KYUNGPOOK Math. J. 52(2012), 397-404
http://dx.doi.org/10.5666/KMJ.2012.52.4.397

On Two Dimensional q-Hölder's Inequality

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Abstract. In this article, the reverse q-Hölder type inequality and two dimensional q-Hölder's inequality are established. We also obtain some q-integral inequalities by using q-Hölder's inequality which give q-Hardy's inequalities as spacial cases.

1. Introduction

Throughout this paper, we will fit $q \in (0,1)$. We denote by I one of the following sets: (1) $\mathbb{R}_{q,+} = \{q^n : n \in \mathbb{Z}\}$; (2) $[0,b]_q = \{bq^n : n \in \mathbb{Z}\}$, $b > 0$; (3) $[a,b]_q = \{bq^k : 0 \leq k \leq n\}$, $b > 0$, $a = bq^n$, $n \in \mathbb{Z}$. Due to restrictions on the number of pages, the basic definitions and theorems of q-integral were omitted, and the reader was referred to [3, 2, 4]. And we note $\int_I f(x)d_qx$ the q-integral of f on the correspondent I.

Let p and p' be two positive reals satisfying $p > 1$ and $1/p + 1/p' = 1$, and f and g be two functions defined on I. Then

$$(1.1)\qquad \left|\int_I f(x)g(x)d_qx\right| \leq \left(\int_I |f(x)|^p d_qx\right)^{\frac{1}{p}} \left(\int_I |g(x)|^{p'} d_qx\right)^{\frac{1}{p'}}.$$

The above inequality was given by Fitouhi and Brahim [3], but the condition $p > 0$ was not added. According to the definition of q-integral, we have $\left|\int_I f(x)g(x)d_qx\right| \leq \int_I |f(x)g(x)|d_qx$. So, the above conditions hold, (1.1) is restated as follows:

$$(1.2)\qquad \int_I |f(x)g(x)|d_qx \leq \left(\int_I |f(x)|^p d_qx\right)^{\frac{1}{p}} \left(\int_I |g(x)|^{p'} d_qx\right)^{\frac{1}{p'}}.$$

Tuna and Kutukcu [5] and Ammi and Torres [1] gave two dimensional Δ-Hölder's inequalities and two dimensional $\diamondsuit_\alpha$-Hölder's inequalities, respectively. Motivated by [5] and [1], we will study the reverse q-Hölder type inequality and two dimensional q-Hölder's inequality. We also obtain some q-integral inequalities by using q-Hölder's inequality which give q-Hardy's inequalities as spacial cases.

Received October 1, 2010; accepted October 9, 2012.
2010 Mathematics Subject Classification: 26A39, 26D15.
Key words and phrases: q-Hölder's inequality, q-Hardy's inequalities, q-integral inequalities.

2. Main results

Theorem 2.1. *For two positive functions f and g satisfying $0 < m \le f^p/g^{p'} \le M < \infty$ on I. If $1/p + 1/p' = 1$ with $p > 1$, we have*

$$\left(\int_I f^p(x)d_qx\right)^{\frac{1}{p}}\left(\int_I g^{p'}(x)d_qx\right)^{\frac{1}{p'}} \le \left(\frac{M}{m}\right)^{\frac{1}{pp'}}\int_I f(x)g(x)d_qx. \tag{2.1}$$

Proof. Since $f^p/g^{p'} \le M$, then $f^{p/p'} \le M^{1/p'}g$. Multiplying by $f > 0$, it follows that

$$f^p = f^{1+\frac{p}{p'}} \le M^{\frac{1}{p'}}fg$$

and so,

$$\left(\int_I f^p(x)d_qx\right)^{\frac{1}{p}} \le M^{\frac{1}{pp'}}\left(\int_I f(x)g(x)d_qx\right)^{\frac{1}{p}}. \tag{2.2}$$

On the other hand, since $m \le f^p/g^{p'}$, then $f \ge m^{1/p}g^{p'/p}$, hence

$$\int_I f(x)g(x)d_qx \ge \int_I m^{\frac{1}{p}}g^{1+\frac{p'}{p}}(x)d_qx = m^{\frac{1}{p}}\int_I g^{p'}(x)d_qx.$$

We obtain that

$$\left(\int_I f(x)g(x)d_qx\right)^{\frac{1}{p'}} \ge m^{\frac{1}{pp'}}\left(\int_I g^{p'}(x)d_qx\right)^{\frac{1}{p'}}. \tag{2.3}$$

Combining (2.2) and (2.3), we have the desired inequality (2.1). The proof is completed. □

Theorem 2.2. *Let $f(x,y)$ $g(x,y)$ and $h(x,y)$ be three functions defined on I^2. If $1/p + 1/p' = 1$ with $p > 1$, we have*

$$\begin{aligned}&\int_I\int_I |h(x,y)f(x,y)g(x,y)|d_qxd_qy\\ &\le \left(\int_I\int_I |h(x,y)||f(x,y)|^p d_qxd_qy\right)^{\frac{1}{p}} \times \left(\int_I\int_I |h(x,y)||g(x,y)|^{p'} d_qxd_qy\right)^{\frac{1}{p'}}.\end{aligned} \tag{2.4}$$

Proof. Inequality (2.4) is trivially true in the case when f or g or h is identically zero. Suppose that

$$\left(\int_I\int_I |h(x,y)||f(x,y)|^p d_qxd_qy\right)\left(\int_I\int_I |h(x,y)||g(x,y)|^{p'} d_qxd_qy\right) \ne 0.$$

Apply the following Young's inequality

$$x^{\frac{1}{p}}y^{\frac{1}{p'}} \le \frac{1}{p}x + \frac{1}{p'}y, \quad x,y \ge 0 \quad \text{and} \quad \frac{1}{p} + \frac{1}{p'} = 1, \quad p > 1,$$

to

$$\alpha(x,y) = \frac{|h(x,y)||f(x,y)|^p}{\int_I\int_I |h(x,y)||f(x,y)|^p d_qx d_qy}$$

and

$$\beta(x,y) = \frac{|h(x,y)||g(x,y)|^{p'}}{\int_I\int_I |h(x,y)||g(x,y)|^{p'} d_qx d_qy}$$

obtaining

$$\frac{|h(x,y)|^{\frac{1}{p}}|f(x,y)||h(x,y)|^{\frac{1}{p'}}|g(x,y)|}{\left(\int_I\int_I |h(x,y)||f(x,y)|^p d_qx d_qy\right)^{\frac{1}{p}}\left(\int_I\int_I |h(x,y)||g(x,y)|^{p'} d_qx d_qy\right)^{\frac{1}{p'}}}$$
$$\leq \frac{1}{p}\frac{|h(x,y)||f(x,y)|^p}{\int_I\int_I |h(x,y)||f(x,y)|^p d_qx d_qy} + \frac{1}{p'}\frac{|h(x,y)||g(x,y)|^{p'}}{\int_I\int_I |h(x,y)||g(x,y)|^{p'} d_qx d_qy}.$$

Integrating both sides of the obtained inequality, we have

$$\begin{aligned}
&\int_I\int_I \left\{ \frac{|h(x,y)|^{\frac{1}{p}}|f(x,y)|}{\left(\int_I\int_I |h(x,y)||f(x,y)|^p d_qx d_qy\right)^{\frac{1}{p}}} \right. \\
&\qquad \left. \times \frac{|h(x,y)|^{\frac{1}{p'}}|g(x,y)|}{\left(\int_I\int_I |h(x,y)||g(x,y)|^{p'} d_qx d_qy\right)^{\frac{1}{p'}}} \right\} d_qx d_qy \\
\leq\ & \int_I\int_I \frac{1}{p}\frac{|h(x,y)||f(x,y)|^p}{\int_I\int_I |h(x,y)||f(x,y)|^p d_qx d_qy} d_qx d_qy \\
&+ \int_I\int_I \frac{1}{p'}\frac{|h(x,y)||g(x,y)|^{p'}}{\int_I\int_I |h(x,y)||g(x,y)|^{p'} d_qx d_qy} d_qx d_qy \\
=\ & \frac{1}{p} + \frac{1}{p'} = 1.
\end{aligned}$$

This directly gives the desired inequality (2.4). The proof is completed. □

Corollary 2.1. *Let $f(x,y)$ $g(x,y)$ and $h(x,y)$ be three functions defined on I^2. Then*

$$\begin{aligned}
\left(\int_I\int_I |h(x,y)f(x,y)g(x,y)| d_qx d_qy\right)^2 \leq\ & \left(\int_I\int_I |h(x,y)||f(x,y)|^2 d_qx d_qy\right) \\
& \times \left(\int_I\int_I |h(x,y)||g(x,y)|^2 d_qx d_qy\right).
\end{aligned} \tag{2.5}$$

Proof. The Cauchy-Schwartz inequality (2.5) is the particular case $p = p' = 2$ of (2.4). □

Theorem 2.3. *Let $K(x,y)$ $f(x)$ and $g(y)$, $\varphi(x)$ and $\psi(y)$ be nonnegative functions. Let*

$$F(x) = \int_I K(x,y)\psi^{-p}(y)d_qy$$

and

$$G(y) = \int_I K(x,y)\varphi^{-p'}(x)d_qx,$$

where $1/p + 1/p' = 1$ with $p > 1$. Then the two inequalities

$$\int_I \int_I K(x,y)f(x)g(y)d_qxd_qy \leq \left(\int_I \varphi^p(x)F(x)f^p(x)d_qx\right)^{\frac{1}{p}} \times \left(\int_I \psi^{p'}(y)G(y)g^{p'}(y)d_qy\right)^{\frac{1}{p'}} \tag{2.6}$$

and

$$\int_I G^{1-p}(y)\psi^{-p}(y)\left(\int_I K(x,y)f(x)d_qx\right)^p d_qy \leq \int_I \varphi^p(x)F(x)f^p(x)d_qx \tag{2.7}$$

hold and are equivalent. Equation (2.7) is the q-Hardy's inequality.

Proof. First, we prove that (2.6) hold. Write

$$\int_I \int_I K(x,y)f(x)g(y)d_qxd_qy = \int_I \int_I K(x,y)f(x)\frac{\varphi(x)}{\psi(y)}g(y)\frac{\psi(y)}{\varphi(x)}d_qxd_qy.$$

Applying q-Hölder's inequality (1.2), we have

$$\int_I \int_I K(x,y)f(x)g(y)d_qxd_qy \leq \left(\int_I \varphi^p(x)F(x)f^p(x)d_qx\right)^{\frac{1}{p}} \times \left(\int_I \psi^{p'}(y)G(y)g^{p'}(y)d_qy\right)^{\frac{1}{p'}}.$$

Now we show that (2.6) is equivalent to (2.7). Suppose that inequality (2.6) is verified. Set

$$g(y) = G^{1-p}(y)\psi^{-p}(y)\left(\int_I K(x,y)f(x)d_qx\right)^{p-1}.$$

Using (2.6) and taking into account that $1/p + 1/p' = 1$ with $p > 1$, we obtain

$$\begin{aligned}
&\int_I G^{1-p}(y)\psi^{-p}(y)\left(\int_I K(x,y)f(x)d_qx\right)^p d_qy = \int_I \int_I K(x,y)f(x)g(y)d_qxd_qy \\
\leq\ & \left(\int_I \varphi^p(x)F(x)f^p(x)d_qx\right)^{\frac{1}{p}} \left(\int_I \psi^{p'}(y)G(y)g^{p'}(y)d_qy\right)^{\frac{1}{p'}} \\
=\ & \left(\int_I \varphi^p(x)F(x)f^p(x)d_qx\right)^{\frac{1}{p}} \left(\int_I G^{1-p}(y)\psi^{-p}(y)\left(\int_I K(x,y)f(x)d_qx\right)^p d_qy\right)^{\frac{1}{p'}}.
\end{aligned}$$

Inequality (2.7) is obtained by dividing both sides of the previous inequality by

$$\left(\int_I G^{1-p}(y)\psi^{-p}(y)\left(\int_I K(x,y)f(x)d_qx\right)^p d_qy\right)^{\frac{1}{p'}}.$$

Reciprocally, suppose that inequality (2.7) is valid. From q-Hardy's inequality we can write that

$$\begin{aligned}
&\int_I\int_I K(x,y)f(x)g(y)d_qxd_qy\\
=\ &\int_I\left(\psi^{-1}(y)G^{-1/p'}(y)\int_I K(x,y)f(x)d_qx\right)\psi(y)G^{1/p'}(y)g(y)d_qy\\
\leq\ &\left(\int_I G^{1-p}(y)\psi^{-p}(y)\left(\int_I K(x,y)f(x)d_qx\right)^p d_qy\right)^{\frac{1}{p}}\left(\int_I \psi^{p}(y)G(y)g^{p'}(y)d_qy\right)^{\frac{1}{p'}}.
\end{aligned}$$

Using (2.7), we get that

$$\begin{aligned}
\int_I\int_I K(x,y)f(x)g(y)d_qxd_qy \ \leq\ &\left(\int_I \varphi^p(x)F(x)f^p(x)d_qx\right)^{\frac{1}{p}}\\
&\times\left(\int_I \psi^{p'}(y)G(y)g^{p'}(y)d_qy\right)^{\frac{1}{p'}},
\end{aligned}$$

which completes the proof. □

As corollaries of Theorem 2.3 we have the following results. Without loss of generality, only take $I=[a,b]_q$ for example.

Corollary 2.2. *Let $h(y)$ $f(x)$ and $g(y)$, $\varphi(x)$ and $\psi(y)$ be nonnegative functions, and $1/p+1/p'=1$ with $p>1$. Setting $H(y)=h(y)\psi^{-p}(y)$, then the two inequalities*

$$\begin{aligned}
\int_a^b\int_a^y h(y)f(x)g(y)d_qxd_qy \ \leq\ &\left(\int_a^b \varphi^p(x)f^p(x)\left(\int_x^b H(y)d_qy\right)d_qx\right)^{\frac{1}{p}}\\
&\times\left(\int_a^b \psi^{p'}(y)g^{p'}(y)h(y)\left(\int_a^y \varphi^{-p'}(y)d_qx\right)d_qy\right)^{\frac{1}{p'}}
\end{aligned}$$

and

$$\begin{aligned}
&\int_a^b H(y)\left(\int_a^y \varphi^{-p'}(y)d_qx\right)^{1-p}\left(\int_a^y f(x)d_qx\right)^p d_qy\\
&\qquad\leq\left(\int_a^b \varphi^p(x)f^p(x)\left(\int_x^b H(y)d_qy\right)d_qx\right)^{\frac{1}{p}}
\end{aligned}$$

hold and are equivalent.

Proof. Use Theorem 2.3 with $K(x,y)=\begin{cases} h(y), & \text{if} \quad x\le y,\\ 0, & \text{if} \quad x>y.\end{cases}$ □

Corollary 2.3. *Let $h(y)$ $f(x)$ and $g(y)$, $\varphi(x)$ and $\psi(y)$ be nonnegative functions, and $1/p+1/p'=1$ with $p>1$. Setting $H(y)=h(y)\psi^{-p}(y)$, then the two inequalities*

$$\begin{aligned}\int_a^b\int_y^b h(y)f(x)g(y)d_qxd_qy &\le \left(\int_a^b \varphi^p(x)f^p(x)\left(\int_a^x H(y)d_qy\right)d_qx\right)^{\frac{1}{p}}\\ &\quad\times\left(\int_a^b \psi^{p'}(y)g^{p'}(y)h(y)\left(\int_y^b \varphi^{-p'}(y)d_qx\right)d_qy\right)^{\frac{1}{p'}}\end{aligned}$$

and

$$\begin{aligned}&\int_a^b H(y)\left(\int_y^b \varphi^{-p'}(y)d_qx\right)^{1-p}\left(\int_y^b f(x)d_qx\right)^p d_qy\\ &\qquad\le\left(\int_a^b \varphi^p(x)f^p(x)\left(\int_a^x H(y)d_qy\right)d_qx\right)^{\frac{1}{p}}\end{aligned}$$

hold and are equivalent.

Proof. Use Theorem 2.3 with $K(x,y)=\begin{cases} 0, & \text{if} \quad x\le y,\\ h(y), & \text{if} \quad x>y.\end{cases}$ □

It is interesting to consider the case when functions $F(x)$ and $G(y)$ of Theorem 2.3 are bounded. We then obtain the following:

Theorem 2.4. *Let $K(x,y)$ $f(x)$ and $g(y)$, $\varphi(x)$ and $\psi(y)$ be nonnegative functions. Let*

$$F(x)=\int_I K(x,y)\psi^{-p}(y)d_qy\le F_1(x)$$

and

$$G(y)=\int_I K(x,y)\varphi^{-p'}(y)d_qx\le G_1(y),$$

where $1/p+1/p'=1$ with $p>1$. Then the two inequalities

$$\begin{aligned}\int_I\int_I K(x,y)f(x)g(y)d_qxd_qy &\le \left(\int_I \varphi^p(x)F_1(x)f^p(x)d_qx\right)^{\frac{1}{p}}\\ &\quad\left(\int_I \psi^{p'}(y)G_1(y)g^{p'}(y)d_qy\right)^{\frac{1}{p'}}\end{aligned}$$

and

$$\int_I G_1^{1-p}(y)\psi^{-p}(y)\left(\int_I K(x,y)f(x)d_qx\right)^p d_qy\le\int_I \varphi^p(x)F_1(x)f^p(x)d_qx$$

hold and are equivalent.

Theorem 2.5. *Let $F, G, L(f,g), M(f)$ and $N(g)$ be positive functions, and $1/p + 1/p' = 1$ with $p > 1$ such that*

$$0 < \int_I M^p(f(x))F^p(x)d_qx < \infty, \qquad 0 < \int_I N^{p'}(g(x))G^{p'}(x)d_qx < \infty.$$

then the two inequalities

$$\begin{aligned}(2.8)\qquad &\int_I\int_I \frac{F(x)G(y)}{L(f(x),g(y))}d_qxd_qy \\ &\leq C\left(\int_I M^p(f(x))F^p(x)d_qx\right)^{\frac{1}{p}} \times \left(\int_I N^{p'}(g(y))G^{p'}(y)d_qy\right)^{\frac{1}{p'}}\end{aligned}$$

and

$$(2.9)\qquad \int_I N^{-p}(g(y))\left(\int_I \frac{F(x)}{L(f(x),g(y))}d_qx\right)^p d_qy \leq C^p\int_I M^p(f(x))F^p(x)d_qx,$$

where C is a constant, are equivalent.

Proof. Suppose that the inequality (2.9) is valid. Then we have

$$\begin{aligned} &\int_I\int_I \frac{F(x)G(y)}{L(f(x),g(y))}d_qxd_qy \\ = \;& \int_I N(g(y))G(y)\left(N^{-1}(g(y))\int_I \frac{F(x)}{L(f(x),g(y))}d_qx\right)d_qy \\ \leq \;& \left(\int_I N^{p'}(g(y))G^{p'}(y)d_qy\right)^{\frac{1}{p'}}\left(\int_I N^{-p}(g(y))\left(\int_I \frac{F(x)}{L(f(x),g(y))}d_qx\right)^p d_qy\right)^{\frac{1}{p}} \\ \leq \;& C\left(\int_I M^p(f(x))F^p(x)d_qx\right)^{\frac{1}{p}}\left(\int_I N^{p'}(g(y))G^{p'}(y)d_qy\right)^{\frac{1}{p'}},\end{aligned}$$

which implies inequality 2.8. Let us now suppose that the inequality (2.8) is valid. By setting $G(y) = N^{-p}(g(y))\left(\int_I \frac{F(x)}{L(f(x),g(y))}d_qx\right)^{\frac{p}{p'}}$ and applying 2.8, then we obtain

$$
\begin{aligned}
&\int_I N^{-p}(g(y))\left(\int_I \frac{F(x)}{L(f(x),g(y))}d_qx\right)^p d_qy \\
= &\int_I\left(\int_I \frac{F(x)}{L(f(x),g(y))}d_qx\right)N^{-p}(g(y))\left(\int_I \frac{F(x)}{L(f(x),g(y))}d_qx\right)^{\frac{p}{p'}} d_qy \\
\le &\ C\left(\int_I M^p(f(x))F^p(x)d_qx\right)^{\frac{1}{p}} \\
&\times\left(\int_I N^{p'}(g(y))N^{-pp'}(g(y))\left(\int_I \frac{F(x)}{L(f(x),g(y))}d_qx\right)^p d_qy\right)^{\frac{1}{p'}} \\
= &\ C\left(\int_I M^p(f(x))F^p(x)d_qx\right)^{\frac{1}{p}}\left(\int_I N^{-p}(g(y))\left(\int_I \frac{F(x)}{L(f(x),g(y))}d_qx\right)^p d_qy\right)^{\frac{1}{p'}},
\end{aligned}
$$

which implies that

$$
\int_I N^{-p}(g(y))\left(\int_I \frac{F(x)}{L(f(x),g(y))}d_qx\right)^p d_qy \le C^p\int_I M^p(f(x))F^p(x)d_qx.
$$

The proof is complete. □

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