Numerical Inversion Technique for the One and Two-Dimensional $\mathcal{L}_2$-Transform Using the Fourier Series and Its Application to Fractional Partial Differential Equations

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Abstract. In this paper, we use a computational algorithm for the inversion of the one and two-dimensional $\mathcal{L}_2$-transform based on the Bromwich's integral and the Fourier series. The new inversion formula can evaluate the inverse of the $\mathcal{L}_2$-transform with considerable accuracy over a wide range of values of the independent variable and can be devised for the functions which are not Laplace transformable and have damping motion in small interval near origin.

1. Introduction

The Laplace-type integral transform called the $\mathcal{L}_2$-transform was introduced by Yurekli and Sadek [16] and is denoted as follows

$$(1.1) \qquad \mathcal{L}_2\{f(t); s\} = \int_0^\infty t e^{-s^2 t^2} f(t) dt,$$

where $f(t)$ is piecewise continuous and of the exponential order α (i.e. $\mid f(t) \mid \leq M e^{\alpha t^2}$ for real number α and positive constant M) and s is complex parameter. Authors [1-4] generalized definition (1-1) for the two-dimensional $\mathcal{L}_2$-transform of the function $f(t_1, t_2)$ by the following relation

$$(1.2) \qquad \mathcal{L}_2^{(s_1, s_2)}\{f(t_1, t_2)\} = \int_0^\infty \int_0^\infty t_1 t_2 e^{-s_1^2 t_1^2 - s_2^2 t_2^2} f(t_1, t_2) dt_1 dt_2,$$

where in the above integral, s_1, s_2 are complex parameters determining a point (s_1, s_2) in the plane of two complex dimensions and $f(t_1, t_2)$ is a real valued function of two real variables and exponential orders α_1, α_2 (i.e. $\mid f(t_1, t_2) \mid \leq M e^{\alpha_1 t_1^2 + \alpha_2 t_2^2}$ for real numbers α_1, α_2 and positive constant M).

Also, the inversion integral formulas for the one and two-dimensional $\mathcal{L}_2$-transform in terms of the Bromwich's integral can be presented [1] as follows

$$(1.3) \qquad \mathcal{L}_2^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2F(\sqrt{s})e^{st^2} ds,$$

$$(1.4) \quad \mathcal{L}_2^{-1}\{F(s_1, s_2)\} = f(t_1, t_2)$$
$$= (\frac{1}{2\pi i})^2 \int_{c_1+i\infty}^{c_1+i\infty} \int_{c_2+i\infty}^{c_2+i\infty} 2^2 F(\sqrt{s_1}, \sqrt{s_2})e^{s_1 t_1^2 + s_2 t_2^2} ds_1 ds_2,$$

where $F(s)$ is analytic for $\Re s^2 > \alpha$ and c can be chosen to satisfy the condition $c > \alpha$, and in the two dimensional case $F(s_1, s_2)$ is analytic for $\Re s_1^2 > \alpha_1$, $\Re s_2^2 > \alpha_2$ and c_1, c_2 can be taken to hold the conditions $c_1 > \alpha_1, c_2 > \alpha_2$.

Some contributions and applications of the one the $\mathcal{L}_2$-transform are shown by Yurekli [17] and authors [1-4]. Yurekli showed the Parseval-Goldstein theorems involving the $\mathcal{L}_2$-transform and the Laplace transform may be used to yield identities involving several well-known integral transforms and infinite integrals of elementary and special functions. Moreover, authors [1-4] presented roles of the one and two-dimensional $\mathcal{L}_2$-transform in linear partial differential equations and system of differential and integral equations and utilized this transform as a useful and supplementary tool for analyzing the systems which the Laplace transform can not easily or never solve them.

Our aim in this article is based on the inversion of the functions which the Laplace transform of them does not exist and can be expressed as functions with damping motions near the zero point. We intend to extend the inversion methods of Dubner and Abate [8] and Crump [5] for the one-dimensional Laplace transform and Moorthy's method [11] for the two-dimensional Laplace transform in terms of the Fourier series which are noted for their accuracy with several different types of functions.

In this regard, in Sections 2 and 3, we introduce numerical inversion techniques for the one and two-dimensional $\mathcal{L}_2$-transform based on the Fourier series and analyze the errors of these computations. These inversions can be obtained at any value of the independent variable by means of a simple series summation and are expressed as the trapezoidal rule of quadrature for infinite-range integral.

In Section 4, the accuracy of these algorithms is shown for the functions which are not Laplace transformable. At the end, in Section 5, applicability of the $\mathcal{L}_2$-transform and its numerical inversion technique for solving fractional partial differential equations are discussed and the main conclusions are drawn.

2. Inversion method for the one-dimensional $\mathcal{L}_2$-transform and error analysis

For the one-dimensional $\mathcal{L}_2$-Transform and inversion integral formula in term of the Bromwich's integral

$$\mathcal{L}_2\{f(t); s\} = \int_0^\infty te^{-s^2t^2} f(t)dt, \quad \mathcal{L}_2^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2F(\sqrt{s})e^{st^2}ds,$$

we can consider the following cosine transform-like and sine transform-like pairs for the real-valued function $f(t)$ and exponential order α

$$(2.1) \qquad f(t) = \frac{4e^{ct^2}}{\pi} \int_0^\infty \Re(F(\sqrt{s})) \cos(wt^2)dw, \quad 2\Re(F(\sqrt{s}))$$

$$= \int_0^\infty te^{-ct^2} f(t) \cos(wt^2)dt,$$

$$(2.2) \qquad f(t) = \frac{-4e^{ct^2}}{\pi} \int_0^\infty \Im(F(\sqrt{s})) \sin(wt^2)dw, \quad 2\Im(F(\sqrt{s}))$$

$$= \int_0^\infty te^{-ct^2} f(t) \sin(wt^2)dt,$$

or, alternatively

$$(2.3) \qquad f(t) = \frac{2e^{ct^2}}{\pi} \int_0^\infty [\Re(F(\sqrt{s})) \cos(wt^2) - \Im(F(\sqrt{s})) \sin(wt^2)]dw,$$

where $s = c + iw$ and c can be any real number greater than α.

We want to obtain the procedure of computing algorithm for the function $f(t)$ in relation (2-3) in terms of the Fourier series. In this sense, we write the Fourier series for a function $g_0(t)$ that is periodic with period $2T$ and equals to $f(t)e^{-ct^2}$ on the interval $(0, 2T)$ and for $n = 0, 1, 2, \cdots, -\infty < t < \infty$, we define $g_n(t)$ by $g_n(t) = f(t)e^{-ct^2}, 2nT \leq t < 2(n+1)T$.

In order to converge the Fourier series to $g_0(t)$ at points of discontinuity, we impose the condition $f(t) = \frac{1}{2}(f(t^+) + f(t^-))$ for all t at which $f(t)$ is discontinuous. Therefore, the Fourier series representation of each $g_n(t)$ is given by

$$(2.4) \qquad g_n(t) = \frac{1}{2}A_{n,0} + \sum_{k=1}^\infty [A_{n,k} \cos(\frac{k\pi t^2}{T}) + B_{n,k} \sin(\frac{k\pi t^2}{T})],$$

where the Fourier coefficients are

$$A_{n,k} = \frac{1}{2T} \int_{2nT}^{2(n+1)T} te^{-ct^2} f(t) \cos(\frac{k\pi t^2}{T})dt,$$

$$B_{n,k} = \frac{1}{2T} \int_{2nT}^{2(n+1)T} te^{-ct^2} f(t) \sin(\frac{k\pi t^2}{T})dt.$$

By summing (2-4) with respect to n and using (2-1) and (2-2), we find that

$$(2.5) \quad \sum_{n=0}^{\infty} g_n(t)$$

$$= \frac{1}{T}\left(\frac{1}{2}F(c) + \sum_{k=1}^{\infty}[2\Re(F(\sqrt{c+\frac{k\pi i}{T}}))\cos(\frac{k\pi t^2}{T}) - 2\Im(F(\sqrt{c+\frac{k\pi i}{T}}))\sin(\frac{k\pi t^2}{T})]\right).$$

Since on the interval $(0, 2T)$, $g_0(t) = f(t)e^{-ct^2}$, therefore from (2-5) we obtain the approximation $\hat{f}(t)$ to the inverse transform in the form

$$(2.6) \quad \hat{f}(t) = \frac{e^{ct^2}}{T}\left(\frac{1}{2}F(c)\right.$$

$$\left. + \sum_{k=1}^{\infty}[2\Re(F(\sqrt{c+\frac{k\pi i}{T}}))\cos(\frac{k\pi t^2}{T}) - 2\Im(F(\sqrt{c+\frac{k\pi i}{T}}))\sin(\frac{k\pi t^2}{T})]\right),$$

where $f(t) = \hat{f}(t) - E_d$ and the error E_d is given by

$$(2.7) \quad\quad E_d = e^{ct^2}\sum_{n=1}^{\infty} g_n(t) = e^{ct^2}\sum_{n=1}^{\infty} e^{-c(2nT+t)^2}f(2nT+t).$$

The relation (2-6) is the desired approximation formula for $f(t)$ and the relation (2-7) is the error of the computation which there are two major sources of error in the approximation besides the round-off error. One of them is the discretization error given by E_d and the other is the truncation error E_t. At first, we intend to find a bound for E_d and then control the truncation error E_t by imposing a condition for choosing N in the series

$$(2.8) \quad\quad \hat{f}_N(t) = \frac{e^{ct^2}}{T}\left(\frac{1}{2}F(c)\right.$$

$$\left. + \sum_{k=1}^{N}[2\Re(F(\sqrt{c+\frac{k\pi i}{T}}))\cos(\frac{k\pi t^2}{T}) - 2\Im(F(\sqrt{c+\frac{k\pi i}{T}}))\sin(\frac{k\pi t^2}{T})]\right).$$

Since $\mid f(t) \mid \le Me^{\alpha t^2}$, it turns out the upper bound for the series (2-7) can be written as

$$(2.9) \quad\quad\quad\quad E_d \le Me^{ct^2}\sum_{n=1}^{\infty} e^{-(c-\alpha)(2nT+t)^2}.$$

It follows that by choosing c sufficiently larger than α for the convergent series (2-9) we can make the E as small as desired. In applying, we expect to choose the value

of parameter c, if the relative error $E_r = \frac{E_d}{Me^{\alpha t^2}}$ to be less than the known value ϵ. For this manner, for $2nT \leq t < 2(n+1)T$ we have the inequality

$$e^{-(c-\alpha)(6T)^2} \leq E_r = \sum_{n=1}^{\infty} e^{-(c-\alpha)(2nT+t)^2},$$

which enable us to choose the parameter c by the relation

$$(2.10) \qquad c = \alpha - \frac{\ln(\epsilon)}{36T^2}.$$

Also, in practical purposes the numerical value of $f(t)$ is desired over a range of t-values, of which the largest is t_{max}. Therefore, parameter T can be chosen by requirement $2T > t_{max}$ (we can find by experimenting when $0.5t_{max} \leq T \leq 0.8t_{max}$, this method gives better results) and parameter α is chosen to be a number slightly larger than $\max\{\Re s_p \mid s_p \text{ is a pole of } F(s)\}$. Finally, by taking the suitable parameters T, α the series (2-6) is summed until it has converged to the desired number of significant figures.

To control the truncation error E_t, the epsilon algorithm proposed by Macdonald [10] is used to accelerate the convergence of the series $S_m = \sum_{n=0}^{m} a_n$ in the following procedure

$$(2.11) \qquad \varepsilon_{p+1}^{(m)} = \varepsilon_{p-1}^{(m+1)} + \frac{1}{\varepsilon_p^{(m+1)} - \varepsilon_p^{(m)}}, \qquad \varepsilon_0^{(m)} = 0, \quad \varepsilon_1^{(m)} = S_m,$$

where the approximate value of the series can be finally shown by $S_\infty = \lim_{p\to\infty} \varepsilon_{2p}^{(m)}$. By applying this algorithm for the sequence of series (2-8), we evaluate $\hat{f}_{p+1}(t)$, $\hat{f}_{p+\frac{p}{4}}(t)$ and select a p for which the difference between $\hat{f}_{p+1}$, $\hat{f}_{p+\frac{p}{4}}$ be negligible.

3. Inversion method for the two-dimensional $\mathcal{L}_2$-transform and error analysis

In this section, we generalize the presented method in previous section for the two-dimensional $\mathcal{L}_2$-Transform. For this extension, we let $s_1 = c_1 + iw_1, s_2 = c_2 + iw_2$. It is obvious that the relation (1-4) can be written in the form

$$(3.1) \qquad f(t_1, t_2)$$
$$= \frac{e^{c_1 t_1^2 + c_2 t_2^2}}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} 4F(\sqrt{c_1 + iw_1}, \sqrt{c_2 + iw_2}) e^{i(w_1 t_1^2 + w_2 t_2^2)} dw_1 dw_2.$$

Also, by using the fact that $f(t_1, t_2)$ is real-value, (3-1) can be reformed as

$$(3.2) \quad f(t_1, t_2) = \frac{e^{c_1 t_1^2 + c_2 t_2^2}}{\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [\Re F(\sqrt{c_1 + iw_1}, \sqrt{c_2 + iw_2}) \cos(w_1 t_1^2 + w_2 t_2^2)$$
$$- \Im F(\sqrt{c_1 + iw_1}, \sqrt{c_2 + iw_2}) \sin(w_1 t_1^2 + w_2 t_2^2)] dw_1 dw_2.$$

At this point, we want to simplify (3-2) further for better form in computations. In this regard, we set

$$\varphi(w_1, w_2) = \Re F(\sqrt{c_1 + iw_1}, \sqrt{c_2 + iw_2}) \cos(w_1 t_1^2 + w_2 t_2^2)$$
$$- \Im F(\sqrt{c_1 + iw_1}, \sqrt{c_2 + iw_2}) \sin(w_1 t_1^2 + w_2 t_2^2).$$

Since $f(t_1, t_2)$ is real, we have $F(\overline{s_1}, \overline{s_2}) = \overline{F(s_1, s_2)}$ which implies $\Re F(\overline{s_1}, \overline{s_2}) = \Re F(s_1, s_2) = \Re F(s_1, s_2)$ and $, \Im F(\overline{s_1}, \overline{s_2}) = \Im \overline{F(s_1, s_2)} = -\Im F(s_1, s_2)$. Therefore

$$\Re\varphi(-w_1, w_2) = \Re\varphi(w_1, w_2), \quad \Im\varphi(w_1, -w_2) = -\Im\varphi(-w_1, w_2),$$

and (3-2) gives rise to

(3.3) $f(t_1, t_2)$

$$= \frac{2e^{c_1 t_1^2 + c_2 t_2^2}}{\pi^2} \{ \int_0^\infty \int_0^\infty [\Re F(\sqrt{c_1 + iw_1}, \sqrt{c_2 + iw_2}) \cos(w_1 t_1^2 + w_2 t_2^2)$$
$$- \Im F(\sqrt{c_1 + iw_1}, \sqrt{c_2 + iw_2}) \sin(w_1 t_1^2 + w_2 t_2^2)] dw_1 dw_2$$
$$+ \int_0^\infty \int_0^\infty [\Re F(\sqrt{c_1 + iw_1}, \sqrt{c_2 - iw_2}) \cos(w_1 t_1^2 - w_2 t_2^2)$$
$$- \Im F(\sqrt{c_1 + iw_1}, \sqrt{c_2 - iw_2}) \sin(w_1 t_1^2 - w_2 t_2^2)] dw_1 dw_2 \}.$$

For description of the inversion method for the two-dimensional $\mathcal{L}_2$-transform, we write the Fourier series for the function $g^{00}(t_1, t_2)$ that is periodic with period $2T$ in t_1, t_2 and is presented by

(3.4) $$g^{00}(t_1, t_2) = e^{-c_1 t_1^2 - c_2 t_2^2} f(t_1, t_2),$$

Also, we can similary define for $j, k = 0, 1, \cdots$ and $-\infty < t_1, t_2 < \infty$

(3.5) $g^{jk}(t_1, t_2) = e^{-c_1 t_1^2 - c_2 t_2^2} f(t_1, t_2), \quad in \quad (2jT, 2(j+1)T) \times (2kT, 2(k+1)T),$

where it is periodic with period $2T$ in t_1 and t_2. Moreover, the Fourier series representation for $g^{jk}(t_1, t_2)$ can be written as

(3.6) $$g^{jk}(t_1, t_2) = \frac{1}{4} a_{00}^{jk} + \frac{1}{2} \sum_{m=1}^\infty (a_{0m}^{jk} \cos(my) + b_{0m}^{jk} \sin(my)$$

$$+ \frac{1}{2} \sum_{n=1}^\infty (a_{n0}^{jk} \cos(nx) + c_{n0}^{jk} \sin(nx)$$

$$+ \frac{1}{2} \sum_{n=1}^\infty \sum_{m=1}^\infty (a_{nm}^{jk} \cos(nx) \cos(my) + b_{nm}^{jk} \cos(nx) \sin(my)$$

$$+ c_{nm}^{jk} \sin(nx) \cos(my) + d_{nm}^{jk} \sin(nx) \sin(my)),$$

where $x = \frac{\pi t_1^2}{T}, y = \frac{\pi t_2^2}{T}$ and the coefficients are given by

$$a_{nm}^{jk} = \frac{1}{4T^2} \int_{2jT}^{2(j+1)T} \int_{2kT}^{2(k+1)T} uv e^{-c_1 u^2 - c_2 v^2} f(u,v) \cos(\frac{n\pi u^2}{T}) \cos(\frac{m\pi v^2}{T}) du dv$$

$$b_{nm}^{jk} = \frac{1}{4T^2} \int_{2jT}^{2(j+1)T} \int_{2kT}^{2(k+1)T} uv e^{-c_1 u^2 - c_2 v^2} f(u,v) \cos(\frac{n\pi u^2}{T}) \sin(\frac{m\pi v^2}{T}) du dv$$

$$c_{nm}^{jk} = \frac{1}{4T^2} \int_{2jT}^{2(j+1)T} \int_{2kT}^{2(k+1)T} uv e^{-c_1 u^2 - c_2 v^2} f(u,v) \sin(\frac{n\pi u^2}{T}) \cos(\frac{m\pi v^2}{T}) du dv$$

$$(3.7) \quad d_{nm}^{jk} = \frac{1}{4T^2} \int_{2jT}^{2(j+1)T} \int_{2kT}^{2(k+1)T} uv e^{-c_1 u^2 - c_2 v^2} f(u,v) \sin(\frac{n\pi u^2}{T}) \sin(\frac{m\pi v^2}{T}) du dv.$$

Now, by substituting integrals of (3-7) in coefficients (3-6), and taking the sum of the resulting series over j, k, we get

$$(3.8) \quad \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} g^{jk}(t_1, t_2)$$

$$= \frac{1}{T^2} [\frac{F(c_1, c_2)}{4}$$

$$+ \sum_{m=1}^{\infty} (2\Re F(\sqrt{c_1}, \sqrt{c_2 + \frac{im\pi}{T}}) \cos(\frac{m\pi t_2^2}{T}) - 2\Im F(\sqrt{c_1}, \sqrt{c_2 + \frac{im\pi}{T}}) \sin(\frac{m\pi t_2^2}{T}))$$

$$+ \sum_{n=1}^{\infty} (2\Re F(\sqrt{c_1 + \frac{in\pi}{T}}, \sqrt{c_2}) \cos(\frac{n\pi t_1^2}{T}) - 2\Im F(\sqrt{c_1 + \frac{in\pi}{T}}, \sqrt{c_2}) \sin(\frac{n\pi t_1^2}{T}))$$

$$+ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (2\Re F(\sqrt{c_1 + \frac{in\pi}{T}}, \sqrt{c_2 + \frac{im\pi}{T}}) \cos(\frac{n\pi t_1^2}{T} + \frac{m\pi t_2^2}{T})$$

$$+ 2\Re F(\sqrt{c_1 + \frac{in\pi}{T}}, \sqrt{c_2 - \frac{im\pi}{T}}) \cos(\frac{n\pi t_1^2}{T} - \frac{m\pi t_2^2}{T})$$

$$- 2\Im F(\sqrt{c_1 + \frac{in\pi}{T}}, \sqrt{c_2 + \frac{im\pi}{T}}) \sin(\frac{n\pi t_1^2}{T} + \frac{m\pi t_2^2}{T})$$

$$- 2\Im F(\sqrt{c_1 + \frac{in\pi}{T}}, \sqrt{c_2 - \frac{im\pi}{T}}) \sin(\frac{n\pi t_1^2}{T} - \frac{m\pi t_2^2}{T}))].$$

It is obvious that the sum of the right-hand side of (3-8) can be an approximation value of $f(t_1, t_2)$ as

(3.9) $\hat{f}(t_1, t_2)$

$$
= \frac{e^{c_1 t_1^2 + c_2 t_2^2}}{T^2} \Big[\frac{F(c_1, c_2)}{4}
$$

$$
+ \sum_{m=1}^{\infty} \Big(2\Re F(\sqrt{c_1}, \sqrt{c_2 + \frac{im\pi}{T}}) \cos(\frac{m\pi t_2^2}{T}) - 2\Im F(\sqrt{c_1}, \sqrt{c_2 + \frac{im\pi}{T}}) \sin(\frac{m\pi t_2^2}{T}) \Big)
$$

$$
+ \sum_{n=1}^{\infty} \Big(2\Re F(\sqrt{c_1 + \frac{in\pi}{T}}, \sqrt{c_2}) \cos(\frac{n\pi t_1^2}{T}) - 2\Im F(\sqrt{c_1 + \frac{in\pi}{T}}, \sqrt{c_2}) \sin(\frac{n\pi t_1^2}{T}) \Big)
$$

$$
+ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \Big(2\Re F(\sqrt{c_1 + \frac{in\pi}{T}}, \sqrt{c_2 + \frac{im\pi}{T}}) \cos(\frac{n\pi t_1^2}{T} + \frac{m\pi t_2^2}{T})
$$

$$
+ 2\Re F(\sqrt{c_1 + \frac{in\pi}{T}}, \sqrt{c_2 - \frac{im\pi}{T}}) \cos(\frac{n\pi t_1^2}{T} - \frac{m\pi t_2^2}{T})
$$

$$
- 2\Im F(\sqrt{c_1 + \frac{in\pi}{T}}, \sqrt{c_2 + \frac{im\pi}{T}}) \sin(\frac{n\pi t_1^2}{T} + \frac{m\pi t_2^2}{T})
$$

$$
- 2\Im F(\sqrt{c_1 + \frac{in\pi}{T}}, \sqrt{c_2 - \frac{im\pi}{T}}) \sin(\frac{n\pi t_1^2}{T} - \frac{m\pi t_2^2}{T}) \Big) \Big],
$$

with the error term $E_d = \hat{f}(t_1, t_2) - f(t_1, t_2)$, where

$$
(3.10) \qquad E_d = e^{c_1 t_1^2 + c_2 t_2^2} \Big\{ \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} g^{jk}(t_1, t_2) + \sum_{j=1}^{\infty} g^{j0}(t_1, t_2) \Big\},
$$

or equivalently

$$
(3.11) \qquad E_d = e^{c_1 t_1^2 + c_2 t_2^2} \Big\{ \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} e^{-c_1(t_1 + 2jT)^2 - c_2(t_2 + 2kT)^2} f(t_1 + 2jT, t_2 + 2kT)
$$

$$
+ \sum_{j=1}^{\infty} e^{-c_1(t_1 + 2jT)^2 - c_2 t_2^2} f(t_1 + 2jT, t_2) \Big\}.
$$

Two major sources of error in approximation of $f(t_1, t_2)$, besides round-off error like the one-dimensional, are the discretization error and the truncation error given by E_d, E_t respectively. At first, by virtue of the exponential order of $f(t_1, t_2)$ we find a bound for E_d

$$
(3.12) \qquad E_d \leq M e^{c_1 t_1^2 + c_2 t_2^2} \Big\{ \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} e^{-(c_1 - \alpha_1)(t_1 + 2jT)^2 - (c_2 - \alpha_2)(t_2 + 2kT)^2}
$$

$$
+ \sum_{j=1}^{\infty} e^{-(c_1 - \alpha_1)(t_1 + 2jT)^2 - (c_2 - \alpha_2)t_2^2} \Big\}.
$$

In (3-12) if we take the parameters c_1 and c_2 sufficiently larger than α_1 and α_2, then the error E_d can be obtained arbitrary small. For this regard, we want to choose the value of parameter c_1, c_2 which the relative error $E_r = \frac{E_d}{M e^{c_1 t_1^2 + c_2 t_2^2}}$ to be less than the known value ϵ. At first, by picking a value for c_1 such that $c_1 > \alpha_1$, the parameter c_2 is chosen until following inequality for $(t_1, t_2) \in (2jT, 2(j+1)T) \times (2kT, 2(k+1)T)$ holds true

$$(3.13) \qquad e^{-(c_1-\alpha_1)(2T)^2 - (c_2-\alpha_2)(6T)^2} + e^{-(c_1-\alpha_1)(6T)^2 - (c_2-\alpha_2)(2T)^2} < \epsilon.$$

Then by finding the suitable T when $0.5 t_{max}^* \leq T \leq 0.8 t_{max}^*$ (where t_{max}^* is the largest value of t_1, t_2 over a range of t-values) we compute the series (3-9) until it converge to the desired number of significant figures.

To control of the truncation error E_t, we can use the epsilon algorithm (2-11) to accelerate the convergence of the series and evaluate the values $\hat{f}_{p+1}(t_1, t_2)$, $\hat{f}_{p+\frac{p}{4}}(t_1, t_2)$ in (3-9) until the difference between them be small.

4. Illustrative examples

The following four examples are used to test the approximation methods developed in Section 2 and Section 3 for one and two-dimensional $\mathcal{L}_2$-transform. These examples are chosen for the functions which are unbounded and the Laplace transform of them does not exist. The format of the examples is as follows:

In the one and two-dimensional cases the first column gives the value of the independent variable t and second column, the exact value of $f(t)$. The third column shows the approximate values of $f(t)$. The fourth column gives the relative error for all computations (see Tables 1-4). Also, all evaluations are performed by the *Maple 13* software and the epsilon algorithm is used to accelerate the convergence of the series.

Example 4.1.
$$\mathcal{L}_2\{\frac{\cos(t)}{t}; s\} = \frac{\sqrt{\pi}}{2s} e^{-\frac{1}{4s^2}}$$

Example 4.2.
$$\mathcal{L}_2\{t + \frac{1}{t}; s\} = \frac{\pi(s^2 + \frac{1}{2})}{2s^3}$$

Example 4.3.
$$\mathcal{L}_2^{(s_1, s_2)}\{\frac{\cos(t_1 t_2)}{t_1 t_2}\} = \frac{\pi}{2\sqrt{1 + 4s_1^2 s_2^2}}$$

Example 4.4.
$$\mathcal{L}_2^{(s_1, s_2)}\{\frac{e^{-(t_1+t_2)}}{t_1 t_2}\} = \frac{\pi}{4s_1 s_2} e^{\frac{1}{4s_1^2} + \frac{1}{4s_2^2}} Erfc(\frac{1}{2s_1}) Erfc(\frac{1}{2s_2})$$

Table1: Numerical test for the Example 4.1, $c = 2, T = 1.5, p = 34$

t	Exact value	Approximate value	Relative error
0.001	999.99950	999.99950	0.43E-06
0.01	99.99500	99.99500	0.47E-06
0.1	9.95004	9.95004	0.33E-06
0.25	3.87564	3.87564	0.12E-07
0.5	1.75516	1.75516	0.23E-07
1	0.54030	0.54030	0.58E-07
1.5	0.047158	0.047158	0.68E-07
2	-0.20807	-0.20807	0.74E-07

Table 2: Numerical test for the Example 4.2, $c = 2.5, T = 2, p = 34$

t	Exact value	Approximate value	Relative error
0.001	1000.00100	1000.00076	0.23E-06
0.01	100.01000	100.00833	0.16E-04
0.1	10.10000	10.09973	0.26E-04
0.25	4.25000	4.24985	0.35E-04
0.5	2.50000	2.49985	0.60E-04
1	2.00000	1.99979	0.10E-03
1.5	2.16666	2.16623	0.19E-03
2	2.50000	2.49989	0.44E-04

Table 3: Numerical test for the Example 4.3, $c_1 = 1, c_2 = 1.5, T = 1.5, p = 43$

t_1	t_2	Exact value	Approximate value	Relative error
0.1	0.1	99.99500	99.99500	0.13E-06
0.1	0.25	39.98750	39.98750	0.44E-06
0.25	0.25	15.96876	15.96876	0.56E-07
0.25	0.5	7.93758	7.93758	0.23E-07
0.5	0.5	3.875649	3.875649	0.63E-07
1	1	0.540302	0.540302	0.43E-07
1.5	1.5	-0.27918	-0.27918	0.62E-07
2	2	-0.163410	-0.163410	0.84E-07

Table 4: Numerical test for the Example 4.4, $c_1 = 3.5, c_2 = 3.3, T = 2, p = 43$

t_1	t_2	Exact value	Approximate value	Relative error
0.1	0.1	81.87307	81.87298	0.11E-05
0.1	0.25	28.18752	28.18742	0.34E-05
0.25	0.25	9.70449	9.70424	0.24E-04
0.25	0.5	3.77893	3.77845	0.13E-03
0.5	0.5	1.47151	1.47135	0.10E-03
1	1	0.13533	0.13524	0.65E-03
1.5	1.5	0.02212	0.02202	0.46E-02
2	2	0.00457	0.00426	0.68E-01

5. Application of the $\mathcal{L}_2$-transform in fractional partial differential equations

In this section for the applicability of the $\mathcal{L}_2$-transform and the introduced numerical inversion technique, using the joint Laplace and $\mathcal{L}_2$-transform we solve a fractional partial differential equation and then in special case of this equation, we apply the proposed algorithm in Section 2 to obtain the solution numerically. This methods can be considered as a promising technique beside the existing methods for solving fractional partial differential equations, see [6, 7] and [9, 13, 14].

Problem 5.1. We consider the fractional disturbance equation in the Caputo sense [12]

$$(5.1) \qquad {}_t^C D_{0^+}^\alpha u(x,t) + \frac{1}{x}\frac{\partial u(x,t)}{\partial x} = f(x), \quad x, t > 0,$$

with Cauchy type initial and boundary conditions $u(x,0) = u(0,t) = 0$.
For this equation, if we choose the function $f(x)$ with damping motion near zero such as functions introduced in examples (4.1) and (4.2), then the Lapalce transform is not suitable for solving this equation. Therefore, by applying the Laplace transform on fractional derivative in the sense of the Caputo for t

$$(5.2) \qquad \mathcal{L}\{{}_t^C D_{0^+}^\alpha u(x,t); s\} = s^\alpha \widetilde{u}(x,s) - s^{\alpha-1} u(x,0),$$

and the $\mathcal{L}_2$-transform with respect to x [1] the transformed equation of (5-1), takes the form

$$(5.3) \qquad \hat{\widetilde{u}}(p,s) = \frac{F(p)}{s^\alpha + 2p^2},$$

where $F(p)$ is the $\mathcal{L}_2$-transform of the function $f(x)$. Now, by using the complex inversion formula for the $\mathcal{L}_2$-transform (1-3) for the above equation and the convolution of the two functions f, g for the $\mathcal{L}_2$-transform [3]

$$f * g = \int_0^x t g(t) f(\sqrt{x^2 - t^2}) dt,$$

we have

$$\widetilde{u}(x,s) = \frac{1}{2}e^{-s^\alpha \frac{x^2}{4}} * f(x).$$

Also, in regard to the inverse Laplace transform of the functions via the Wright functions [12]

$$\mathcal{L}^{-1}\{e^{-s^\alpha \frac{x^2}{4}}\} = \frac{1}{t}W(-\alpha, 0; -\frac{x^2}{4}t^{-\alpha}),$$

the explicit solution of the Cauchy type problem (5-1) is given by

(5.4) $$u(x,t) = \int_0^x \tau G^\alpha(x^2 - \tau^2, t)f(\tau)d\tau,$$

where the Green function G^α has the form

(5.5) $$G^\alpha(x,t) = \frac{1}{2t}W(-\alpha, 0; -\frac{x}{4}t^{-\alpha}).$$

In special case, when we set $\alpha = 1$ in (5-1) we obtain the standard Shankar equation [15]

(5.6) $$\frac{\partial u(x,t)}{\partial t} + \frac{1}{x}\frac{\partial u(x,t)}{\partial x} = f(x), \quad x, t > 0,$$

which in this case by inverting the transformed equation

(5.7) $$\hat{\widetilde{u}}(p,s) = \frac{F(p)}{s + 2p^2},$$

with respect to s , the remaining function $\widetilde{u}(p,t) = F(p)e^{-2p^2 t}$ can be inverted in term of the Heaviside function as follows

(5.8) $$u(x,t) = \mathcal{L}_2^{-1}\{F(p); x^2 - 2t\}H(x^2 - 2t).$$

Now, to obtain the fundamental solution of the equation (5-5) numerically, the inverse of the $\mathcal{L}_2$-transform in (5-7) can be obtained by the proposed algorithm in section 2 for any desired point x, t.

6. Conclusions

In this work, we showed a general numerical algorithm for inversion of the $\mathcal{L}_2$-transform in terms of the Fourier series with a good accuracy. The $\mathcal{L}_2$-transform and the proposed inversion algorithm enable us to solve the partial differential equations or fractional partial differential equations with boundary conditions which the Laplace transform is not suitable for solving these equations.

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