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MINIMAL CLOZ-COVERS AND BOOLEAN ALGEBRAS

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Abstract. In this paper, we first show that for any space X, there is a Boolean subalgebra $\mathcal{G}(z_X)$ of $R(X)$ containg $\mathcal{G}(X)$. Let X be a strongly zero-dimensional space such that $z_\beta^{-1}(X)$ is the minimal cloz-coevr of X, where $(E_{cc}(\beta X), z_\beta)$ is the minimal cloz-cover of βX. We show that the minimal cloz-cover $E_{cc}(X)$ of X is a subspace of the Stone space $S(\mathcal{G}(z_X))$ of $\mathcal{G}(z_X)$ and that $E_{cc}(X)$ is a strongly zero-dimensional space if and only if $\beta E_{cc}(X)$ and $S(\mathcal{G}(z_X))$ are homeomorphic. Using these, we show that $E_{cc}(X)$ is a strongly zero-dimensional space and $\mathcal{G}(z_X) = \mathcal{G}(X)$ if and only if $\beta E_{cc}(X) = E_{cc}(\beta X)$.

1. Introduction

All spaces in this paper are Tychonoff spaces and βX denotes the Stone-Čech compactification of a space X .

Iliadis constructed the absolute of Hausdorff spaces, which is the minimal extremally disconnected cover of Hausdorff spaces and they turn out to be the perfect onto projective covers([5]). To generalize extremally disconnected spaces, basically disconnected spaces, quasi-F spaces and cloz-spaces have been introduced and their minimal covers have been studied by various aurthors([2], [3], [4], [9]). In these ramifications, minimal covers of compact spaces can be nisely characterized.

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In particular, Henriksen, Vermeer and Woods ([3]) introduced the notion of cloz-spaces and they showed that every compact space X has a minimal cloz-cover $(E_{cc}(X), z_X)$. Open questions in the theory of cloz-spaces concerns with the minimal cloz-covers of non-compact spaces and the relation between $E_{cc}(\beta X)$ and $E_{cc}(X)$([3]). For this problem, we have partial answers in [6] and [7]. Indeed, it is shown that for a weakly Lindelöf space X, $\beta E_{cc}(X) = E_{cc}(\beta X)$([4], [6]) and every spaces has a minimal cloz-cover([7]).

In this paper, we first show that for any space X, there is a Boolean subalgebra $\mathcal{G}(z_X)$ of $R(X)$ such that $\mathcal{G}(X) \subseteq \mathcal{G}(z_X)$. Let X be a strongly zero-dimensional space such that $z_\beta^{-1}(X)$ is the minimal cloz-cover of X. We show that $E_{cc}(X)$ is a subspace of the Stone space $S(\mathcal{G}(z_X))$ of $\mathcal{G}(z_X)$ and that $\beta E_{cc}(X)$ is a zero-dimensional space if and only if $\beta E_{cc}(X)$ and $S(\mathcal{G}(z_X))$ are homeomorphic. Finally, we show that $E_{cc}(X)$ is a strongly zero-dimensional space and $\mathcal{G}(z_X) = \mathcal{G}(X)$ if and only if $\beta E_{cc}(X) = E_{cc}(\beta X)$..

For the terminology, we refer to [1] and [8].

2. Minimal cloz-covers and Boolean algebras

The set $\mathcal{R}(X)$ of all regular closed sets in a space X, when partially ordered by inclusion, becomes a complete Boolean algebra, in which the join, meet, and complementation operations are defined as follows : for any $A \in \mathcal{R}(X)$ and any $\{A_i \mid i \in I\} \subseteq \mathcal{R}(X)$,

$\vee\{A_i \mid i \in I\} = cl_X(\cup\{A_i \mid i \in I\})$,

$\wedge\{A_i \mid i \in I\} = cl_X(int_X(\cap\{A_i \mid i \in I\}))$, and

$A' = cl_X(X - A)$

and a sublattice of $\mathcal{R}(X)$ is a subset of $\mathcal{R}(X)$ that contains $\emptyset$, X and is closed under finite joins and meets.

Recall that a map $f : Y \longrightarrow X$ is called *a covering map* if it is a continuous, onto, perfect, and irreducible map.

Lemma 2.1. *([6])*

(1) *Let $f : Y \longrightarrow X$ be a covering map. Then the map $\psi : R(Y) \longrightarrow R(X)$, defined by $\psi(A) = A \cap X$, is a Boolean isomorphism and the inverse map ψ^{-1} of ψ is given by*

$$\psi^{-1}(B) = cl_Y(f^{-1}(int_X(B))) = cl_Y(int_Y(f^{-1}(B))).$$

(2) *Let X be a dense subspace of a space K. Then the map $\phi : R(K) \longrightarrow R(X)$, defined by $\phi(A) = A \cap X$, is a Boolean isomorphism and the inverse map ϕ^{-1} of ϕ is given by $\phi^{-1}(B) = cl_K(B)$.*

Definition 2.2. Let X be a space.
(1) A cozero-set C in X is said to be *a complemented cozero-set in* X if there is a cozero-set D in X such that $C \cap D = \emptyset$ and $C \cup D$ is a dense subset of X. In case, $\{C, D\}$ is called *a complemented pair of cozero-sets in* X.
(2) Let $\mathcal{G}(X) = \{cl_X(C) \mid C$ is a complemented cozero-set in $X\}$.

Let X be a space and $Z(X)^{\#} = \{cl_X(int_X(A)) \mid A$ is a zero-set in $X\}$. Suppose that $\{C, D\}$ is a complemented pair of cozero-sets in X. Then $cl_X(C) = cl_X(X - D)$ and since $cl_X(X - D) \in Z(X)^{\#}$, $cl_X(C) \in Z(X)^{\#}$. Hence $\mathcal{G}(X) = \{A \in Z(X)^{\#} \mid A' \in Z(X)^{\#}\}$ and $\mathcal{G}(X)$ is a Boolean subalgebra of $R(X)$.

Since X is C^*-embedded in βX, by Lemma 2.1., $\mathcal{G}(X)$ and $\mathcal{G}(\beta X)$ are Boolean isomorphic.

Definition 2.3. ([3]) A space X is called *a cloz-space* if every element of $\mathcal{G}(X)$ is a clopen set in X.

A space X is a cloz-space if and only if βX is a cloz-space([3]).

Definition 2.4. Let X be a space.
(1) A pair (Y, f) is called *a cloz-cover of* X if Y is a cloz-space and $f : Y \longrightarrow X$ is a covering map.
(2) A cloz-cover (Y, f) of X is called *a minimal cloz-cover of* X if for any cloz-cover (Z, g) of X, there is a covering map $h : Z \longrightarrow Y$ with $f \circ h = g$.

Let $\mathcal{B}$ be a Boolean subalgebra of $R(X)$. Let $S(\mathcal{B}) = \{\alpha \mid \alpha$ is a $\mathcal{B}$-ultrafilter$\}$ and for any $B \in \mathcal{B}$, let $\Sigma_B^{\mathcal{B}} = \{\alpha \in S(\mathcal{B}) \mid B \in \alpha\}$. Then the space $S(\mathcal{B})$, equipped with the topology for which $\{\Sigma_B^{\mathcal{B}} \mid B \in \mathcal{B}\}$ is a base, called *the Stone-space of* $\mathcal{B}$. Then $S(\mathcal{B})$ is a compact, zero-dimensional space([8]).

Henriksen, Vermeer and Woods showed that every compact space has the minimal cloz-cover. Let X be a compact space, $\mathcal{S}(\mathcal{G}(X))$ the Stone-space of $\mathcal{G}(X)$ and $E_{cc}(X) = \{(\alpha, x) \mid x \in \cap\{A \mid A \in \alpha\}\}$ the subspace of the product space $\mathcal{S}(\mathcal{G}(X)) \times X$. Then $(E_{cc}(X), z_X)$ is the minimal cloz-cover of X, where $z_X((\alpha, x)) = x$([3]).

A space X is called *a weakly Lindelöf space* if for any open cover $\mathcal{U}$ of X, there is a countable subfamily $\mathcal{V}$ of $\mathcal{U}$ such that $\cup\{V \mid V \in \mathcal{V}\}$ is a dense subset of X.

Let X be a weakly Lindelöf space. Then $E_{cc}(X)$ is the subspace $\{(\alpha, x) \in \mathcal{S}(\mathcal{G}(\beta X)) \times X \mid x \in \cap\{A \mid A \in \alpha\}\}$ of $\mathcal{S}(\mathcal{G}(\beta X)) \times X$ and $E_{cc}(X)$ is a dense C^*-embedded subspace of $E_{cc}(\beta X)$, that is, $\beta E_{cc}(X) = E_{cc}(\beta X)$([4], [6]). Moreover, it was shown that every space has a minimal cloz-cover([7]).

Let X be a space. Since $\mathcal{G}(X)$ and $\mathcal{G}(\beta X)$ are Boolean isomorphic, $S(\mathcal{G}(X))$ and $S(\mathcal{G}(\beta X))$ are homeomorphic.

Let X, Y be spaces and $f : Y \longrightarrow X$ a map. For any $U \subseteq X$, let $f_U : f^{-1}(U) \longrightarrow U$ denote the restriction and co-restriction of f with respect to $f^{-1}(U)$ and U, respectively.

For any space X, let $(E_{cc}(\beta X), z_\beta)$ denote the minimal cloz-cover of βX.

Lemma 2.5. *([6]) Let X be a space. If $z_\beta^{-1}(X)$ is a cloz-space, then $(z_\beta^{-1}(X), z_{\beta_X})$ is the minimal cloz-cover of X.*

For any covering map $f : Y \longrightarrow X$, let $\mathcal{G}(f) = \{cl_Y(int_X(f(A))) \mid A \in \mathcal{G}(Y)\}$. Since $R(E_{cc}(X))$ and $R(X)$ are isomorphic and $\mathcal{G}(E_{cc}(X))$ is a Boolean algebra, by Lemma 2.1, $\mathcal{G}(z_X)$ is a Boolean algebra.

Definition 2.6. Let X be a space and $\mathcal{B}$ a sublattice of $R(X)$. Then a $\mathcal{B}$-filter $\mathcal{F}$ is called *fixed* if $\cap\{F \mid F \in \mathcal{F}\} \neq \emptyset$.

Let X be a space and for any $\mathcal{G}(z_X)$-ultrafilter α, let $\alpha_c = \{A \in \mathcal{G}(E_{cc}(X)) \mid z_X(A) \in \alpha\}$.

Proposition 2.7. *Let X be a space. Then we have the following :*
(1) *$\mathcal{G}(z_X)$ is a Boolean subalgebra of $R(X)$ such that $\mathcal{G}(X) \subseteq \mathcal{G}(z_X)$.*
(2) *Suppose that α is a fixed $\mathcal{G}(z_X)$-ultrafilter. Then α_c is a fixed $\mathcal{G}(E_{cc}(X))$-ultrafilter.*

Proof. (1) Clearly, $\mathcal{G}(z_X)$ is a Boolean subalgebra of $R(X)$. Let $\{C, D\}$ be a complemented pair of cozero-sets in X. Since z_X is a continuous map, $z_X^{-1}(C)$ and $z_X^{-1}(D)$ are cozero-sets in $E_{cc}(X)$ such that $z_X^{-1}(C) \cap z_X^{-1}(D) = \emptyset$ and $z_X^{-1}(C) \cup z_X^{-1}(D)$ is dense in $E_{cc}(X)$, because z_X is irreducible. That is, $\{z_X^{-1}(C), z_X^{-1}(D)\}$ is a complemented pair of cozero-sets in $E_{cc}(X)$. Since $z_X(cl_{E_{cc}(X)}(z_X^{-1}(C))) = cl_X(C) \in \mathcal{G}(z_X)$, $\mathcal{G}(X) \subseteq \mathcal{G}(z_X)$.

(2) Clearly, α_c is a $\mathcal{G}(E_{cc}(X))$-filter. Suppose that $A \in \mathcal{G}(E_{cc}(X)) - \alpha_c$. Then $z_X(A) \notin \alpha$. Since α is a $\mathcal{G}(z_X)$-ultrafilter, there is a $C \in \alpha$ such that $C \wedge z_X(A) = \emptyset$. By Lemma 2.1., $A \wedge cl_{E_{cc}(X)}(z_X^{-1}(int_X(C))) = \emptyset$ and by (1), $cl_{E_{cc}(X)}(z_X^{-1}(int_X(C))) \in \mathcal{G}(E_{cc}(X))$.
Since $z_X(cl_{E_{cc}(X)}(z_X^{-1}(int_X(C)))) = C \in \alpha$, $cl_{E_{cc}(X)}(z_X^{-1}(int_X(C))) \in \alpha_c$. Hence α_c is a $\mathcal{G}(E_{cc}(X))$-ultrafilter. Since α is fixed, there is an $x \in \cap\{B \mid B \in \alpha\}$. Then $\{A \cap z_X^{-1}(x) \mid A \in \alpha_c\}$ has a family of closed sets in $z_X^{-1}(x)$ with the finite intersection property. Since $z_X^{-1}(x)$ is a compact subset of $E_{cc}(X)$, $\cap\{A \cap z_X^{-1}(x) \mid A \in \alpha_c\} \neq \emptyset$ and hence $\cap\{B \mid B \in \alpha_c\} \neq \emptyset$. Thus α_c is a fixed $\mathcal{G}(E_{cc}(X))$-ultrafilter. □

Let X be a space and $ccX = \{\alpha \mid \alpha \text{ is a fixed } \mathcal{G}(z_X)\text{-ultrafilter}\}$ the subspace of $S(\mathcal{G}(z_X))$.

If $\mathcal{G}(X)$ is a base for closed sets in X or $\mathcal{G}(E_{cc}(X))$ is a base for closed sets in $E_{cc}(X)$, then for any fixed $\mathcal{G}(z_X)$-ultrafilter α, $\mid \cap\{B \mid B \in \alpha_c\} \mid = 1$. Indeed, if X is a zero-dimensional space, then $\mathcal{G}(X)$ is a base for closed sets in X. Using this, we have the following :

Proposition 2.8. *Let X be a space, $\mathcal{G}(X)$ a base for closed sets in X and $Y = \{(\alpha, x) \mid x \in \cap\alpha\}$ the subspace of the product space $S(\mathcal{G}(z_X)) \times X$. Then the map $t : Y \longrightarrow ccX$, defined by $t((\alpha, x)) = \alpha$, is a homeomorphism.*

For any zero-dimensional space X, define a map $h_X : ccX \longrightarrow E_{cc}(X)$ by $h_X(\alpha) = \cap\alpha_c$. In the following, let $\Sigma_B = \Sigma_B^{\mathcal{G}(z_X)}$ for all $B \in \mathcal{G}(z_X)$.

We recall that a space X is called *a strongly zero-dimensional space* if βX is a zero-dimensional space.

Theorem 2.9. *Let X be a strongly zero-dimensional space such that $z_\beta^{-1}(X)$ is a cloz-space. Then $h_X : ccX \longrightarrow E_{cc}(X)$ is a homeomorphism.*

Proof. Let $\alpha, \delta \in ccX$. Suppose that $\alpha \neq \delta$. Then there are A, B in $\mathcal{G}(E_{cc}(X))$ such that $z_X(A) \in \alpha$, $z_X(B) \in \delta$ and $z_X(A) \wedge z_X(B) = \emptyset$. Then $A \in \alpha_c$, $B \in \delta_c$ and by Lemma 2.1., $z_X(A) \wedge z_X(B) = z_X(A \wedge B) = \emptyset$ and $A \wedge B = \emptyset$. Since $E_{cc}(X)$ is a cloz-space, A and B are clopen sets in $E_{cc}(X)$ and hence $A \cap B = \emptyset$. Note that $h_X(\alpha) \in A$ and $h_X(\delta) \in B$. Hence $h_X(\alpha) \neq h_X(\delta)$ and so h_X is an one-to-one map.

Let $y \in E_{cc}(X)$ and $\gamma = \{z_X(C) \mid y \in C \in \mathcal{G}(E_{cc}(X))\}$. Clearly, γ is a $\mathcal{G}(z_X)$-filter and $\cap\{B \mid B \in \gamma\} \neq \emptyset$. Let $D \in \mathcal{G}(E_{cc}(X))$ such that $z_X(D) \notin \gamma$. Then $y \notin D$. Since $y \in D' = E_{cc}(X) - D$, $z_X(D') =$

$z_X(D)' \in \gamma$ and hence γ is a $\mathcal{G}(z_X)$-ultrafilter. Hence $\gamma \in ccX$ and $h_X(\gamma) = y$. Thus h_X is an onto map.

Let F be a closed set in $E_{cc}(X)$. By Lemma 2.5., $E_{cc}(X) = z_\beta^{-1}(X)$ and $(\gamma, x) \in E_{cc}(X) - F$. Since X is a strongly zero-dimensional space, $\mathcal{G}(E_{cc}(\beta X))$ is a base for closed sets in $E_{cc}(\beta X)$. Hence there is an $A \in \mathcal{G}(\beta X)$ such that $\gamma \in \Sigma_A^{\mathcal{G}(\beta X)}$ and $(\Sigma_A^{\mathcal{G}(\beta X)} \times U) \cap F = \emptyset$ for some clopen neighborhood U of x in βX. Let $V = (\Sigma_A^{\mathcal{G}(\beta X)} \times U) \cap E_{cc}(X)$. Since V is a clopen set in $E_{cc}(X)$, $V \in \mathcal{G}(E_{cc}(X))$ and hence $\mathcal{G}(E_{cc}(X))$ is a base for closed sets in $E_{cc}(X)$.

Let $E \in \mathcal{G}(E_{cc}(X))$. Suppsoe that $\mu \in ccX - h_X^{-1}(E)$. Then $h_X(\mu) = \cap\mu_c \notin E$ and so $E \notin \mu_c$. By the definition of μ_c, $z_X(E) \notin \mu$ and hence $\mu \notin \Sigma_{z_X(E)}$. Thus $\Sigma_{z_X(E)} \subseteq h_X^{-1}(E)$.

Suppose that $\theta \in h_X^{-1}(E)$. Then $h_X(\theta) \in E$ and hence for any $A \in \theta_c$, $A \cap E \neq \emptyset$. Since $E_{cc}(X)$ is a cloz-space, $A \wedge E \neq \emptyset$ for all $A \in \theta_c$. Since θ_c is a $\mathcal{G}(E_{cc}(X))$-ultrafilter, $E \in \theta_c$ and so $z_X(E) \in \theta$. Since $\theta \in \Sigma_{z_X(E)}$, $\Sigma_{z_X(E)} = h_X^{-1}(E)$.

Since h_X is an one-to-one, onto map and $\mathcal{G}(E_{cc}(X))$ is a base for closed sets in $E_{cc}(X)$, h_X is a homeomorphism. □

Let $c_X = z_X \circ h_X$. Then we have the following :

Corollary 2.10. *Let X be a strongly zero-dimensional space such that $z_\beta^{-1}(X)$ is a cloz-space. Then (ccX, c_X) is the minimal cloz-cover of X and $c_X(\alpha) = \cap\alpha$ for all $\alpha \in ccX$.*

Let X be a strongly zero-dimensional space such that $z_\beta^{-1}(X)$ is a cloz-space. Since βX is a zero-dimensional space, $B(\beta X) = \{B \mid B$ is a clopen set in $\beta X\}$ is a Boolean subalgeba of $\mathcal{G}(z_X)$. Since $\mathcal{G}(z_X)$ and $\{cl_{\beta X}(A) \mid A \in \mathcal{G}(z_X)\}$ is Boolean homeomorphic, the map $m : S(\mathcal{G}(z_X)) \longrightarrow S(B(\beta X))$, defined by $m(\alpha) = \alpha \cap B(\beta X)$, is a covering map. Since $n : S(B(\beta X)) \longrightarrow \beta X$, defined by $n(\alpha) = \cap\{A \mid A \in \alpha\}$, is a covering. Hence $f : S(\mathcal{G}(z_X)) \longrightarrow \beta X$, defined by $f(\alpha) = \cap\{cl_{\beta X}(A) \mid A \in \alpha\}$, is a covering map([8]).

Theorem 2.11. *Let X be a strongly zero-dimensional space such that $z_\beta^{-1}(X)$ is a cloz-space. Then $E_{cc}(X)$ is a strongly zero-dimensional space if and only if there is a homeomorphism $k_X : \beta E_{cc}(X) \longrightarrow S(\mathcal{G}(z_X))$ such that $k_X \circ \beta_{E_{cc}(X)} = j \circ h_X^{-1}$, where $j : ccX \longrightarrow S(\mathcal{G}(z_X))$ is the dense embedding.*

Proof. (⇒). By Theorem 2.9., $\beta(ccX) = \beta E_{cc}(X)$ and since $S(\mathcal{G}(z_X))$ is a compactification of ccX, there is a continuous map $k_X : \beta E_{cc}(X) \longrightarrow S(\mathcal{G}(z_X))$ such that $k_X \circ \beta_{E_{cc}(X)} = j \circ h_X^{-1}$. Since $\beta E_{cc}(X)$ and $S(\mathcal{G}(z_X))$ are compact spaces and $\beta_{E_{cc}(X)}$ and j are dense embeddings, k_X is a covering map.

Let $p \neq q$ in $\beta E_{cc}(X)$. Since $\beta E_{cc}(X)$ is a zero-dimensional space, there is a clopen set B in $\beta E_{cc}(X)$ such that $p \in B$ and $q \notin B$. Then $B \cap E_{cc}(X) \in \mathcal{G}(E_{cc}(X))$. Note that $f \circ k_X \circ \beta_{E_{cc}(X)} = \beta_X \circ z_X$ and $f \circ k_X$ is a covering map. Then $z_X(B \cap E_{cc}(X)) = f(k_X(B)) \cap \beta X$ and $z_X(B \cap E_{cc}(X)) \in \mathcal{G}(z_X)$. Hence $f(\Sigma_{z_X(B\cap E_{cc}(X))}) = f(k_X(B))$. Since f is a covering map and $\Sigma_{z_X(B\cap E_{cc}(X))}, k_X(B)$ are regular closed sets in $S(\mathcal{G}(z_X))$, by Lemma 2.1., $\Sigma_{z_X(B\cap E_{cc}(X))} = k_X(B)$. Since $k_X(p) \in k_X(B)$, $k_X(p) \in \Sigma_{z_X(B\cap E_{cc}(X))}$. Similarly, $k_X(q) \in \Sigma_{z_X(B'\cap E_{cc}(X))}$. Note that

$$\begin{aligned}
&\Sigma_{z_X(B\cap E_{cc}(X))} \cap \Sigma_{z_X(B'\cap E_{cc}(X))}\\
&= \Sigma_{z_X(B\cap E_{cc}(X)) \wedge z_X(B'\cap E_{cc}(X))}\\
&= \Sigma_{z_X((B\cap E_{cc}(X)) \wedge (B'\cap E_{cc}(X)))}\\
&= \Sigma_{\emptyset} = \emptyset.
\end{aligned}$$

Hence $k_X(p) \neq k_X(q)$ and so k_X is an one-to-one map. Thus k_X is a homeomorphism.

(⇐) Since $S(\mathcal{G}(z_X))$ is a zero-dimensional space, $\beta E_{cc}(X)$ is a zero-dimensional space. □

Let X be a strongly zero-dimensional space. Then $\beta E_{cc}(X) = E_{cc}(\beta X)$ if and only if $E_{cc}(X)$ is $z^{\#}$-embedded in $E_{cc}(\beta X)$, that is, for any $A \in Z(E_{cc}(X))^{\#}$, there is a $B \in Z(E_{cc}(\beta X))^{\#}$ such that $A = B \cap E_{cc}(X)$. Morever, if $\beta E_{cc}(X) = E_{cc}(\beta X)$, then $z_X : E_{cc}(X) \longrightarrow X$ is $z^{\#}$-irreducible, that is, $z_X(Z(E_{cc}(X))^{\#}) \subseteq Z(X)^{\#}$ ([7]).

For any strongly zero-dimensional space X, $S(\mathcal{G}(\beta X))$ and βX are zero-dimensional space and hence $E_{cc}(\beta X)$ is a zero-dimensional space. Using these, we have the following :

Corollary 2.12. *Let X be a strongly zero-dimensional space such that $z_\beta^{-1}(X)$ is a cloz-space. Then $E_{cc}(X)$ is a strongly zero-dimensional space and $\mathcal{G}(z_X) = \mathcal{G}(X)$ if and only if $\beta E_{cc}(X) = E_{cc}(\beta X)$.*

Proof. (⇒) Since $E_{cc}(X)$ is a strongly zero-dimensional space, by Theorem 2. 11., $S(\mathcal{G}(z_X)) = \beta E_{cc}(X)$. That is, k_X is a homeomorphism. Since $S(\mathcal{G}(z_X))$ is a cloz-spce, there is a covering map $g : S(\mathcal{G}(z_X)) \longrightarrow$

$E_{cc}(\beta X)$ such that $z_\beta \circ g = f$. Suppose that $\alpha \neq \delta$ in $S(\mathcal{G}(z_X))$. Then there is an $A \in \mathcal{G}(z_X)$ such that $\alpha \in \Sigma_A$ and $\delta \in \Sigma_{A'}$. Since $cl_{E_{cc}(\beta X)}(z_\beta^{-1}(A)) \in \mathcal{G}(E_{cc}(\beta X))$, $cl_{E_{cc}(\beta X)}(z_\beta^{-1}(A))$ is a clopen set in $E_{cc}(\beta X)$ and $cl_{E_{cc}(\beta X)}(z_\beta^{-1}(A')) = E_{cc}(\beta X) - cl_{E_{cc}(\beta X)}(z_\beta^{-1}(A))$. Since $g(\alpha) \in cl_{E_{cc}(\beta X)}(z_\beta^{-1}(A))$ and $g(\delta) \in cl_{E_{cc}(\beta X)}(z_\beta^{-1}(A'))$, $g(\alpha) \neq g(\delta)$ and g is a homeomorphism. Thus $\beta E_{cc}(X) = E_{cc}(\beta X)$.

($\Leftarrow$) Clearly $\beta E_{cc}(X)$ is a zero-dimensional space. Since $\beta E_{cc}(X) = E_{cc}(\beta X)$, $z_X : E_{cc}(X) \longrightarrow X$ is $z^\#$-irreducible. Since $\mathcal{G}(E_{cc}(X)) = \{A \in Z(E_{cc}(X))^\# \mid A' \in Z(E_{cc}(X))^\#\}$, $z_X(\mathcal{G}(E_{cc}(X))) \subseteq \{z_X(A) \in Z(X)^\# \mid z_X(A)' \in Z(X)^\#\}$. Hence $\mathcal{G}(z_X) = \mathcal{G}(X)$. □

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