EVERY LINK IS A BOUNDARY OF A COMPLETE BIPARTITE GRAPH $K_{2,n}$

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ABSTRACT. A voltage assignment on a graph was used to enumerate all possible 2-cell embeddings of a graph onto surfaces. The boundary of the surface which is obtained from 0 voltage on every edges of a very special diagram of a complete bipartite graph $K_{m,n}$ is surprisingly the (m, n) torus link. In the present article, we prove that every link is the boundary of a complete bipartite multi-graph $K_{m,n}$ for which voltage assignments are either -1 or 1 and that every link is the boundary of a complete bipartite graph $K_{2,n}$ for which voltage assignments are either $-1, 0$ or 1 where edges in the diagram of graphs may be linked but not knotted.

1. Introduction

Let L be a link in $\mathbb{S}^3$. A compact orientable surface $\mathcal{F}$ is a *Seifert surface* of L if the boundary of $\mathcal{F}$ is isotopic to L. The existence of such a surface was first proven by Seifert using an algorithm on a diagram of L, named after him as *Seifert's algorithm* [10]. Some of Seifert surfaces feature some extra structures. For example, Rudolph has introduced several plumbing Seifert surfaces [8,9]. These surfaces have been studied extensively for the fibredness of links and surfaces [4, 11]. The third author proved the existence of basket surfaces, flat plumbing surfaces
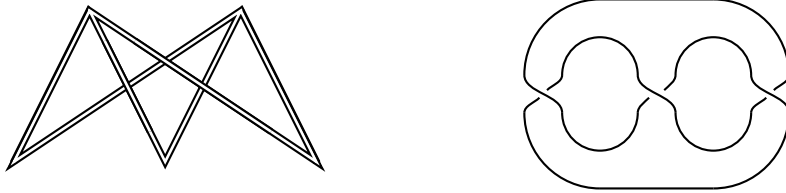
FIGURE 1. A diagram of the complete bipartite graph $K_{2,3}$ whose boundary is the torus knot $T(2,3)$.

and flat plumbing basket surfaces of a given link L using the induced graphs of canonical Seifert surfaces [6] and he also showed that every link is a boundary of a flat string surface [7] using the induced graph of the link. Since the induced graphs are bipartite, these articles build a bridge between the Seifert surfaces and bipartite graphs as one can find a Seifert surface which is the surface obtained from the $(+1, -1)$ voltage assignment on edges of bipartite graphs. However, these graphs are not complete in general. In a very recent paper, Baader introduced *ribbon diagrams for strongly quasipositive links* [2] to show that every (m, n) torus link is a boundary of a surface which is obtained from the 0 voltage assignment on all edges of the complete bipartite graphs $K_{m,n}$ where the diagram of the complete bipartite graph $K_{m,n}$ is chosen to be in a very special form as explained as the standard diagram.

The main interest of this article can be explained as follows. If the edges from the top left vertex lie over those of top right vertex of a complete bipartite graph, such a diagram is called *the standard diagram* of the complete bipartite graph. It is known that the boundary of the surface which is obtained from 0 voltage on every edges of the standard diagram of the complete bipartite graph $K_{m,n}$ is the (m, n) torus link. An example of the surface which is obtained from 0 voltage on every edges of the standard diagram of the complete bipartite graph $K_{2,3}$ is illustrated in the left hand side of Figure 1. If we flip the top three bands down, one can easily obtain the knot in the right hand side of Figure 1 which is the trefoil, or the $(2, 3)$ torus knot.

However, if we are allowed to use different diagrams of $K_{m,n}$ instead of the standard diagram, we can obtain links other than torus links.
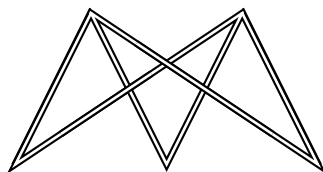
FIGURE 2. A different diagram of the complete bipartite graph $K_{2,3}$ whose boundary is the figure-eight knot.

For example, the boundaries of the surfaces which are obtained from 0 voltage on every edges of all other non-standard diagrams of $K_{2,3}$ (one that all three crossings are changed is in fact a standard diagram if we look at it from behind) are in fact the figure-eight knot. A typical one is given in Figure 2.

These examples motivate us to raise the following questions which address the focus of this article.

QUESTION 1.1. *Given a link L, is there a graph diagram $D(K_{m,n})$ of a complete bipartite graph $K_{m,n}$ such that the link L is a boundary of $D(K_{m,n})$ where all voltage assignments on the edges of $K_{m,n}$ are 0?*

A weaker version of Question 1.1 which we are studying is given as follows.

QUESTION 1.2. *Given a link L, is there a graph diagram $D(K_{m,n})$ of a complete bipartite graph $K_{m,n}$ such that the link L is a boundary of $D(K_{m,n})$ where all voltage assignments on the edges of $K_{m,n}$ are either $0, 1$ or -1?*

In the present article, we give some partial answers for Question 1.1 and 1.2 as in the following theorems.

THEOREM 1.3. *For a given link L, there exists a graph diagram $D(K_{m,n})$ of a complete bipartite multi-graph $K_{m,n}$ such that the link L is a boundary of $D(K_{m,n})$ where all voltage assignments on the edges of $K_{m,n}$ are either 1 or -1.*

THEOREM 1.4. *For a given link L, there exists a graph diagram $D(K_{2,n})$ of a complete bipartite graph $K_{2,n}$ such that the link L is a*

boundary of $D(K_{2,n})$ where all voltage assignments on the edges of $K_{2,n}$ are either $0, 1$ or -1.

The outline of this paper is as follows. In section 2, we review some preliminary definitions in graph theory and knot theory. In section 3, we first examine the boundary of the complete bipartite graphs $K_{1,n}, K_{2,2}$ and $K_{2,3}$, then we prove main theorems.

2. Preliminaries

A Seifert surface $\mathcal{F}_L$ of an oriented link L obtained by applying Seifert's algorithm to a link diagram $D(L)$ as shown in Figure 3 (i) is called a *canonical Seifert surface*. From such a canonical Seifert surface, we construct an induced graph $\Gamma(\mathcal{F}_L)$ by collapsing discs to vertices and half twist bands to signed edges as illustrated in Figure 3 (ii). From arbitrary Seifert surfaces, these processes can be done too. Since the link L is tame and its Seifert surface $\mathcal{F}_L$ is compact, the induced graph $\Gamma(\mathcal{F}_L)$ is finite. By considering the local orientation as indicated on each vertices in Figure 3 (ii), $\Gamma(\mathcal{F}_L)$ is a bipartite graph. For bipartite graphs, it is easy to see that the length of a closed path is always even. For general terminology for knots and graphs, we refer to [1, 5]. It is fairly easy to see that the number of Seifert circles (half twisted bands), denoted by $s(\mathcal{F}_L)(c(\mathcal{F}_L))$, is the cardinality of the vertex set, $V(\Gamma(\mathcal{F}_L))$ (edge set $E(\Gamma(\mathcal{F}_L))$, respectively). A spanning tree T of $\Gamma(\mathcal{F}_L)$ is depicted in Figure 3 (iii). The number of edges of a spanning tree of a connected graph with n vertices is $n - 1$. One can see that the length of the path joining both end vertices of $e \in \Gamma(\mathcal{F}_L)$ is odd.

However, in the case of 7_5, there exists a spanning tree which is a path, thus, any alternating signing κ on the spanning path will satisfy the condition of Theorem 2.1. To obtain a flat plumbing basket surface from Seifert surfaces, we need to find a spanning tree T with a coloring $\kappa : E(T) \to \{+, -\}$ such that for any $e \in E(G) - T$, there exists a path P joining both end vertices of e in T whose coloring is alternating.

THEOREM 2.1. *([6]) For a connected bipartite graph Γ, there exist a spanning tree T and a vertex v such that for any $e \in \Gamma - T$, the unique path P_e in T joining both end vertices of e has alternating signs with respect to the depth coloring $\kappa_v : E(T) \to \{+, -\}$.*
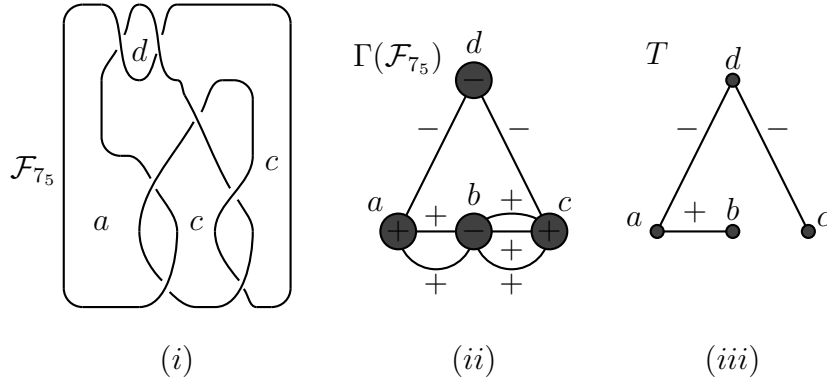
FIGURE 3. (i) A knot 7_5 and its Seifert surface $\mathcal{F}_{7_5}$ whose discs are named a, b, c, d, (ii) its corresponding signed induced graph $\Gamma(\mathcal{F}_{7_5})$ and (iii) a spanning tree T of $\Gamma(\mathcal{F}_{7_5})$.
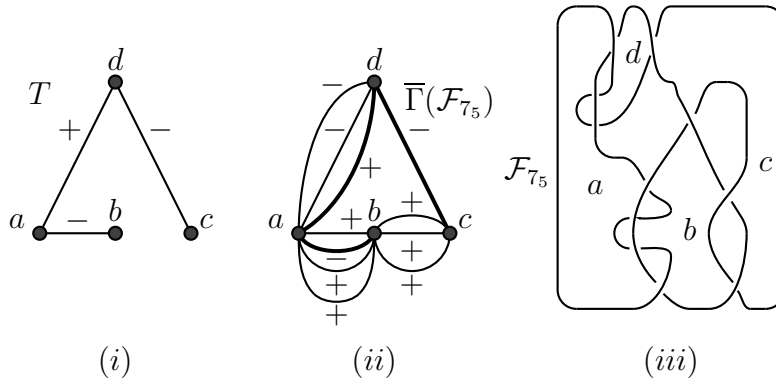

FIGURE 4. (i) a coloring κ on T of $\Gamma(F_{7_5})$ such that for any $e \in E(\Gamma(F_{7_5})) - T$, the unique path in T joining both ends of e has an alternating signs., (ii) a new induced graph $\overline{\Gamma}(\mathcal{F}_{7_5})$ after applying type II Reidemeister move and (iii) a new knot diagram of 7_5 and Seifert surface $\mathcal{F}_{7_5}$ corresponding to $\overline{\Gamma}(\mathcal{F}_{7_5})$.

Let us deal with general cases for the flat plumbing basket number of L by using canonical Seifert surface $\mathcal{F}_L$. Let Γ be the induced graph of a canonical Seifert surface $\mathcal{F}_L$ of L where $|V(\Gamma)| = s(\mathcal{F}_L) = n$ and $|E(\Gamma)| = c(\mathcal{F}_L) = m$. Using Theorem 2.1, there is a spanning tree T and a coloring κ on T such that for any $e \in E(\Gamma) - T$, the unique path in T
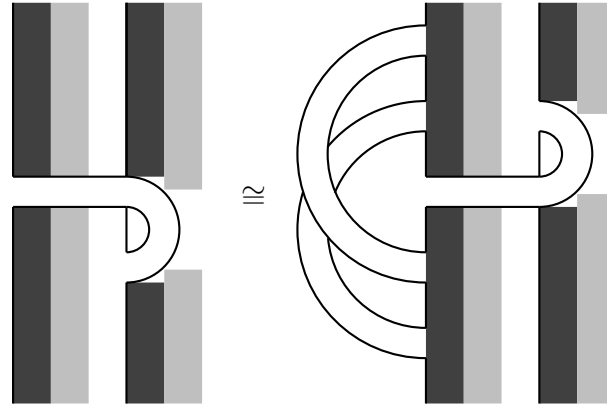
FIGURE 5. How to change the sign of a twisted band by adding two flat annuli.

joining both ends of e has alternating signs. Let $\mu : E(\Gamma) \to \{+, -\}$ be coloring representing the signs of edges in Γ. Let

$$\mathcal{B} = \{e \in T \mid \mu(e) \neq \kappa(e)\}.$$

First if an edge e in T belongs to $\mathcal{B}$, then we have to isotop the link by a type II Reidemeister move as shown in Figure 4 (iii). Since we can completely reverse the sign of all edges in the spanning tree T, we may assume that the total number of type II Reidemeister moves in the process is less than or equal to $\left\lceil \dfrac{n-1}{2} \right\rceil$. Let γ be the minimum of the cardinality of the set $\mathcal{B}$ and $n - |\mathcal{B}| - 1$. Now we set D the disc corresponding to the spanning tree T as depicted in Figure 4 (iii). Let

$$\mathcal{C} = \{e \in E(\Gamma) - T \mid \mu(e) \neq \sum_{f \in P_e} \kappa(f)\}.$$

If an edge e in $E(\Gamma) - T$ belongs to $\mathcal{C}$, then we can plumb a flat annulus along a curve α corresponding to the path P_e in the spanning tree T. Otherwise, we need to add three flat annuli to make the half twisted band presented by the edge e as shown in Figure 5 [3]. By plumbing all egdes in $E(\Gamma) - T$ as described, we have a flat plumbing basket surface of L. Then by summarizing above description of flat plumbing surface of L, we obtain the following theorem.

THEOREM 2.2. ([6]) *Let Γ be an induced graph of canonical Seifert surface $\mathcal{F}$ of a link L with the vertex set $V(\Gamma)$ and the edge set $E(\Gamma)$ where $V(\Gamma) = s(\mathcal{F}) = n$ and $E(|\Gamma|) = c(\mathcal{F}) = m$. Let T be a spanning tree of Γ and κ a coloring on T chosen in Theorem 2.1. Let γ be the minimum of $|\mathcal{B}|$ and $n - |\mathcal{B}| - 1$ and let δ be $|\mathcal{C}|$. Then the flat plumbing basket number of L is bounded above by $3(m - n) + 2(\gamma - \delta) + 3$, i.e.,*

$$fpbk(L) \leq 3(m - n) + 2(\gamma - \delta) + 3.$$

Voltage assignments

Originally voltage assignments were used to construct covering graphs. For terminology and further reading regarding voltage assignments and voltage graphs, we refer to [5]. In particular, $\mathbb{Z}_2$ voltage assignment can be used to construct embeddings of graphs into surfaces, where 0 represents a flat band and 1 represents a twisted band between vertices.

Furthermore, to obtain surface we cap off each boundary by a 2-dimensional disc. This process is called a two cell embeddings of graph. To determine the genera of surfaces in which the given graph, it does not matter the direction of twisted band whether it is positive or negative because it only matters the homeomorphic types of surfaces.

3. Boundaries of complete bipartite graphs

In this section, we first look at all possible boundaries of complete bipartite graphs, $K_{1,n}$, $K_{2,2}$ and $K_{2,3}$. And then we prove main theorems.

3.1. Boundaries of the complete bipartite graphs $K_{1,n}$, $K_{2,2}$ and $K_{2,3}$. We first look at the boundaries of the complete bipartite graph $K_{1,n}$ to obtain the following theorem.

THEOREM 3.1. *The boundaries of the complete bipartite graph $K_{1,n}$ are unknot for all possible diagrams and arbitrary integral twists.*

Next, we examine the boundaries of the complete bipartite graph of $K_{2,2}$. All possible diagrams of $K_{2,2}$ are all isotopic to an unknotted single band. Even if we use different voltage assignments, we can only obtain very limited number of links. For the boundaries of the complete bipartite graph $K_{2,2}$, we obtain the following theorem.
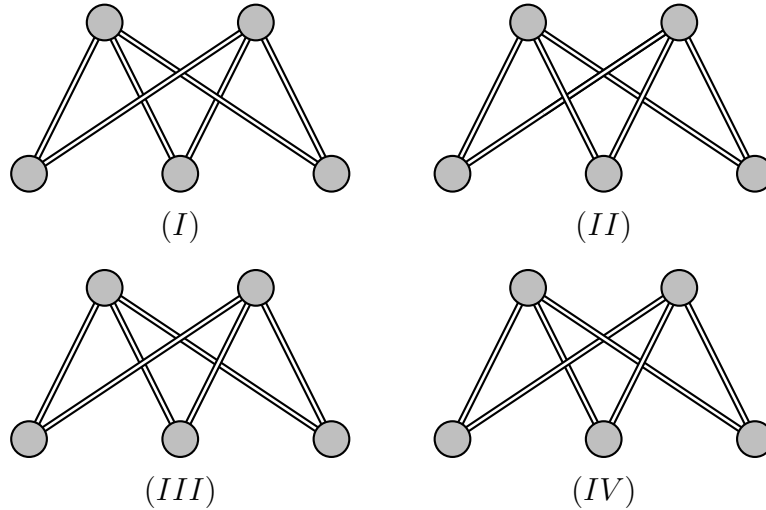
FIGURE 6. Four different modifications of the standard diagram of $K_{2,3}$ used in Table 1

THEOREM 3.2. *The boundaries of the complete bipartite graph $K_{2,2}$ are closed two braids for all possible diagrams and arbitrary integral twists.*

For the boundaries of the complete bipartite graph $K_{2,3}$, we only examine four different cases of the modifications of the standard diagram of $K_{2,3}$ which are given in Figure 6. We obtain Table 1 where $\#$ presents the connected sum, $+O$ presents linking a trivial knot by linking number 1 and $*$ presents the link in Figure 7. All detailed isotopies we used to obtain the table are omitted but one can easily follow it.

3.2. Proof of Theorem 1.3. One might prove the theorem by an induction on the crossing number of L. However, it is very simple to observe that the Reidemeister move II can produce two edges between any two vertices of an induced diagram of link L without changing link type. Therefore, if an induced diagram of L is a subgraph of a complete bipartite multi-graph $K_{m,n}$, then L can be isotop only using Reidemeister move II such that the link L is a boundary of $D(K_{m,n})$ where all voltage assignments on the edges of the complete bipartite multi-graph $K_{m,n}$ are either 1 or -1.

Voltage assignment	Diagram I Name of link	Diagram II Name of link	Diagram III Name of link	Diagram IV Name of link
$(0,0,0)$	4_1	4_1	3_1	4_1
$(1,0,0)$	3_1	3_1	4_1	3_1
$(0,1,0)$	3_1	3_1	4_1	3_1
$(0,0,1)$	5_2	5_2	4_1	3_1
$(1,1,0)$	3_1	3_1	5_1	4_1^2
$(1,0,1)$	4_1^2	4_1^2	5_1^2	$2_1^2\#3_1$
$(0,1,1)$	4_1^2	4_1^2	5_1^2	4_1^2
$(1,1,1)$	4_1^2+O	4_1^2+O	6_2^3	4_1^2+O
$(-1,0,0)$	5_2	5_2	O	5_2
$(0,-1,0)$	5_2	5_2	O	3_1
$(0,0,-1)$	3_1	3_1	O	5_2
$(-1,-1,0)$	7_8^2	4_1^2	$O+O$	4_1^2
$(-1,0,-1)$	4_1^2	4_1^2	$O+O$	7_8^2
$(0,-1,-1)$	4_1^2	7_8^2	$O+O$	4_1^2
$(-1,-1,-1)$	6_3^3	6_3^3	$O+O+O$	6_3^3
$(1,1,-1)$	2_1^2+O	$2_1^2\#4_1^2$	$2_1^2\#2_1^2$	$2_1^2\#4_1^2$
$(1,-1,1)$	$2_1^2\#4_1^2$	$2_1^2\#4_1^2$	$2_1^2\#2_1^2$	2_1^2+O
$(-1,1,1)$	$2_1^2\#4_1^2$	2_1^2+O	$2_1^2\#2_1^2$	$2_1^2\#4_1^2$
$(-1,-1,1)$	$*$	$2_1^2\#2_1^2$	2_1^2+O	$2_1^2\#2_1^2$
$(-1,1,-1)$	$2_1^2\#2_1^2$	$2_1^2\#2_1^2$	2_1^2+O	See c)
$(1,-1,-1)$	$2_1^2\#2_1^2$	$*$	2_1^2+O	$2_1^2\#2_1^2$
$(1,0,-1)$	2_1^2	6_2^2	2_1^2	2_1^2
$(1,-1,0)$	$2_1^2\#3_1$	6_2^2	2_1^2	$O+O$
$(0,1,-1)$	2_1^2	$2_1^2\#3_1$	2_1^2	6_2^2
$(0,-1,1)$	6_2^2	$2_1^2\#3_1$	2_1^2	$O+O$
$(-1,0,1)$	6_2^2	2_1^2	2_1^2	2_1^2
$(-1,1,0)$	$2_1^2\#3_1$	2_1^2	2_1^2	6_2^2

TABLE 1. The boundaries of the complete bipartite graph $K_{2,3}$ of diagram I, II, III and IV in Figure 6
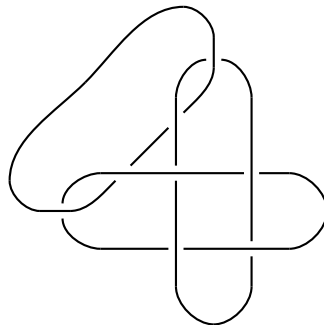

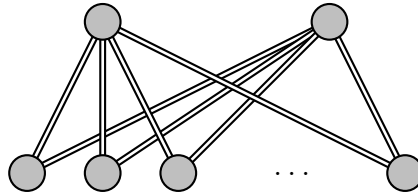FIGURE 7. A link represented as $*$ in Table 1

FIGURE 8. An induced diagram of the complete bipartite multi-graph $K_{2,n}$ whose boundary is the trivial link of n components
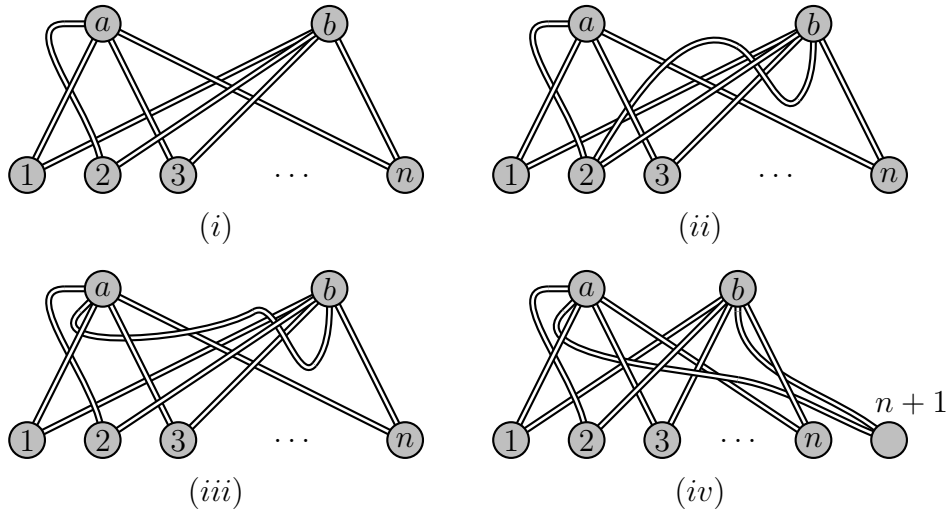

FIGURE 9. Handle slides moves show how to modify $K_{2,n}$ plus one handle to $K_{2,n+1}$ whose boundary is the link L

3.3. Proof of Theorem 1.4. For a given link L, let us induct on the number of edges in an induced diagram $D(L)$ of L. If the number of edges in $D(L)$ is zero, then the link L is the trivial link with n components, and it can be represented by a boundary of the complete bipartite multi-graph $K_{2,n}$ where the signs of edges connected to the left top vertex are $+$ and the signs of edges connected to the right top vertex are 0 as illustrated in Figure 8.

Suppose that if the number of edges in an induced graph $D(L)$ is less than n, then the theorem holds. If we remove an edge e from $D(L)$, by

the induction hypothesis, there exists an n such that the link with the boundary $D(L) - \{e\}$ is a boundary of a complete bipartite graph $K_{2,n}$ as depicted in Figure 9 (i). Then, the link L is a boundary of a surface F which is obtained from the boundary of a complete bipartite graph $K_{2,n}$ by adding an half twisted 1-handle as illustrated in Figure 9 (ii). One can see that both ends of the 1 handle can be slidden to each of two vertices a, b as shown in Figure 9 (iii). Then we can isotop the 1 handle as given in Figure 9 (iv) to have that the link L is a boundary of a complete bipartite graph $K_{2,n+1}$.

Acknowledgments

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