\int_0^t \|\sigma_s\| d|\eta|(s) \Big]^{kp'} < \infty.
\end{aligned}$$

Letting $\lambda \to -iq$ through $\mathbb{C}_+$, we have the result by the dominated convergence theorem. □

Theorem 3.6. *For $x \in C^r[0,t]$ let*

$$F(x) = \exp\Big\{ \int_0^t \theta(s,x(s)) d\eta(s) \Big\}. \tag{3.6}$$

Then under the assumptions given as in Theorem 3.5, $T_q^{(p)}[F|X_t](y,\vec{\xi}_n)$ exists for $y \in C^r[0,t]$ and $\vec{\xi}_n \in \mathbb{R}^{(n+1)r}$, and it is given by

$$T_q^{(p)}[F|X_t](y,\vec{\xi}_n) = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} T_q^{(p)}[F_k|X_t](y,\vec{\xi}_n) \tag{3.7}$$

where $T_q^{(p)}[F_k|X_t](y,\vec{\xi_n})$ *is given as in Theorem 3.5.*

Proof. By the Maclaurin series of the exponential function, we have

$$F(x)=1+\sum_{k=1}^{\infty}\frac{1}{k!}F_k(x)$$

and for $\lambda\in\mathbb{C}_+$ or $\lambda=-iq$, $y\in C^r[0,t]$ and $\vec{\xi_n}=(\xi_0,\xi_1,\cdots,\xi_n)\in\mathbb{R}^{(n+1)r}$ we have

$$\begin{aligned}&1+\sum_{k=1}^{\infty}\frac{1}{k!}\bigg|k!\sum_{q_1+\cdots+q_n=k}\prod_{l=1}^{n}A(l,\lambda,y,\xi_{l-1},\xi_l,q_l)\bigg|\\ \le\ &1+\sum_{k=1}^{\infty}\frac{1}{k!}\bigg[\int_0^t\|\sigma_s\|d|\eta|(s)\bigg]^k=\exp\bigg\{\int_0^t\|\sigma_s\|d|\eta|(s)\bigg\}\end{aligned}$$

where A is given as in Theorems 3.2 and 3.5. Hence we have for $\lambda\in\mathbb{C}_+$

$$T_\lambda[F|X_t](y,\vec{\xi_n})=1+\sum_{k=1}^{\infty}\frac{1}{k!}T_\lambda[F_k|X_t](y,\vec{\xi_n})\tag{3.8}$$

since the convergence of (3.8) is uniform with respect to both λ and y. For $1\le p\le\infty$ let $T_q^{(p)}[F|X_t](y,\vec{\xi_n})$ be given by the right hand side of (3.7). For $\lambda\in\mathbb{C}_+$ we have

$$|T_\lambda[F|X_t](y,\vec{\xi_n})-T_q^{(p)}[F|X_t](y,\vec{\xi_n})|\le 2\exp\bigg\{\int_0^t\|\sigma_s\|d|\eta|(s)\bigg\}$$

and hence when $1<p\le\infty$, for $\frac{1}{p}+\frac{1}{p'}=1$ and $\rho>0$, we have

$$\begin{aligned}&\int_{C^r}|T_\lambda[F|X_t](\rho y,\vec{\xi_n})-T_q^{(p)}[F|X_t](\rho y,\vec{\xi_n})|^{p'}dw_\varphi^r(y)\\ \le\ &\int_{C^r}\bigg[2\exp\bigg\{\int_0^t\|\sigma_s\|d|\eta|(s)\bigg\}\bigg]^{p'}dw_\varphi^r(y)=2^{p'}\exp\bigg\{p'\int_0^t\|\sigma_s\|d|\eta|(s)\bigg\}<\infty.\end{aligned}$$

Letting $\lambda\to-iq$ through $\mathbb{C}_+$, we have the result by the dominated convergence theorem. □

For $\nu\in\mathcal{M}(\mathbb{R}^r)$ define ψ on $\mathbb{R}^r$ by

$$\psi(\vec{u})=\int_{\mathbb{R}^r}\exp\{i\langle\vec{u},\vec{v}\rangle\}d\nu(\vec{v}).\tag{3.9}$$

Then we have for $\vec{\xi_n}=(\xi_0,\xi_1,\cdots,\xi_n)\in\mathbb{R}^{(n+1)r}$ and $\lambda>0$

$$\psi(y(t)+\lambda^{-\frac{1}{2}}(x(t)-[x](t))+[\vec{\xi_n}](t))=\psi(y(t)+\xi_n)\tag{3.10}$$

and

$$|\psi(y(t)+\xi_n)| \le \|\nu\|. \tag{3.11}$$

By (3.10), (3.11), Theorems 3.2, 3.5 and 3.6, we have the following theorem.

Theorem 3.7. *Let G_k and G be given by*

$$G_k(x) = F_k(x)\psi(x(t))$$

and

$$G(x) = F(x)\psi(x(t)) \tag{3.12}$$

for $x \in C^r[0,t]$, where F_k, F and ψ are given by (3.2), (3.6) and (3.9), respectively. Then, under the assumptions and notations given as in Theorems 3.2, 3.5 and 3.6, we have for $y \in C^r[0,t]$, nonzero real q and $\vec{\xi_n} = (\xi_0, \xi_1, \cdots, \xi_n) \in \mathbb{R}^{(n+1)r}$

$$T_q^{(p)}[G_k|X_t](y, \vec{\xi_n}) = \psi(y(t)+\xi_n) T_q^{(p)}[F_k|X_t](y, \vec{\xi_n})$$

and

$$\begin{aligned} T_q^{(p)}[G|X_t](y, \vec{\xi_n}) &= \psi(y(t)+\xi_n) T_q^{(p)}[F|X_t](y, \vec{\xi_n}) \\ &= \psi(y(t)+\xi_n) + \sum_{k=1}^{\infty} \frac{1}{k!} T_q^{(p)}[G_k|X_t](y, \vec{\xi_n}). \end{aligned}$$

Remark 3.8.

- If F_k, G_k, F and G are defined on r-dimensional Wiener space, then we can obtain the same results in Theorems 3.2, 3.5, 3.6 and 3.7 with $\xi_0 = \vec{0} \in \mathbb{R}^r$ in the expression of $\vec{\xi_n}$.
- If some of the $p_{l,j}$s are in the set $\{t_0, t_1, \cdots, t_n\}$, we can obtain all the results in the present section with minor modifications.
- If $\eta = \mu + \sum_{l=1}^{n} \sum_{j=1}^{r_l} w_{l,j} \delta_{p_{l,j}}$ and some of the r_ls are ∞, then, using the following version of the $\aleph_0$-nomial formula [7, p.41]

  $$\left(\sum_{p=0}^{\infty} b_p\right)^n = \sum_{h=0}^{\infty} \sum_{q_0+q_1+\cdots+q_h=n, q_h \neq 0} \frac{n!}{q_0! q_1! \cdots q_h!} b_0^{q_0} b_1^{q_1} \cdots b_h^{q_h}, \tag{3.13}$$

  we can show that $T_q^{(p)}[G|X_t](y, \vec{\xi_n})$ exists in Theorem 3.7.

4. Time-independent conditional Fourier-Feynman transform

In the present section we evaluate the conditional analytic Fourier-Feynman transform of G given Y_t, where Y_t and G are given by (2.4) and (3.12), respectively. For the purpose we need the following lemma.

Lemma 4.1. *For $\lambda > 0$, $\vec{v} \in \mathbb{R}^r$, $y \in C^r[0,t]$ and $\xi_{n-1} \in \mathbb{R}^r$ let*

$$\begin{aligned}\Psi(n,\lambda,y,\xi_{n-1},\vec{v},\vec{v}_n,\vec{h}_n,\vec{s}_n) &= \left[\frac{\lambda}{2\pi(t-t_{n-1})}\right]^{\frac{r}{2}} \int_{\mathbb{R}^r} D(n,y,\xi_{n-1},\xi_n,\vec{v}_n,\vec{h}_n,\vec{s}_n) \\ &\quad \times \exp\left\{i\langle\xi_n,\vec{v}\rangle - \frac{\lambda}{2}\frac{\|\xi_n-\xi_{n-1}\|^2_{\mathbb{R}^r}}{t-t_{n-1}}\right\}d\xi_n\end{aligned}$$

where $\vec{v}_n$, $\vec{h}_n$, $\vec{s}_n$ and $D(n,y,\xi_{n-1},\xi_n,\vec{v}_n,\vec{h}_n,\vec{s}_n)$ are given as in Theorem 3.2 with $l = n$. Then we have

$$\begin{aligned}&\Psi(n,\lambda,y,\xi_{n-1},\vec{v},\vec{v}_n,\vec{h}_n,\vec{s}_n) \\ &= \exp\left\{i\langle\xi_{n-1},\vec{v}\rangle + i\sum_{u=0}^{r_n}\sum_{v=1}^{j_u+1}\langle y(s_{n,u,v})+\xi_{n-1},\vec{v}_{n,u,v}\rangle\right. \\ &\quad \left. -\frac{t-t_{n-1}}{2\lambda}\left\|\vec{v}+\sum_{u=0}^{r_n}\sum_{v=1}^{j_u+1}\frac{s_{n,u,v}-t_{n-1}}{t-t_{n-1}}\vec{v}_{n,u,v}\right\|^2_{\mathbb{R}^r}\right\}.\end{aligned} \tag{4.1}$$

Proof. For $\lambda > 0$, we have by the change of variable theorem and (3.4)

$$\begin{aligned}&\Psi(n,\lambda,y,\xi_{n-1},\vec{v},\vec{v}_n,\vec{h}_n,\vec{s}_n) \\ &= \left[\frac{\lambda}{2\pi(t-t_{n-1})}\right]^{\frac{r}{2}}\int_{\mathbb{R}^r}\exp\left\{i\sum_{u=0}^{r_n}\sum_{v=1}^{j_u+1}\left\langle y(s_{n,u,v})+\xi_{n-1}+\frac{s_{n,u,v}-t_{n-1}}{t-t_{n-1}}(\xi_n\right.\right. \\ &\quad \left.\left. -\xi_{n-1}),\vec{v}_{n,u,v}\right\rangle + i\langle\xi_{n-1},\vec{v}\rangle + i\langle\xi_n-\xi_{n-1},\vec{v}\rangle - \frac{\lambda}{2}\frac{\|\xi_n-\xi_{n-1}\|^2_{\mathbb{R}^r}}{t-t_{n-1}}\right\}d\xi_n \\ &= \exp\left\{i\langle\xi_{n-1},\vec{v}\rangle + i\sum_{u=0}^{r_n}\sum_{v=1}^{j_u+1}\langle y(s_{n,u,v})+\xi_{n-1},\vec{v}_{n,u,v}\rangle - \frac{t-t_{n-1}}{2\lambda}\left\|\vec{v}+\sum_{u=0}^{r_n}\right.\right. \\ &\quad \left.\left.\sum_{v=1}^{j_u+1}\frac{s_{n,u,v}-t_{n-1}}{t-t_{n-1}}\vec{v}_{n,u,v}\right\|^2_{\mathbb{R}^r}\right\}\end{aligned}$$

which completes the proof. □

Theorem 4.2. *Let the assumptions and notations be given as in Theorem 3.2. Suppose that Y_t is given by (2.4). Then for $\lambda \in \mathbb{C}_+$, $y \in C^r[0,t]$ and $\vec{\xi}_{n-1} = (\xi_0,\xi_1,\cdots,\xi_{n-1}) \in \mathbb{R}^{nr}$, $T_\lambda[F_k|Y_t](y,\vec{\xi}_{n-1})$ exists and it is given by*

$$T_\lambda[F_k|Y_t](y,\vec{\xi}_{n-1}) = k!\sum_{q_1+\cdots+q_n=k}\left[\prod_{l=1}^{n-1}A(l,\lambda,y,\xi_{l-1},\xi_l,q_l)\right]B(n,\lambda,y,\vec{0},\xi_{n-1},q_n)$$

where for $\vec{v} \in \mathbb{R}^r$, $B(n, \lambda, y, \vec{v}, \xi_{n-1}, q_n)$ is given by the expression of $A(n, \lambda, y, \xi_{n-1}, \xi_n, q_n)$ replacing $D(n, y, \xi_{n-1}, \xi_n, \vec{v}_n, \vec{h}_n, \vec{s}_n)$ by $\Psi(n, \lambda, y, \xi_{n-1}, \vec{v}, \vec{v}_n, \vec{h}_n, \vec{s}_n)$ which is given by (4.1).

Proof. For $\vec{\xi}_{n-1} = (\xi_0, \xi_1, \cdots, \xi_{n-1}) \in \mathbb{R}^{nr}$, let $(\vec{\xi}_{n-1}, \xi_n) = (\xi_0, \xi_1, \cdots, \xi_{n-1}, \xi_n) \in \mathbb{R}^{(n+1)r}$, where $\xi_n \in \mathbb{R}^r$. Then for $\lambda > 0$ and $y \in C^r[0, t]$ we have by Theorem 3.2

$$\begin{aligned}
K_{F_k}^{\lambda}(y, \vec{\xi}) &= \left[\frac{\lambda}{2\pi(t - t_{n-1})}\right]^{\frac{r}{2}} \int_{\mathbb{R}^r} I_{F_k}^{\lambda}(y, (\vec{\xi}_{n-1}, \xi_n)) \exp\left\{-\frac{\lambda \|\xi_n - \xi_{n-1}\|_{\mathbb{R}^r}^2}{2(t - t_{n-1})}\right\} d\xi_n \\
&= k! \left[\frac{\lambda}{2\pi(t - t_{n-1})}\right]^{\frac{r}{2}} \int_{\mathbb{R}^r} \left[\sum_{q_1 + \cdots + q_n = k} \prod_{l=1}^{n} A(l, \lambda, y, \xi_{l-1}, \xi_l, q_l)\right] \\
&\quad \times \exp\left\{-\frac{\lambda \|\xi_n - \xi_{n-1}\|_{\mathbb{R}^r}^2}{2(t - t_{n-1})}\right\} d\xi_n \\
&= k! \sum_{q_1 + \cdots + q_n = k} \left[\prod_{l=1}^{n-1} A(l, \lambda, y, \xi_{l-1}, \xi_l, q_l)\right] \left[\frac{\lambda}{2\pi(t - t_{n-1})}\right]^{\frac{r}{2}} \\
&\quad \times \int_{\mathbb{R}^r} A(n, \lambda, y, \xi_{n-1}, \xi_n, q_n) \exp\left\{-\frac{\lambda \|\xi_n - \xi_{n-1}\|_{\mathbb{R}^r}^2}{2(t - t_{n-1})}\right\} d\xi_n \\
&= k! \sum_{q_1 + \cdots + q_n = k} \left[\prod_{l=1}^{n-1} A(l, \lambda, y, \xi_{l-1}, \xi_l, q_l)\right] B(n, \lambda, y, \vec{0}, \xi_{n-1}, q_n)
\end{aligned}$$

where the last equality follows from Lemma 4.1. Now, by the Morera's theorem, we have the theorem. □

For $\lambda \in \mathbb{C}_+$ it is easy to prove

$$(4.2) \qquad |\Psi(n, \lambda, y, \xi_{n-1}, \vec{v}, \vec{v}_n, \vec{h}_n, \vec{s}_n)| \leq 1$$

where Ψ is given by (4.1). Applying the same method used in the proof of Theorems 3.5, 3.6 with (4.2), we can prove the following theorem.

Theorem 4.3. *Let the assumptions and notations be given as in Theorems 3.2, 3.5, 3.6 and 4.2. Furthermore, let q be a nonzero real number and $1 \leq p \leq \infty$. Then for $y \in C^r[0, t]$ and $\vec{\xi}_{n-1} = (\xi_0, \xi_1, \cdots, \xi_{n-1}) \in \mathbb{R}^{nr}$, $T_q^{(p)}[F_k|Y_t](y, \vec{\xi}_{n-1})$ and $T_q^{(p)}[F|Y_t](y, \vec{\xi}_{n-1})$ exist and they are given by*

$$\begin{aligned}
T_q^{(p)}[F_k|Y_t](y, \vec{\xi}_{n-1}) &= k! \sum_{q_1 + \cdots + q_n = k} \left[\prod_{l=1}^{n-1} A(l, -iq, y, \xi_{l-1}, \xi_l, q_l)\right] \\
&\quad \times B(n, -iq, y, \vec{0}, \xi_{n-1}, q_n)
\end{aligned}$$

and

$$T_q^{(p)}[F|Y_t](y, \vec{\xi}_{n-1}) = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} T_q^{(p)}[F_k|Y_t](y, \vec{\xi}_{n-1}).$$

Theorem 4.4. *Under the assumptions and notations given as in Theorems 3.2, 3.7, 4.2 and 4.3, we have for* $y \in C^r[0,t]$ *and* $\vec{\xi}_{n-1} = (\xi_0, \xi_1, \cdots, \xi_{n-1}) \in \mathbb{R}^{nr}$

$$\begin{aligned} T_q^{(p)}[G_k|Y_t](y, \vec{\xi}_{n-1}) &= k! \sum_{q_1+\cdots+q_n=k} \bigg[\prod_{l=1}^{n-1} A(l, -iq, y, \xi_{l-1}, \xi_l, q_l)\bigg] \\ &\quad \times \int_{\mathbb{R}^r} B(n, -iq, y, \vec{v}, \xi_{n-1}, q_n) \exp\{i\langle y(t), \vec{v}\rangle\} d\nu(\vec{v}) \end{aligned}$$

and

$$\begin{aligned} T_q^{(p)}[G|Y_t](y, \vec{\xi}_{n-1}) &= \int_{\mathbb{R}^r} \exp\bigg\{ i\langle y(t)+\xi_{n-1}, \vec{v}\rangle + \frac{t-t_{n-1}}{2qi}\|\vec{v}\|_{\mathbb{R}^r}^2 \bigg\} d\nu(\vec{v}) \\ &\quad + \sum_{k=1}^{\infty} \frac{1}{k!} T_q^{(p)}[G_k|Y_t](y, \vec{\xi}_{n-1}). \end{aligned}$$

Proof. For $\vec{\xi}_{n-1} = (\xi_0, \xi_1, \cdots, \xi_{n-1}) \in \mathbb{R}^{nr}$, let $(\vec{\xi}_{n-1}, \xi_n) = (\xi_0, \xi_1, \cdots, \xi_{n-1}, \xi_n) \in \mathbb{R}^{(n+1)r}$, where $\xi_n \in \mathbb{R}^r$. Then for $\lambda > 0$ we have by Theorem 3.7

$$\begin{aligned} & K_{G_k}^{\lambda}(y, \vec{\xi}_{n-1}) \\ = & \bigg[\frac{\lambda}{2\pi(t-t_{n-1})}\bigg]^{\frac{r}{2}} \int_{\mathbb{R}^r} I_{G_k}^{\lambda}(y, (\vec{\xi}_{n-1}, \xi_n)) \exp\bigg\{-\frac{\lambda\|\xi_n - \xi_{n-1}\|_{\mathbb{R}^r}^2}{2(t-t_{n-1})}\bigg\} d\xi_n \\ = & k! \bigg[\frac{\lambda}{2\pi(t-t_{n-1})}\bigg]^{\frac{r}{2}} \int_{\mathbb{R}^r} \psi(y(t)+\xi_n) \bigg[\sum_{q_1+\cdots+q_n=k} \prod_{l=1}^{n} A(l, \lambda, y, \xi_{l-1}, \xi_l, q_l)\bigg] \\ & \times \exp\bigg\{-\frac{\lambda\|\xi_n - \xi_{n-1}\|_{\mathbb{R}^r}^2}{2(t-t_{n-1})}\bigg\} d\xi_n \\ = & k! \sum_{q_1+\cdots+q_n=k} \bigg[\prod_{l=1}^{n-1} A(l, \lambda, y, \xi_{l-1}, \xi_l, q_l)\bigg] \bigg[\frac{\lambda}{2\pi(t-t_{n-1})}\bigg]^{\frac{r}{2}} \int_{\mathbb{R}^r} \exp\{i\langle y(t), \vec{v}\rangle\} \\ & \times \int_{\mathbb{R}^r} A(n, \lambda, y, \xi_{n-1}, \xi_n, q_n) \exp\bigg\{ i\langle \xi_n, \vec{v}\rangle - \frac{\lambda\|\xi_n - \xi_{n-1}\|_{\mathbb{R}^r}^2}{2(t-t_{n-1})}\bigg\} d\xi_n d\nu(\vec{v}). \end{aligned}$$

Now by Theorem 3.2 and Lemma 4.1 we have

$$\begin{aligned} & \bigg[\frac{\lambda}{2\pi(t-t_{n-1})}\bigg]^{\frac{r}{2}} \int_{\mathbb{R}^r} A(n, \lambda, y, \xi_{n-1}, \xi_n, q_n) \\ & \times \exp\bigg\{ i\langle \xi_n, \vec{v}\rangle - \frac{\lambda\|\xi_n - \xi_{n-1}\|_{\mathbb{R}^r}^2}{2(t-t_{n-1})}\bigg\} d\xi_n = B(n, \lambda, y, \vec{v}, \xi_{n-1}, q_n) \end{aligned}$$

where $B(n, \lambda, y, \vec{v}, \xi_{n-1}, q_n)$ is given as in Theorem 4.2. We also have by (3.4)

$$\begin{aligned}
& \left[\frac{\lambda}{2\pi(t-t_{n-1})}\right]^{\frac{r}{2}} \int_{\mathbb{R}^r} \psi(y(t)+\xi_n) \exp\left\{-\frac{\lambda\|\xi_n-\xi_{n-1}\|^2_{\mathbb{R}^r}}{2(t-t_{n-1})}\right\} d\xi_n \\
= & \left[\frac{\lambda}{2\pi(t-t_{n-1})}\right]^{\frac{r}{2}} \int_{\mathbb{R}^r} \exp\{i\langle y(t)+\xi_{n-1}, \vec{v}\rangle\} \int_{\mathbb{R}^r} \exp\left\{i\langle \xi_n - \xi_{n-1}, \vec{v}\rangle\right. \\
& \left. -\frac{\lambda\|\xi_n-\xi_{n-1}\|^2_{\mathbb{R}^r}}{2(t-t_{n-1})}\right\} d\xi_n d\nu(\vec{v}) \\
= & \int_{\mathbb{R}^r} \exp\left\{i\langle y(t)+\xi_{n-1}, \vec{v}\rangle - \frac{t-t_{n-1}}{2\lambda}\|\vec{v}\|^2_{\mathbb{R}^r}\right\} d\nu(\vec{v}).
\end{aligned}$$

By the Morera's theorem and the dominated convergence theorem, we have the theorem. □

Remark 4.5.

- If F_k, G_k, F and G are defined on r-dimensional Wiener space, then we can obtain the same results in Theorems 4.2, 4,3 and 4.4 with $\xi_0 = \vec{0} \in \mathbb{R}^r$ in the expression of $\vec{\xi}_{n-1}$.
- If $\eta = \mu$ or $\eta = \sum_{l=1}^{n} \sum_{j=1}^{r_l} w_{l,j}\delta_{p_{l,j}}$, we can obtain more simple expressions in Theorems 4.2, 4.3 and 4.4.
- If some of the $p_{l,j}$s are in the set $\{t_0, t_1, \cdots, t_n\}$, we can obtain all the results in the present section with minor modifications.
- If $\eta = \mu + \sum_{l=1}^{n} \sum_{j=1}^{r_l} w_{l,j}\delta_{p_{l,j}}$ and some of the r_ls are ∞, then, using (3.13), we can show that $T_q^{(p)}[G|Y_t](y, \vec{\xi}_{n-1})$ exists in Theorem 4.4.

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