KYUNGPOOK Math. J. 52(2012), 359-373
http://dx.doi.org/10.5666/KMJ.2012.52.4.359

Block LU Factorization for the Coupled Stokes Equations by Spectral Element Discretization

Xiangfan Piao, Philsu Kim and Sang Dong Kim*
Department of Mathematics, Kyungpook National University, Taegu, 702-701, Korea
e-mail : `piaoxf76@hanmail.net`, `kimps@knu.ac.kr` *and* `skim@knu.ac.kr`

Abstract. The block LU factorization is used to solve the coupled Stokes equations arisen from an optimal control problem subject to Stokes equations. The convergence of the spectral element solution is proved. Some numerical evidences are provided for the model coupled Stokes equations. Moreover, as an application, this algorithm is performed for an optimal control problem.

1. Introduction

We consider the coupled Stokes equations on a bounded and connected open subset Ω of $\mathbb{R}^d$ $(d = 2, 3)$ with a Lipschitz boundary Γ, which occurs from an optimal control problem subject to the Stokes equations. These coupled Stokes equations consist of two vector momentum equations and two continuity equations. These momentum equations are connected by primal- and adjoint- velocity variables. Because of this connection, it is more difficult to solve the coupled Stokes equations compared with the standard Stokes equations numerically. Many works have been devoted to solve an optimal control problem related to Stokes equations theoretically or numerically(see [2], [3], [4], [15], [21] and etc). Lately, a mixture use of finite element methods and FOSLS was studied in [18]. Uzawa algorithm for the coupled Stokes equations was analyzed in [17] analytically. But most numerical works for an optimal control problem are done by finite element methods even if spectral element methods(=:SEM) are popular and accurate.

In the necessity of developing a numerical method, we follow a common approach to solving the Stokes problem by carrying out block LU factorization (see [10], [12], [22] for example) which decouples the velocity and pressure variables. For this purpose, we rearrange the equations and variables of velocity and pressure and

* Corresponding Author.
Received October 4, 2012; accepted October 26, 2012.
2010 Mathematics Subject Classification: 49J20, 76D05, 65M70.
Key words and phrases: LU factorization, coupled stokes equations, optimal control, spectral element method.
This work was supported by KRF under contract number C00094.

then the coupled Stokes equations will be written as a vector formulation. The Gaussian elimination gets rid of the continuity equations so that the coupled Stokes equations can be solved by decoupling the velocity and pressure variables. Hence, it is possible to apply the backward substitution algorithm for solving the coupled Stokes equations by the pressure variables first and then the velocity variables. For Navier-Stokes equations, more effective way was introduced to avoid solving Helmholtz equations for each iteration many times(for more detail, see [1], [9], [12], [19] and etc).

For discretization of the coupled Stokes equations in this paper, SEM will be employed because it is one of popular and accurate methods among other numerical methods(see [6], [7], [11] for example). In relation with an optimal control problem, spectral methods are used for its discretization (see [8], [13] for example).

In section 2, the validity of block LU factorization is provided for the coupled Stokes equations. The backward substitution algorithm for the coupled Stokes equations in Galerkin formulation is presented. In section 3, the SEM discretization is introduced for the backward substitution algorithm developed in section 2. The spectral convergence is also provided. In section 4, some numerical evidences are presented to support the convergence results for a model problem. Further, the developed algorithm is applied to an optimal control problem. Finally, we mention the conclusion in last section.

2. Coupled Stokes

We begin with some notations on Sobolev spaces. Let $\mathbf{H}_0^1(\Omega) := H_0^1(\Omega)\times H_0^1(\Omega)$ and $\mathbf{L}_0^2(\Omega) := L_0^2(\Omega) \times L_0^2(\Omega)$ where $H_0^1(\Omega)$ is the standard Sobolev space whose norm and seminorm are denoted as $\|\cdot\|_1$ and $|\cdot|_1$, respectively. Let $L_0^2(\Omega)$ be the subspace of $L^2(\Omega)$ with mean zero. The L^2- inner product and norm are denoted by $(\cdot,\cdot)$ and $\|\cdot\|$. With the standard dual space $H^{-1}(\Omega)$ of $H_0^1(\Omega)$ equipped with the dual norm $\|\cdot\|_{-1}$, let $\mathbf{H}^{-1}(\Omega) := H^{-1}(\Omega) \times H^{-1}(\Omega)$.

Let us consider the coupled Stokes equations as follows; Find $(\mathbf{u},\mathbf{v}) \in [\mathbf{H}_0^1(\Omega)]^2$ and $(p, q) \in \mathbf{L}_0^2(\Omega)$ such that

$$\begin{cases} -\nu\Delta\mathbf{u} + \nabla p + \dfrac{1}{\delta}\mathbf{v} = \mathbf{g} & \text{in } \Omega, \\ -\nu\Delta\mathbf{v} + \nabla q - \mathbf{u} = -\widehat{\mathbf{u}} & \text{in } \Omega, \\ \nabla\cdot\mathbf{u} = 0 & \text{in } \Omega, \\ \nabla\cdot\mathbf{v} = 0 & \text{in } \Omega, \\ \mathbf{u} = \mathbf{v} = 0 & \text{on } \Gamma, \\ \displaystyle\int_\Omega p\, d\Omega = \int_\Omega q\, d\Omega = 0, \end{cases} \tag{2.1}$$

for given functions $\widehat{\mathbf{u}}$ and $\mathbf{g}$ in $\mathbf{H}^{-1}(\Omega)$.

One may note that (2.1) is the consequence of minimizing the L^2 quadratic functional

$$\mathcal{J}(\mathbf{u},\mathbf{f}) = \frac{1}{2}\|\mathbf{u}-\widehat{\mathbf{u}}\|^2 + \frac{\delta}{2}\|\mathbf{f}\|^2 \tag{2.2}$$

subject to

$$\begin{cases} -\nu\Delta\mathbf{u} + \nabla p = \mathbf{f} + \mathbf{g} & \text{in } \Omega, \\ \nabla\cdot\mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = 0 & \text{on } \Gamma, \\ \displaystyle\int_\Omega p\, d\Omega = 0, \end{cases} \tag{2.3}$$

where $\widehat{\mathbf{u}}$ is a given target velocity and δ is a positive penalty parameter (see [15]). We further assume that $\mathbf{f},\mathbf{g} \in \mathbf{H}^{-1}(\Omega)$. Multiplying the second equation in (2.1) by $\frac{1}{\delta}$ and the last equation by $\frac{1}{\sqrt{\delta}}$ and then scaling q by $\sqrt{\delta}q$ one more time, one may have the equivalent coupled Stokes equations as

$$\begin{cases} -\nu\Delta\mathbf{u} + \nabla p + \dfrac{1}{\delta}\mathbf{v} = \mathbf{g} & \text{in } \Omega, \\ -\dfrac{\nu}{\delta}\Delta\mathbf{v} + \dfrac{1}{\sqrt{\delta}}\nabla q - \dfrac{1}{\delta}\mathbf{u} = -\dfrac{1}{\delta}\widehat{\mathbf{u}} & \text{in } \Omega, \\ \nabla\cdot\mathbf{u} = 0 & \text{in } \Omega, \\ \dfrac{1}{\sqrt{\delta}}\nabla\cdot\mathbf{v} = 0 & \text{in } \Omega. \end{cases} \tag{2.4}$$

Let us define the following scaled Laplace $\tilde{\Delta}$, gradient $\tilde{\nabla}$ and divergence $\tilde{\nabla}\cdot$ matrix operators as

$$\tilde{\Delta} := \begin{bmatrix} \Delta & 0 \\ 0 & \frac{1}{\delta}\Delta \end{bmatrix}, \quad \tilde{\nabla} := \begin{bmatrix} \nabla & 0 \\ 0 & \frac{1}{\sqrt{\delta}}\nabla \end{bmatrix}, \quad \tilde{\nabla}\cdot := \begin{bmatrix} \nabla\cdot & 0 \\ 0 & \frac{1}{\sqrt{\delta}}\nabla\cdot \end{bmatrix}, \tag{2.5}$$

and introduce variable vectors and a matrix as

$$\mathbf{U} = \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix}, \quad \mathbf{P} = \begin{bmatrix} p \\ q \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} \mathbf{g} \\ -\frac{1}{\delta}\widehat{\mathbf{u}} \end{bmatrix}, \quad \mathsf{C} = \begin{bmatrix} 0 & \frac{1}{\delta} \\ -\frac{1}{\delta} & 0 \end{bmatrix}, \tag{2.6}$$

where $\mathbf{u} = (u_1, u_2)^T$ and $\mathbf{v} = (v_1, v_2)^T$. The notation $(\cdot)^T$ means the transpose of a vector or matrix.

Then (2.4) becomes: for a given $\mathbf{F} \in [\mathbf{H}^{-1}(\Omega)]^2$, find $(\mathbf{U},\mathbf{P}) \in [\mathbf{H}_0^1(\Omega)]^2 \times \mathbf{L}_0^2(\Omega)$ satisfying

$$\begin{cases} -\nu\tilde{\Delta}\mathbf{U} + \tilde{\nabla}\mathbf{P} + \mathsf{C}\mathbf{U} = \mathbf{F} & \text{in } \Omega, \\ \tilde{\nabla}\cdot\mathbf{U} = 0 & \text{in } \Omega, \\ \mathbf{U} = 0 & \text{on } \Gamma. \end{cases} \tag{2.7}$$

Following the proof of Theorem 5.1 in [14], one may show the existence of unique solution $(\mathbf{U},\mathbf{P}) \in [\mathbf{H}_0^1(\Omega)]^2 \times \mathbf{L}_0^2(\Omega)$ of (2.7). For reader's convenience, we will provide its proof. It is well-known that if $\mathbf{Q} = (q_1, q_2)^{\mathrm{T}} \in \mathbf{L}_0^2(\Omega)$, then there exists a unique vector $\mathbf{W} := (\mathbf{w}_1, \mathbf{w}_2)^{\mathrm{T}} \in [\mathbf{H}_0^1(\Omega)]^2 := \mathbf{H}_0^1(\Omega) \times \mathbf{H}_0^1(\Omega)$ satisfying

$$\tilde{\nabla} \cdot \mathbf{W} = \mathbf{Q}, \quad |\mathbf{W}|_1 \le C\|\mathbf{Q}\|, \tag{2.8}$$

where $|\mathbf{W}|_1^2 = |\mathbf{w}_1|_1^2 + |\mathbf{w}_2|_1^2$ and $\|\mathbf{Q}\|^2 = \|q_1\|^2 + \|q_2\|^2$. Hence

$$\frac{(\mathbf{Q}, \tilde{\nabla} \cdot \mathbf{W})}{|\mathbf{W}|_1} = \frac{\|\mathbf{Q}\|^2}{|\mathbf{W}|_1} \ge (1/C)\|\mathbf{Q}\|,$$

where C is a positive constant. This implies the LBB(see[5],[14]) condition holds. Let $\underline{\mathcal{A}} : [\mathbf{H}_0^1(\Omega)]^2 \times [\mathbf{H}_0^1(\Omega)]^2 \to \mathbb{R}$ be the bilinear form which is defined as

$$\underline{\mathcal{A}}(\mathbf{U}, \mathbf{V}) := (\nu \tilde{\nabla} \mathbf{U}, \tilde{\nabla} \mathbf{V}) + (\mathsf{C}\mathbf{U}, \mathbf{V}). \tag{2.9}$$

By using the definition of the scaled gradient $\tilde{\nabla}$, the fact $0 < \delta < 1$ and the Poincare inequality, it can be shown as

$$\underline{\mathcal{A}}(\mathbf{U}, \mathbf{U}) = \nu(\nabla \mathbf{u}, \nabla \mathbf{u}) + \frac{\nu}{\delta}(\nabla \mathbf{v}, \nabla \mathbf{v}) + \frac{1}{\delta}\Big((\mathbf{v}, \mathbf{u}) - (\mathbf{u}, \mathbf{v})\Big) \ge C_1 \|\mathbf{U}\|_1^2, \tag{2.10}$$

where C_1 is a positive constant depending on ν, δ and Ω. The continuity of $\underline{\mathcal{A}}(\cdot,\cdot)$ also holds

$$|\underline{\mathcal{A}}(\mathbf{U}, \mathbf{W})| \le C_2 \|\mathbf{U}\|_1 \|\mathbf{W}\|_1, \tag{2.11}$$

where C_2 is a positive constant depending on ν, δ and Ω.

In order to use the block LU factorizations, with rearrangement of the equations and variables, one may rewrite (2.7) as a more convenient coupled Stokes system. For this purpose, let us define some necessary operators $\mathcal{H}$ and $\mathcal{D}_t, (t = x, y)$ as

$$\mathcal{H} := \begin{bmatrix} -\nu\Delta & \frac{1}{\delta} \\ -\frac{1}{\delta} & -\frac{\nu}{\delta}\Delta \end{bmatrix}, \quad \mathcal{D}_x := \begin{bmatrix} \partial_x & 0 \\ 0 & \frac{1}{\sqrt{\delta}}\,\partial_x \end{bmatrix}, \quad \mathcal{D}_y := \begin{bmatrix} \partial_y & 0 \\ 0 & \frac{1}{\sqrt{\delta}}\,\partial_y \end{bmatrix}, \tag{2.12}$$

and variable vectors as

$$\mathbf{P} = \begin{bmatrix} p \\ q \end{bmatrix}, \quad \mathbf{U}_i = \begin{bmatrix} u_i \\ v_i \end{bmatrix}, \quad \mathbf{F}_i = \begin{bmatrix} g_i \\ -\frac{1}{\delta}\widehat{u}_i \end{bmatrix}, \quad i = 1, 2. \tag{2.13}$$

Hence (2.7) becomes; for a given $\mathbf{F}_i \in \mathbf{H}^{-1}(\Omega)$, $(i = 1, 2)$

$$\begin{bmatrix} \mathcal{H} & 0 & \mathcal{D}_x \\ 0 & \mathcal{H} & \mathcal{D}_y \\ \mathcal{D}_x & \mathcal{D}_y & 0 \end{bmatrix} \begin{bmatrix} \mathbf{U}_1 \\ \mathbf{U}_2 \\ \mathbf{P} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_1 \\ \mathbf{F}_2 \\ \mathbf{0} \end{bmatrix}. \tag{2.14}$$

Note that the coupled elliptic operator $\mathcal{H}$ has its domain as the space $\mathbf{H}_0^1(\Omega)$ because it works on the velocity u_i and v_i in $H_0^1(\Omega)$. One may easily find that $\mathcal{H}$ has a bounded inverse $\mathcal{H}^{-1}$ on $\mathbf{H}^{-1}(\Omega)$ (see [16] for example or the arguments in (2.10) and (2.11)).

It is possible to define the following elementary operator $\mathcal{L}_1$ from $\underline{\mathcal{S}} := \mathbf{H}^{-1}(\Omega) \times \mathbf{H}^{-1}(\Omega) \times \{\mathbf{0}\}$ to $\underline{\mathcal{T}} := \mathbf{H}^{-1}(\Omega) \times \mathbf{H}^{-1}(\Omega) \times \mathbf{L}^2(\Omega)$ by

$$\mathcal{L}_1 := \begin{bmatrix} \mathcal{I} & 0 & 0 \\ 0 & \mathcal{I} & 0 \\ -\mathcal{D}_x\mathcal{H}^{-1} & -\mathcal{D}_y\mathcal{H}^{-1} & \mathcal{I} \end{bmatrix}. \tag{2.15}$$

Lemma 2.1. *The operator $\mathcal{L}_1$ is invertible. Moreover, we have*

$$\|\mathcal{L}_1\| = \sup_{\mathbf{U}\in\underline{\mathcal{S}}} \frac{\|\mathcal{L}_1\mathbf{U}\|_{\underline{\mathcal{T}}}}{\|\mathbf{U}\|_{\underline{\mathcal{S}}}} \geq 1.$$

Proof. Let $\mathbf{U} := (\mathbf{F}, \mathbf{G}, \mathbf{0})^T \in \underline{\mathcal{S}}$. Then $\|\mathcal{L}_1\mathbf{U}\|^2_{\underline{\mathcal{T}}} = \|\mathbf{F}\|^2_{-1} + \|\mathbf{G}\|^2_{-1} + \|\mathbf{H}\|^2 = \|\mathbf{U}\|^2_{\underline{\mathcal{S}}} + \|\mathbf{H}\|^2$ where $\mathbf{H}$ is the third component of vector $\mathcal{L}_1\mathbf{U}$. Hence it follows that $\|\mathcal{L}_1\mathbf{U}\|_{\underline{\mathcal{T}}} \geq \|\mathbf{U}\|_{\underline{\mathcal{S}}}$ for any vector $\mathbf{U} \in \underline{\mathcal{S}}$. Thus we have the conclusion. □

Applying the elementary operator $\mathcal{L}_1$ to (2.14) yields

$$\begin{bmatrix} \mathcal{H} & 0 & \mathcal{D}_x \\ 0 & \mathcal{H} & \mathcal{D}_y \\ 0 & 0 & \mathcal{E} \end{bmatrix} \begin{bmatrix} \mathbf{U}_1 \\ \mathbf{U}_2 \\ \mathbf{P} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_1 \\ \mathbf{F}_2 \\ \mathbf{F}_3 \end{bmatrix}, \tag{2.16}$$

where the operator $\mathcal{E}$ is given by

$$\mathcal{E} = -(\mathcal{D}_x\mathcal{H}^{-1}\mathcal{D}_x + \mathcal{D}_y\mathcal{H}^{-1}\mathcal{D}_y) \tag{2.17}$$

and $\mathbf{F}_3$ is given by

$$\mathbf{F}_3 = -(\mathcal{D}_x\mathcal{H}^{-1}\mathbf{F}_1 + \mathcal{D}_y\mathcal{H}^{-1}\mathbf{F}_2). \tag{2.18}$$

Due to Lemma 2.1, (2.14) is now equivalent to (2.16). Hence (2.16) has a unique solution $(\mathbf{U}_1, \mathbf{U}_2, \mathbf{P})$ if (2.14) has a unique solution. In fact, (2.14) has a unique solution because (2.7) does. One may note that the unique solution $\mathbf{U}_1, \mathbf{U}_2$ and $\mathbf{P}$ can be written concisely as

$$\mathbf{P} = \mathcal{E}^{-1}\big(\mathcal{D}_x\mathcal{H}^{-1}\mathbf{F}_1 + \mathcal{D}_y\mathcal{H}^{-1}\mathbf{F}_2\big) \tag{2.19}$$

and

$$\mathbf{U}_i = \mathcal{H}^{-1}\Big(\mathbf{F}_i - \mathcal{D}_i\mathcal{E}^{-1}\big(\mathcal{D}_x\mathcal{H}^{-1}\mathbf{F}_1 + \mathcal{D}_y\mathcal{H}^{-1}\mathbf{F}_2\big)\Big), \tag{2.20}$$

where $i = 1, 2$, $\mathcal{D}_1 = \mathcal{D}_x$ and $\mathcal{D}_2 = \mathcal{D}_y$. We write the above statements as theorem below.

Theorem 2.2. *Let Ω be a bounded and connected open subset of $\mathbb{R}^d$ with Lipschitz continuous boundary Γ. For a given $\mathbf{F}_i \in \mathbf{H}^{-1}(\Omega)$, there exists a unique solution $(\mathbf{U}_1, \mathbf{U}_2, \mathbf{P}) \in \mathbf{H}_0^1(\Omega) \times \mathbf{H}_0^1(\Omega) \times \mathbf{L}_0^2(\Omega)$ of (2.16). Therefore, the operator $\mathcal{E}$ is invertible.*

Now, for the solutions to (2.16), one may get $\mathbf{U}_i$, $(i = 1, 2)$ simultaneously by solving coupled Stokes equations if $\mathbf{P}$ is known. Since $\mathcal{H}^{-1} : \mathbf{H}^{-1}(\Omega) \to \mathbf{H}_0^1(\Omega)$, note that $\big(\mathcal{D}_t\mathcal{H}^{-1}\mathcal{D}_t\mathbf{P}, \mathbf{Q}\big) = -\big(\mathcal{H}^{-1}\mathcal{D}_t\mathbf{P}, \mathcal{D}_t\mathbf{Q}\big)$. Then we define the bilinear form $\mathcal{B}(\cdot, \cdot)$ on the space $\mathbf{L}_0^2(\Omega) \times \mathbf{L}_0^2(\Omega)$ as

$$\mathcal{B}(\mathbf{P}, \mathbf{Q}) := \big(\mathcal{H}^{-1}\mathcal{D}_x\mathbf{P}, \mathcal{D}_x\mathbf{Q}\big) + \big(\mathcal{H}^{-1}\mathcal{D}_y\mathbf{P}, \mathcal{D}_y\mathbf{Q}\big) \tag{2.21}$$

and the bilinear form $\mathcal{A}(\cdot, \cdot)$ on the space $\mathbf{H}_0^1(\Omega) \times \mathbf{H}_0^1(\Omega)$ as

$$\mathcal{A}(\mathbf{U}, \mathbf{W}) := (\tilde{\tilde{\nabla}}\mathbf{U}, \tilde{\tilde{\nabla}}\mathbf{W}) + (\mathsf{C}\mathbf{U}, \mathbf{W}), \tag{2.22}$$

where

$$\tilde{\tilde{\nabla}} = \begin{bmatrix} \sqrt{\nu}\nabla & 0 \\ 0 & \frac{\sqrt{\nu}}{\sqrt{\delta}}\nabla \end{bmatrix}.$$

The backward substitution algorithm to solve (2.16) in weak sense can be written as;

Algorithm 2.3(Continuous case)

Step 1. *Find $\mathbf{P} \in \mathbf{L}_0^2(\Omega)$ satisfying $\mathcal{B}(\mathbf{P}, \mathbf{Q}) = -(\mathcal{D}_x\mathcal{H}^{-1}\mathbf{F}_1 + \mathcal{D}_y\mathcal{H}^{-1}\mathbf{F}_2, \mathbf{Q})$ for any $\mathbf{Q} \in \mathbf{L}_0^2(\Omega)$.*

Step 2. *Find $\mathbf{U}_i \in \mathbf{H}_0^1(\Omega)$ satisfying $\mathcal{A}(\mathbf{U}_i, \mathbf{W}) = (\mathbf{F}_i - \mathcal{D}_i\mathbf{P}, \mathbf{W})$ for any $\mathbf{W} \in \mathbf{H}_0^1(\Omega)$ where $i = 1, 2$, $\mathcal{D}_1 = \mathcal{D}_x$ and $\mathcal{D}_2 = \mathcal{D}_y$.*

Proposition 2.4. *There are positive constants C_1 and C_2 satisfying*

$$C_1\|\mathbf{U}\|_1^2 \le \mathcal{A}(\mathbf{U}, \mathbf{U}), \quad \Big|\mathcal{A}(\mathbf{U}, \mathbf{W})\Big| \le C_2\|\mathbf{U}\|_1\,\|\mathbf{W}\|_1, \ \textit{for all } \mathbf{U}, \mathbf{W} \in \mathbf{H}_0^1(\Omega), \tag{2.23}$$

and

$$C_1\|\mathbf{P}\|^2 \le \mathcal{B}(\mathbf{P}, \mathbf{P}), \quad \Big|\mathcal{B}(\mathbf{P}, \mathbf{Q})\Big| \le C_2\|\mathbf{P}\|\|\mathbf{Q}\|, \quad \textit{for all} \quad \mathbf{Q}, \mathbf{P} \in \mathbf{L}_0^2(\Omega). \tag{2.24}$$

Proof. We provide the validity of (2.24). First note that (2.23) easily comes from

$$C_1\|\mathbf{U}\|_1^2 \le (\mathcal{H}\mathbf{U}, \mathbf{U}), \quad \text{and} \quad |(\mathcal{H}\mathbf{U}, \mathbf{V})| \le C_2\|\mathbf{U}\|_1\|\mathbf{V}\|_1 \tag{2.25}$$

where C_1 and C_2 are positive constants depending on ν, δ or Ω. With obvious notations in the following proof, for $\mathbf{P} \in \mathbf{L}_0^2(\Omega)$, let $\mathbf{U} := \mathcal{H}^{-1}\mathcal{D}_t\mathbf{P}$ which is in $\mathbf{H}_0^1(\Omega)$. Then it follows that by using (2.25)

$$\|\mathcal{D}_t\mathbf{P}\|_{-1} = \sup_{\mathbf{V}\in\mathbf{H}_0^1(\Omega)} \frac{(\mathcal{H}\mathbf{U},\mathbf{V})}{\|\mathbf{V}\|_1} \le C_2\|\mathbf{U}\|_1 = C_2\|\mathcal{H}^{-1}\mathcal{D}_t\mathbf{P}\|_1, \tag{2.26}$$

$$\|\mathcal{D}_t\mathbf{P}\|_{-1} = \sup_{\mathbf{V}\in\mathbf{H}_0^1(\Omega)} \frac{(\mathcal{H}\mathbf{U},\mathbf{V})}{\|\mathbf{V}\|_1} \ge C_1\|\mathbf{U}\|_1 \tag{2.27}$$

and

$$\|\mathcal{D}_t\mathbf{P}\|_{-1} = \sup_{\mathbf{V}\in\mathbf{H}_0^1(\Omega)} \frac{(\mathcal{D}_t\mathbf{P},\mathbf{V})}{\|\mathbf{V}\|_1} = \sup_{\mathbf{V}\in\mathbf{H}_0^1(\Omega)} \frac{(\mathbf{P},\mathcal{D}_t\mathbf{V})}{\|\mathbf{V}\|_1} \le \|\mathbf{P}\|. \tag{2.28}$$

Further, using (2.25) and (2.26) we have

$$\begin{aligned} \|\mathcal{D}_t\mathbf{P}\|_{-1}^2 &\le C\|\mathcal{H}^{-1}\mathcal{D}_t\mathbf{P}\|_1^2 \\ &= C\|\mathbf{U}\|_1^2 \le C(\mathbf{U},\mathcal{H}\mathbf{U}) = C(\mathcal{H}^{-1}\mathcal{D}_t\mathbf{P},\mathcal{D}_t\mathbf{P}), \end{aligned} \tag{2.29}$$

where C is an absolute positive constant depend only on ν, δ or Ω.

Now, since $\mathbf{P} \in \mathbf{L}_0^2(\Omega)$, using the same argument for (2.8), it follows that

$$\begin{aligned} \|\mathcal{D}_x\mathbf{P}\|_{-1} + \|\mathcal{D}_y\mathbf{P}\|_{-1} &= \sup_{\mathbf{U}\in\mathbf{H}_0^1(\Omega)} \frac{(\mathcal{D}_x\mathbf{P},\mathbf{U})}{\|\mathbf{U}\|_1} + \sup_{\mathbf{V}\in\mathbf{H}_0^1(\Omega)} \frac{(\mathcal{D}_y\mathbf{P},\mathbf{V})}{\|\mathbf{V}\|_1} \\ &\ge \sup_{\mathbf{W}:=(\mathbf{U},\mathbf{V})^{\mathrm{T}}\in[\mathbf{H}_0^1(\Omega)]^2} \frac{(\mathbf{P},\mathcal{D}_x\mathbf{U}) + (\mathbf{P},\mathcal{D}_y\mathbf{V})}{\|\mathbf{U}\|_1 + \|\mathbf{V}\|_1} \\ &= \sup_{\mathbf{W}\in[\mathbf{H}_0^1(\Omega)]^2} \frac{(\mathbf{P},\tilde{\nabla}\cdot\mathbf{W})}{\|\mathbf{W}\|_1} \\ &\ge C\|\mathbf{P}\|. \end{aligned} \tag{2.30}$$

Now, combining (2.29) and (2.30) completes the coercivity part in (2.24). Now we provide the continuity part of (2.24). For $\mathbf{P}$ and $\mathbf{Q}$ in $\mathbf{L}_0^2(\Omega)$, let $\mathbf{U}^t := \mathcal{H}^{-1}\mathcal{D}_t\mathbf{P}$ and $\mathbf{W}^t := \mathcal{H}^{-1}\mathcal{D}_t\mathbf{Q}$ without confusion below. Then, using (2.25), (2.27) and (2.28), we have

$$\begin{aligned} \mathcal{B}(\mathbf{P},\mathbf{Q}) &= (\mathcal{H}^{-1}\mathcal{D}_x\mathbf{P},\mathcal{D}_x\mathbf{Q}) + (\mathcal{H}^{-1}\mathcal{D}_y\mathbf{P},\mathcal{D}_y\mathbf{Q}) \\ &\le (\mathbf{U}^x,\mathcal{H}\mathbf{W}^x) + (\mathbf{U}^y,\mathcal{H}\mathbf{W}^y) \\ &\le C\Big(\|\mathbf{U}^x\|_1\|\mathbf{W}^x\|_1 + \|\mathbf{U}^y\|_1\|\mathbf{W}^y\|_1\Big) \\ &\le C\Big(\|\mathcal{D}_x\mathbf{P}\|_{-1}\|\mathcal{D}_x\mathbf{Q}\|_{-1} + \|\mathcal{D}_y\mathbf{P}\|_{-1}\|\mathcal{D}_y\mathbf{Q}\|_{-1}\Big) \\ &\le C\|\mathbf{P}\|\|\mathbf{Q}\|. \end{aligned}$$

These arguments complete the proof. □

Due to Lemma 2.1, it follows that the variational formulations in the algorithm (2.3) corresponding to (2.16) have a unique solution $(\mathbf{P}, \mathbf{U}_1, \mathbf{U}_2) \in \mathbf{L}_0^2(\Omega) \times \mathbf{H}_0^1(\Omega) \times \mathbf{H}_0^1(\Omega)$.

3. Spectral element approach

By using $P_N - P_{N-2}$ method for the approximation of the Stokes problem in [20], we present the spectral element approach for Algorithm (3.1) on the square domain $\Omega := I \times I$ where $I := [0, 1]$. Assume that the interval I has uniform knots $\{t_j\}_{j=0}^{E}$ arranged by $0 =: t_0 < t_1 < \cdots < t_{E-1} < t_E := 1$ with $h = t_j - t_{j-1}$ for each $j = 1, 2, \cdots, E$ and that the same degree N of Legendre polynomial on each subinterval $I_j := [t_{j-1}, t_j]$ of x and y directions is employed.

We define $\mathcal{P}_N^h$ as the subspace of $L^2(I)$ whose nodal basis $\{\phi_\mu\}$ at LGL(or LG) points are piecewise Lagrange polynomials of degree N on I_j with proper supports like I_j or $I_j \cup I_{j+1}$. For one dimensional function spaces, let $\mathcal{P}_{N,h}^s := H_s^1(I) \cap \mathcal{P}_N^h$ where s is either 0 or 1. In particular, if $s = 1$, we use $\mathcal{P}_{N,h}^s$ should be understood as $\mathcal{P}_N^h$. Define $\mathbf{P}_{N,h}^s := \mathcal{P}_{N,h}^s \times \mathcal{P}_{N,h}^s$ whose basis functions are given by

$$\underline{\phi}_p(t) := \begin{cases} [\phi_p(t), 0]^T & \text{for} \quad 1 \le p \le \mathfrak{N}, \\ [0, \phi_{p-\mathfrak{N}}(t)]^T & \text{for} \quad \mathfrak{N} < p \le 2\mathfrak{N}, \end{cases}$$

where $\mathfrak{N}$ is the total number of one dimensional global LGL points for velocity or LG points for pressure. Note that the total numbers of LGL nodes and LG nodes are not same. For two dimensional function spaces, let $[[\mathcal{P}_{N,h}^s]] := \mathcal{P}_{N,h}^s \otimes \mathcal{P}_{N,h}^s$ whose basis functions are given by tensor products of one-dimensional piecewise Lagrange polynomials. Then let $[[\mathbf{P}_{N,h}^s]] := [[\mathcal{P}_{N,h}^s]] \times [[\mathcal{P}_{N,h}^s]]$. For example, the function $\mathbf{U}_{N,h} = [\mathbf{u}_{N,h}, \mathbf{v}_{N,h}]^T \in [[\mathbf{P}_{N,h}^s]]$ should be understood as $\mathbf{u}_{N,h}(x,y), \mathbf{v}_{N,h}(x,y) \in [[\mathcal{P}_{N,h}^s]]$. With the notations $\phi_{\tilde{\mu}}(x,y) := \phi_\mu(x)\phi_\nu(y)$ $(\tilde{\mu} = \mu + \mathfrak{N}(\nu - 1))$, basis functions of $[[\mathbf{P}_{N,h}^s]]$ are given by

$$\underline{\Phi}_{\tilde{\mu}}(x,y) := \begin{cases} [\phi_{\tilde{\mu}}(x,y), 0]^T & \text{for} \quad 1 \le \tilde{\mu} \le \mathfrak{N}^2, \\ \left[0, \phi_{\tilde{\mu}-\mathfrak{N}^2}(x,y)\right]^T & \text{for} \quad \mathfrak{N}^2 < \tilde{\mu} \le 2\mathfrak{N}^2, \end{cases}$$

where $\mathfrak{N}^2$ is the total number of two-dimensional global LGL(or LG) points.

Then the variational formulations on the discrete spaces corresponding to (2.16) become as follows:

Algorithm 3.1(Discrete case)

Step 1. *Find* $\mathbf{P}_{N-2,h} \in [[\mathbf{P}_{N-2,h}^1]] \cap \mathbf{L}_0^2(\Omega)$ *satisfying* $\mathcal{B}(\mathbf{P}_{N-2,h}, \mathbf{Q}_{N-2,h}) = -(\mathbf{Q}_{N-2,h}, \mathcal{D}_x \mathcal{H}^{-1} \mathbf{F}_1 + \mathcal{D}_y \mathcal{H}^{-1} \mathbf{F}_2)$, *for all* $\mathbf{Q}_{N-2,h} \in [[\mathbf{P}_{N-2,h}^1]] \cap \mathbf{L}_0^2(\Omega)$.

Step 2. Find $\mathbf{U}^i_{N,h} \in [[\mathbf{P}^0_{N,h}]]$ *satisfying* $\mathcal{A}(\mathbf{U}^i_{N,h}, \mathbf{W}_{N,h}) = (\mathbf{F}_i - \mathcal{D}_i \mathbf{P}_{N-2,h}, \mathbf{W}_{N,h})$, *for all* $\mathbf{W}_{N,h} \in [[\mathbf{P}^0_{N,h}]]$ *where* $i = 1, 2$, $\mathcal{D}_1 = \mathcal{D}_x$ *and* $\mathcal{D}_2 = \mathcal{D}_y$.

Now we are at a position to provide convergence analysis by following the standard techniques (see [6, 7] for example) which employ orthogonality properties of two bilinear forms and the results of interpolation operators.

Theorem 3.2. *Let* $\mathbf{P} \in \mathbf{L}^2_0(\Omega) \cap \mathbf{H}^{k-1}(\Omega)$ *and* $\mathbf{U}_i \in \mathbf{H}^1_0(\Omega) \cap \mathbf{H}^k(\Omega)$ *be the exact solutions in Algorithm 2.3, where* $\mathbf{H}^s(\Omega) := H^s(\Omega) \times H^s(\Omega)$ *and* $H^s(\Omega)$*(s is a positive integer) is the standard Sobolev space. Let* $\mathbf{P}_{N-2,h} \in [[\mathbf{P}^1_{N-2,h}]] \cap \mathbf{L}^2_0(\Omega)$ *and* $\mathbf{U}^i_{N,h} \in [[\mathbf{P}^0_{N,h}]]$ *be the spectral element solutions in Algorithm 3.1. Then it follows that*

$$\|\mathbf{P} - \mathbf{P}_{N-2,h}\| \leq C h^{(\min\{N-1,k-1\})} (N-2)^{1-k} \|\mathbf{P}\|_{k-1} \tag{3.1}$$

and

$$\begin{aligned} \|\mathbf{U}_i - \mathbf{U}^i_{N,h}\|_1 \leq C(& h^{(\min\{N,k-1\})} N^{1-k} \|\mathbf{U}_i\|_k \\ & + h^{(\min\{N-1,k-1\})} (N-2)^{1-k} \|\mathbf{P}\|_{k-1}), \end{aligned} \tag{3.2}$$

where C *is a positive constant independent of* h *and* N.

Proof. Using the orthogonality and coercivity of the bilinear form $\mathcal{B}(\cdot, \cdot)$, it follows that for any $\mathbf{Q}_{N-2,h} \in [[\mathbf{P}_{N-2,h}]] \cap \mathbf{L}^2_0(\Omega)$

$$\|\mathbf{P} - \mathbf{P}_{N-2,h}\| \leq C \inf_{\mathbf{Q}_{N-2,h} \in [[\mathbf{P}_{N-2,h}]] \cap \mathbf{L}^2_0(\Omega)} \|\mathbf{P} - \mathbf{Q}_{N-2,h}\|,$$

where C is a positive constant only depending on ν, δ and Ω. Then, using the approximation property of interpolation operator at LG nodes(see [6, 7]), we have

$$\|\mathbf{P} - \mathbf{P}_{N-2,h}\| \leq C h^{(\min\{N-1,k-1\})} (N-2)^{1-k} \|\mathbf{P}\|_{k-1}.$$

This completes (3.1). Now, we need to prove the error estimate for the velocity part. First, note that

$$\mathcal{A}(\mathbf{U}_i - \mathbf{U}^i_{N,h}, \mathbf{W}_{N,h}) = (\mathcal{D}_i(\mathbf{P}_{N-2,h} - \mathbf{P}), \mathbf{W}_{N,h}) \quad \text{for all } \mathbf{W}_{N,h} \in [[\mathbf{P}^0_{N,h}]].$$

Hence, for any $\mathbf{W}_{N,h} \in [[\mathbf{P}^0_{N,h}]]$, we have

$$\begin{aligned} \mathcal{A}(\mathbf{U}^i_{N,h} & - \mathbf{W}_{N,h}, \mathbf{U}^i_{N,h} - \mathbf{W}_{N,h}) \\ & = \mathcal{A}(\mathbf{U}_i - \mathbf{W}_{N,h}, \mathbf{U}^i_{N,h} - \mathbf{W}_{N,h}) + \mathcal{A}(\mathbf{U}^i_{N,h} - \mathbf{U}_i, \mathbf{U}^i_{N,h} - \mathbf{W}_{N,h}) \\ & = \mathcal{A}(\mathbf{U}_i - \mathbf{W}_{N,h}, \mathbf{U}^i_{N,h} - \mathbf{W}_{N,h}) + (\mathcal{D}_i(\mathbf{P} - \mathbf{P}_{N-2,h}), \mathbf{U}^i_{N,h} - \mathbf{W}_{N,h}). \end{aligned} \tag{3.3}$$

Combining (3.3),(2.28) and Proposition 2.4, we have

$$\begin{aligned} C_1 \|\mathbf{U}^i_{N,h} - \mathbf{W}_{N,h}\|_1 & \leq C_2 \|\mathbf{U}_i - \mathbf{W}_{N,h}\|_1 + \sup_{\mathbf{V}_{N,h} \in [[\mathbf{P}^0_{N,h}]]} \frac{|(\mathcal{D}_i(\mathbf{P} - \mathbf{P}_{N-2,h}), \mathbf{V}_{N,h})|}{\|\mathbf{V}_{N,h}\|_1} \\ & \leq C_2 \|\mathbf{U}_i - \mathbf{W}_{N,h}\|_1 + \|\mathbf{P} - \mathbf{P}_{N-2,h}\|. \end{aligned} \tag{3.4}$$

Using (3.4), the triangular inequality, the approximation property of interpolation operator at LGL nodes and (3.1), one may have

$$\begin{aligned}\|\mathbf{U}_i - \mathbf{U}_{N,h}\|_1 &\le C \inf_{\mathbf{W}_{N,h}\in[[\mathbf{P}^0_{N,h}]]} \|\mathbf{U}_i - \mathbf{W}_{N,h}\|_1 + \|\mathbf{P} - \mathbf{P}_{N-2,h}\| \\ &\le C(h^{(\min\{N,k-1\})} N^{1-k}\|\mathbf{U}_i\|_k + h^{(\min\{N-1,k-1\})}(N-2)^{1-k}\|\mathbf{P}\|_{k-1}),\end{aligned}$$

where C is a positive constant. This completes the prove. □

4. Numerical experiments

The goal of this section is not only to provide numerical supports for Theorem 3.2 but also to apply Algorithm 3.1 to a model optimal control problem. The numerical domain for experiments is the unit square $\Omega := [0,1]\times[0,1]$. The LGL and LG points are used for velocity and pressure respectively. That is, let $\{\underline{\Phi}_\mu(x,y)\}_{\mu=1}^{M_v}$ be the Lagrange polynomial basis using LGL points for velocity and let $\{\underline{\Psi}_\mu(x,y)\}_{\mu=1}^{M_p}$ be the Lagrange polynomial basis using LG points for pressure.

The associated linear system corresponding to the bilinear form $\mathcal{B}(\cdot,\cdot)$ for pressure in Algorithm 3.1 can be expressed by

$$\widehat{\mathbf{B}}_{N^2}\underline{P} = \mathbf{b}, \tag{4.1}$$

where the matrix $\widehat{\mathbf{B}}_{N^2}$ and the load vector $\mathbf{b}$ can be approximated by numerical integration such that

$$\widehat{\mathbf{B}}_{N^2}(i,j) = (\mathcal{H}^{-1}\mathcal{D}_x\underline{\Psi}_i, \mathcal{D}_x\underline{\Psi}_j)_{LG} + (\mathcal{H}^{-1}\mathcal{D}_y\underline{\Psi}_i, \mathcal{D}_y\underline{\Psi}_j)_{LG}, \tag{4.2}$$

$$\mathbf{b}(j) = (\mathcal{H}^{-1}\mathbf{F}_1, \mathcal{D}_x\underline{\Psi}_j)_{LG} + (\mathcal{H}^{-1}\mathbf{F}_2, \mathcal{D}_y\underline{\Psi}_j)_{LG}, \tag{4.3}$$

respectively. Here, $(\cdot,\cdot)_{LG}$ denotes LG- numerical integration for $(\cdot,\cdot)$ (see [6], [11] for example). The associated linear system corresponding to the bilinear form $\mathcal{A}(\cdot,\cdot)$ for velocity in Algorithm 3.1 can be expressed by

$$\widehat{\mathbf{H}}_{N^2}\underline{U}^i = \mathbf{d}^i, \quad i = 1,2, \tag{4.4}$$

where the matrix $\widehat{\mathbf{H}}_{N^2}$ and the load vector $\mathbf{d}^k$ can be also approximated by LGL-numerical integration

$$\widehat{\mathbf{H}}_{N^2}(i,j) = \mathcal{A}_N(\underline{\Phi}_i, \underline{\Phi}_j), \quad \mathbf{d}^k(i) = (\mathbf{F}_k - \mathcal{D}_k\mathbf{P}_{N,h}, \underline{\Phi}_j)_{LGL}, \tag{4.5}$$

where $\mathcal{A}_N(\cdot,\cdot)$ and $(\cdot,\cdot)_{LGL}$ denote the numerical integrations using LGL-quadrature for $\mathcal{A}(\cdot,\cdot)$ and $(\cdot,\cdot)$ respectively (see [6], [11] for example). The numerical integrations for velocity and pressure have been employing 16 points Gauss-Lobatto-quadrature rule and 14 points Gauss-quadrature rule on each direction of subelement, respectively. All computations were performed by using the command $x = A\backslash b$ in Matlab to solve linear systems.

Table 1: Errors, convergence rates for different parameters ν, δ when $N = 4$.

(ν, δ)	h	$Err_1(N,h)$	$Rate$	$Err_0(N,h)$	$Rate$
(10,1)	2^{-1}	5.59e-002	-	4.71e-001	-
	4^{-1}	4.38e-003	3.67	2.30e-002	4.36
	6^{-1}	9.34e-004	3.81	5.26e-003	3.63
	8^{-1}	3.22e-004	3.71	1.95e-003	3.45
	10^{-1}	1.44e-004	3.60	9.30e-004	3.33
(1,1)	2^{-1}	7.33e-002	-	6.81e-002	-
	4^{-1}	1.44e-002	2.35	1.25e-002	2.44
	6^{-1}	4.35e-003	2.95	3.69e-003	3.01
	8^{-1}	1.86e-003	2.96	1.55e-003	3.01
	10^{-1}	9.56e-004	2.97	7.93e-004	3.01
$(1, 10^{-4})$	2^{-1}	3.25e-001	-	3.73e+000	-
	4^{-1}	1.03e-002	4.98	1.39e-001	4.74
	6^{-1}	3.02e-003	3.02	2.70e-002	4.04
	8^{-1}	1.30e-003	2.94	8.58e-003	3.99
	10^{-1}	6.70e-004	2.95	3.54e-003	3.96

For numerical tests, we will take the equation in [21] to show spectral convergence of the approximate solutions by Algorithm 3.1. The other equation will be used for addressing a control problem as an application of Algorithm 3.1.

Example 1. The first example is taken as in [21] with the given C^∞ exact solutions

$$\begin{aligned} u_1 = v_1 &= \sin^2(\pi x)\sin(\pi y)\cos(\pi y), \\ u_2 = v_2 &= -\sin^2(\pi y)\sin(\pi x)\cos(\pi x), \\ p = q &= \sin(2\pi x)\sin(2\pi y), \end{aligned}$$

so that the functions $\mathbf{g}$ and $\widehat{\mathbf{u}}$ are easily calculated, which are in fact

$$\mathbf{g} = -\nu\Delta\mathbf{u} + \nabla p + \frac{1}{\delta}\mathbf{v}, \quad \widehat{\mathbf{u}} = \mathbf{u} - \sqrt{\delta}\nabla q + \nu\Delta\mathbf{v}.$$

In this example, we verify the spectral convergence by calculating the H^1 errors between the spectral element solution $\mathbf{U}_{N,h}$ and the analytic solution $\mathbf{U}$ and the L^2- errors between the spectral element solution $\mathbf{P}_{N-2,h}$ and the exact solution $\mathbf{P}$. For this, let

$$Err_1(N,h) = \|\mathbf{U} - \mathbf{U}_{N,h}\|_1 \quad \text{and} \quad Err_0(N,h) := \|\mathbf{P} - \mathbf{P}_{N-2,h}\|.$$

Table 2: Errors, convergence rates for different parameters ν, δ when $N = 6$.

(ν, δ)	h	$Err_1(N,h)$	$Rate$	$Err_0(N,h)$	$Rate$
(10,1)	2^{-1}	1.20e-003	-	8.79e-003	-
	4^{-1}	2.45e-005	5.61	1.37e-004	6.00
	6^{-1}	2.48e-006	5.65	1.54e-005	5.40
	8^{-1}	5.10e-007	5.50	3.39e-006	5.26
	10^{-1}	1.54e-007	5.37	1.07e-006	5.18
(1,1)	2^{-1}	2.67e-003	-	1.85e-003	-
	4^{-1}	1.18e-004	4.51	9.76e-005	4.24
	6^{-1}	1.58e-005	4.95	1.28e-005	5.01
	8^{-1}	3.80e-006	4.97	3.03e-006	5.01
	10^{-1}	1.25e-006	4.98	9.89e-007	5.01
$(1, 10^{-4})$	2^{-1}	2.00e-003	-	6.21e-002	-
	4^{-1}	8.16e-005	4.62	6.95e-004	6.48
	6^{-1}	1.11e-005	4.92	6.21e-005	5.96
	8^{-1}	2.67e-006	4.95	1.13e-005	5.94
	10^{-1}	8.81e-007	4.97	3.01e-006	5.91

In Tables 1-2, the errors $Err_1(N,h), Err_0(N,h)$ and the convergence rate $Rate$ which is defined as

$$Rate \quad := \quad \frac{\log((Err_i(N,h_1))/Err_i(N,h_2))}{\log(h_1/h_2)}, \qquad i = 0, 1$$

are demonstrated for $h = \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}$ and $N = 4, 6$. The expected convergence according to Theorem 3.2 is $O(h^3)$ and $O(h^5)$ for $N = 4$ and 6, respectively. The numerical results in Tables 1-2 for two cases ($\nu = 10, \delta = 1$) and ($\nu = 1, \delta = 1$) are exactly matched with the Theorem 3.2 as expected. But the numerical results for pressure with ($\nu = 1, \delta = 10^{-4}$) shows the convergence rate about 4 if $N = 4$ and 6 if $N = 6$, which are better results than expected in Theorem 3.2.

Figure 1 presents the convergence curves of the errors, $Err_1(N,h), Err_0(N,h)$, in logarithmic scales for error direction, for $N = 4, 6, 8, 10, 12$ and $h = \frac{1}{2}, \frac{1}{4}$. The results of Figure 1 reveal that both errors have an exponential growth which are exactly matched with Theorem 3.2.

Example 2. This example is for applying the developed algorithm to a model optimal control problem. The computations were done for two elements($h = \frac{1}{2}$) and $N = 12$. The target velocity $\widehat{\mathbf{u}}(x,y) = (\hat{u}_1(x,y), \hat{u}_2(x,y))$ is chosen as examples in [15, 18] and $\mathbf{g} = 0$, where

$$\hat{u}_1(x,y) = \frac{d}{dy}(g(x)g(y)), \quad \hat{u}_2(x,y) = -\frac{d}{dx}(g(x)g(y)),$$

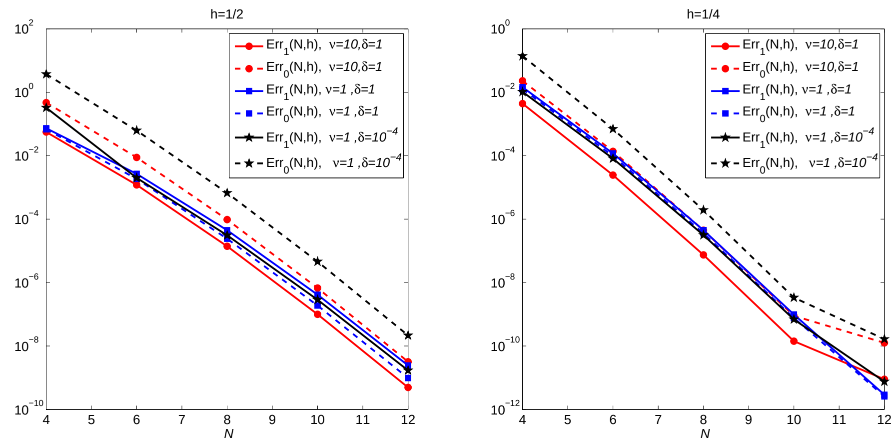

Figure 1: Errors for different parameters.

with $g(t) = (1 - \cos(\pi t))(1 - t)^2$.

In Table 3, we report L^2 errors between the spectral element solution $\mathbf{u}_{N,h}$ and the target velocity $\widehat{\mathbf{u}}$ and the L^2- norms of the optimal control $\mathbf{f}_{N,h}$ plus the magnitude of the functional $\mathcal{J}(\mathbf{u}_{N,h}, \mathbf{f}_{N,h})$ as δ varies from 1 to 10^{-6} for a given viscosity $\nu = 1, 10^{-1}$ and 10^{-2}. As expected, the L^2- errors $\|\mathbf{u}_{N,h} - \widehat{\mathbf{u}}\|$ decreases as δ decreases.

Hence, according to numerical demonstration, one may use the block LU factorization to deal with a control problem in the sense of convergence.

5. Conclusion

In this article we consider the backward substitution algorithm to deal with the coupled stokes equations by rearranging momentum equations and variables. The direct LU factorization requires solving Helmholz equations for pressure variables many times. This is one of disadvantages. To overcome disadvantages, one may adopt the splitting approaches used in Navier-Stokes equations([1], [9], [12] and [19]) for the coupled stokes equations by the usage of the penalty parameter δ in (2.1) like a role of time step in Navier-Stokes equations. This approach will be dealt with in a coming paper. The convergence theorem of our algorithm is provided and several numerical performances have been shown to justify the convergence theorem.

Table 3: The values $\|\mathbf{u}_{N,h}-\widehat{\mathbf{u}}\|$, $\|\mathbf{f}_{N,h}\|$ and cost functional $\mathcal{J}(\mathbf{u}_{N,h},\mathbf{f}_{N,h})$ when $E=2, N=12$.

ν	δ	$\|\mathbf{u}_{N,h}-\widehat{\mathbf{u}}\|$	$\|\mathbf{f}_{N,h}\|$	$\mathcal{J}(\mathbf{u}_{N,h},\mathbf{f}_{N,h})$
1	10^0	1.27e-001	2.37e-003	8.12e-003
	10^{-1}	1.27e-001	3.36e-002	8.09e-003
	10^{-2}	1.23e-001	2.23e-001	7.85e-003
	10^{-3}	9.51e-002	1.75e+000	6.05e-003
	10^{-4}	3.14e-002	5.32e+000	1.91e-003
	10^{-5}	5.21e-003	6.98e+000	2.57e-004
	10^{-6}	9.02e-004	7.36e+000	2.75e-005
10^{-1}	10^0	1.23e-001	2.29e-002	7.85e-003
	10^{-1}	9.51e-002	1.75e-001	6.05e-003
	10^{-2}	3.14e-002	5.32e-001	1.91e-003
	10^{-3}	5.21e-003	6.98e-001	2.57e-004
	10^{-4}	9.02e-004	7.36e-001	2.75e-005
	10^{-5}	1.90e-004	7.47e-001	2.81e-006
	10^{-6}	4.64e-005	7.52e-001	2.84e-007
10^{-2}	10^0	3.14e-002	5.32e-002	1.91e-003
	10^{-1}	5.21e-003	6.98e-002	2.57e-004
	10^{-2}	9.02e-004	7.36e-002	2.75e-005
	10^{-3}	1.90e-004	7.47e-002	2.81e-006
	10^{-4}	4.64e-005	7.52e-002	2.84e-007
	10^{-5}	7.67e-006	7.55e-002	2.86e-008
	10^{-6}	8.30e-007	7.56e-002	2.86e-009

References

[1] J. Blair Perot, *An analysis of the fractional step method*, J. Comput. Phys., **108**(1993), 51-58.

[2] P. Bochev, *Least-squares methods for optimal control*, Nonlinear Anal., **30**(1997), 1875-1885.

[3] P. Bochev and M. Gunzburger, *Least-squares finite element methods for optimization and control problems for the Stokes equations*, Comput. Math. Appl., **48**(2004), 1035-1057.

[4] P. Bochev and M. Gunzburger, *Least-squares finite element methods for optimality systems arising in optimization and control problems*, SIAM J. Numer. Anal., **43**(2006), 2517-2543.

[5] F. Brezzi and M. Fortin, *Mixed and hybrid finite element methods*, Springer Ser. Comput. Math. **15**, Springer-Verlag, New York, (1991).

[6] C. Canuto, M. Y. Hussaini, A. Quarteroni and T. A. Zang, *Spectral Methods:Fundamentals in Single Domains*, Springer-Verlag, Berlin Heidelberg, (2006).

[7] C. Canuto, M. Y. Hussaini, A. Quarteroni and T.A. Zang, *Spectral Methods:Evolution to Complex Geometries and Applications to Fluid Dynamics*, Springer-Verlag, Berlin Heidelberg, (2007).

[8] Y. Chen, N. Yi and W. Liu, *A Legendre-Galerkin spectral method for optimal control problems governed by elliptic equations*, SIAM J. Numer. Anal., **46**(2008), 2254-2275.

[9] W. Couzy, *Spectral Element Discretization of the Unsteady Navier-Stokes Equations and Its Iterative Solution on Parallel Computers*, Thèse No. 1380 École Polytechnique Fédérale de Lausanne, (1995).

[10] P. Dartzi and L. Harvard, *A new approximate LU factorization scheme for the Reynolds-averaged Navier-Stokes equations*, AIAA J., **26**(1988), 163-171.

[11] M. O. Deville, P. F. Fischer and E. H. Mund, *High-Order Methods for Incompressible Fluid Flow*, Cambridge Monographs on Applied and Computational Mathematics, Cambridge Univ. Press. Cambridge, UK, (2002).

[12] P. F. Fischer, *An overlapping Schwarz method for spectral element solution of the incompressible Navier-Stokes equations*, J. Comp. Phys., **133**(1997), 84-101.

[13] R. Ghanem and H. Sissaoui, *A posteriori error estimate by a spectral method of an elliptic optimal control problem*, J. Comput. Math. Optim., **2**(2006), 111-135.

[14] V. Girault and P. A. Raviart, *Finite Element Methods for Navier-Stokes Equations: Theory and Algorithms*, Springer-Verlag, New York, (1986).

[15] M. Gunzburger, *Perspectives in Flow Control and Optimization*, SIAM, Philadelphia, (2003).

[16] V. Hutson and J. S. Pym, *Applications of Functional Analysis and Opertor Theory*, Academic Press Inc. New York, (1980).

[17] S. D. Kim, *Uzawa algorithms for coupled Stokes equations from the optimal control problem*, Calcolo., **46**(2009), 37-47.

[18] S. D. Kim, S. Ryu and H. Lee, *First-Order System Least-Squares Methods for an Optimal Control Problem by the Stokes Flow*, SIAM J. Numer. Anal., **47**(2009), 1524-1545.

[19] Y. Maday, A. T. Patera, and E. M. Rønquist, *An operator-integration-factor splitting method for time-dependent problems: Application to incompressible fluid flow*, J. Sci. Comput., **5**(1990), 263-292.

[20] Y. Maday, A. T. Patera, and E. M. Rønquist, *The $P_N - P_{N-2}$ method for the approximation of the Stokes problem*, Tech. Rep. 92009, Rept. of Mech. Engr., MIT., (1992).

[21] A. Rösch and B. Vexler, *Optimal control of the stokes equations: A priori error analysis for finite element discretization with postprocessing*, SIAM J. Numer. Anal., **44**(2006), 1903-1920.

[22] O. Shigeru and K. Kunio, *An approximate LU factorization method for the compressible Navier-Stokes equations*, J. Comput. Phys., **63**(1986), 157-167.