A Form-finding of Planar Tensegrity Structures

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Abstract A form-finding procedure is presented for planar tensegrity structures. Notably, a simple decision criteria is proposed to select the desirable candidate position vector from the unitary matrix produced by the eigenvalue decomposition of force density matrix. The soundness of the candidate position vector guarantees faster convergence and produces a desirable form of tensegrity without any member having zero-length. Several numerical examples are provided to demonstrate the capability of the proposed form-finding process.

Keywords: Tensegrity Structure, Form-finding, Decomposition, Decision Criteria, Self-equilibrium

1. INTRODUCTION

The tensegrity structure provides us one of the most fascinating and surprising structural form compared to the conventional structures. Young artist Kenneth Snelson made a new sculpture in the autumn of 1948. Instantly, his mentor Richard Buckminster Fuller did have some inspirations from his sculpture. A story of tensegrity structure began in Public. Later, the terminology "Tensegrity" is coined by Fuller around 1960. He used two words: Tensional and Integrity. He also described the tensegrity structure as "islands of compression in an ocean of tension". This expression can give us a clear image of tensegrity structure. More precise definition is then provided by Motro (2003): A tensegrity system is a system in a stable self-equilibrated state comprising a discontinuous set of compressed components inside a continuum of tensioned components. In fact, the definition of tensegrity structure now becomes broader in its meaning because the tensegrity system is continuously facing with some engineering realities (Mitsos 2012).

The form-finding of tensegrity structures requires certain knowledge on the fundamental concept of structural mechanics and matrix theory. Three different approaches can be adopted for the form-findings of tensegrity structure, which could be classified as: (1) intuitive method, (2) analytical method and (3) numerical method. As described by Motro (2003), the intuitive method can deal with the tensegrity having only a few members and so it is not recommended for large and complex tensegrity structures. However it does not require any sophisticated mathematical backgrounds. As an example, Snelson did produce many types of tensegrity structure using this kind of method. Analytical methods have been developed by means of the satisfaction of self-stress status and symbolic method using matrix notation is used in this type of form-finding process. It can deal with the tensegrity structures having the large number of member. Finally, the numerical method provides more general form-finding process for tensegrity structures and it usually uses a few and necessary information of target tensegrity structure and can trace the self-stressed status of tensegrity structures using iterative method in general.

A detailed review on the form-finding methods for tensegrity structures can consult to the articles written by Tibert and Pellegrino (2003), Motro (2003), Schenk (2005) and Hernández Juan and Mirats Tur (2009).

In this study, we intend to provide a numerical method to find a certain form of planar tensegrity structures in self-equilibrium. In particular, we propose a decision criteria how to choose the candidate position vector of tensegrity structures using a decomposition method. Numerical test suite is provided to demonstrate how to produce the geometry of tensegrity in self-equilibrium and the force densities of tensegrity members using the proposed technique.

This paper is organized in the followings: In Section 2, basic definitions on tensegrity structure and notations used in this paper are summarized. Force equilibrium of tensegrity is adopted in this study and the force density method is briefly explained in Section 3. The necessary rank conditions are described in Section 4. Then, the proposed form-finding process and its prerequisites are described in Sections 5 and 6. The decision criterion is provided here. In Section 7, several numerical examples of tensegrity structures showing different rigidity and stability levels are analyzed to check the performance of the proposed form-finding process. Finally, some conclusions are drawn in Section 8.

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2. BASIC DEFINITIONS

A cable-strut tensegrity graph $G(V, E)$ is a graph on vertex (or node) set $V = \{v_1, v_2, \ldots, v_n\}$ and edge (or element) set $E = C \cup S = \{1, 2, \ldots, m\}$ which is partitioned into two sets C and S which are called cables and struts respectively. The elements of E is called as member of the tensegrity. Then, the tensegrity can be defined in a d-dimensional Euclidean space ($\mathbb{R}^d$). The configuration of tensegrity is defined by the set of vertex, more precisely using position vector of vertices $\boldsymbol{p} = \{\boldsymbol{p}_1^T, \boldsymbol{p}_2^T, \ldots, \boldsymbol{p}_n^T\} \in \mathbb{R}^{nd}$. With the given configuration, it is possible to define the length map via the function (f: $\mathbb{R}^{nd} \to \mathbb{R}^m$):

$$f(\boldsymbol{p}_1^T, \boldsymbol{p}_2^T, \ldots, \boldsymbol{p}_n^T) = \left(\ldots, |\boldsymbol{p}_i - \boldsymbol{p}_j|^2, \ldots\right). \tag{1}$$

Then the rigidity matrix $\mathbf{R}(\boldsymbol{p}) \in \mathbb{R}^{m \times nd}$ can be defined. The transpose of the rigidity matrix $(\boldsymbol{R}(\boldsymbol{p})^T)$ is also known as the equilibrium matrix. We can assign a force density (or stress) to each member. In any case, the stress must comply with the equilibrium condition which is in the state that the resultant force at each node must be zero:

$$\sum_{j=1}^{n} q_{ij}(\boldsymbol{x_i} - \boldsymbol{x}_j) = 0. \tag{2}$$

In matrix form, the above equation can be rewritten as follows

$$\mathbf{R}(\boldsymbol{p})^T \mathbf{q} = 0 \tag{3}$$

where the cables have the positive stress q>0 and the struts have negative value of stress q<0. It should be noted that we do not allow zero stress values (q=0) for any member in this study.

It is also possible to build another equilibrium equation in the following form:

$$\boldsymbol{p}^T \boldsymbol{\Omega} = 0 \tag{4}$$

where $\boldsymbol{\Omega} \in \mathbb{R}^{n \times n}$ is the stress matrix.

The structural characteristics of tensegrity can be possibly explained by using these matrices. In this study, those matrices such as the rigidity matrix (or equilibrium matrix) and stress matrix (or force density matrix) can be formulated by using the force density method (Schek 1974).

The notations used in this study are summarized in Table 1.

Table 1. Notations used in study

Symbol	Force density	Size
n	Number of node	*Variable*
m	Number of member	*Variable*
d	Dim. of Euclidean space	*2 (present)*
V	Set of vertex	n
C	Set of cable	m_c
S	Set of strut	m_s
E	Set of edge	$m = m_c + m_s$

$\boldsymbol{x}_a$	Position vector of node a	d
$\mathbf{R}(\boldsymbol{x})$	Rigidity matrix	$m \times nd$
$\boldsymbol{\Omega}$	Stress matrix	$n \times n$
$\mathbf{q}$	Force density vector	m
$\mathbf{A}$	Equilibrium matrix	$nd \times m$
$\mathbf{C}$	Connectivity matrix	$m \times n$
$\mathbf{Q}$	Diag($\mathbf{q}$)	$m \times m$
$\mathbf{D}$	Force density matrix	$n \times n$
$\boldsymbol{\Lambda}$	Eigenvalue matrix	$n \times n$
$\boldsymbol{\Phi}$	Eigenvector matrix	$n \times n$
$\boldsymbol{\Sigma}$	Singular value matrix	$nd \times m$
$\boldsymbol{\Upsilon}$	Unitary matrix	$nd \times nd$
$\mathbf{W}$	Unitary matrix	$m \times m$

3. FORCE DENSITY METHOD

3.1 Force equilibrium

This study is mainly concern with the form-finding of tensegrity structure and it normally requires the *equilibrium* of structure. In two-dimensional space, let us consider the free node *i* which is connected to the nodes *j* and *k* as illustrated in Figure 1.
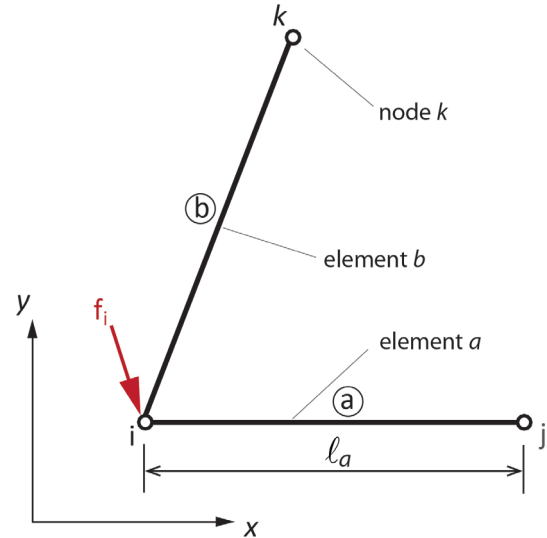

Figure 1. Equilibrium in x-y plane

The force equilibrium equation at the node *i* in the x and y directions can be written as

$$\begin{aligned}(x_i - x_j)q_a + (x_i - x_k)q_b &= f_i^x \\ (y_i - y_j)q_a + (y_i - y_k)q_b &= f_i^y\end{aligned} \tag{5}$$

where $q_a = \left(\frac{t_a}{\ell_a}\right)$ is the force density of the element a, t_a is the internal force of the element a, ℓ_a is the length of the element a, f_i^x is the external load applied at the node *i* in the *x*-direction.

3.2 Equilibrium matrix

Since external force is not considered in the form-finding process, the equilibrium equation of (5) can be written in the matrix form

$$\mathbf{Aq} = 0 \qquad (6)$$

where the equilibrium matrix $\mathbf{A} \in \mathbb{R}^{nd \times m}$ is in the following form

$$\mathbf{C}^T\mathbf{U}\mathbf{q} = 0 \; ; \; \mathbf{C}^T\mathbf{V}\mathbf{q} = 0 \qquad (7)$$

in which the component of the connectivity matrix (or incidence matrix) $\mathbf{C} \in \mathbb{R}^{m \times n}$ is determined by the following relationship:

$$C_{ep} = C(e, p) = \begin{cases} 1 \; for \; p = i(e) \\ -1 \; for \; p = j(e) \\ 0 \quad otherwise \end{cases} \qquad (8)$$

where p are two nodes associated with the element e and $\mathbf{U}, \mathbf{V} = \mathrm{diag}(\mathbf{u}, \mathbf{v}) = \mathrm{diag}(\mathbf{C}\,\mathbf{x}, \mathbf{C}\,\mathbf{y}) \in \mathbb{R}^{m \times m}$ is the diagonal matrix and $\mathbf{q} \in \mathbb{R}^m$ is the force density vector.

3.3 Force density matrix

Since the following relationship is valid

$$\mathbf{U}\mathbf{q} = \mathbf{Q}\mathbf{u} \; ; \; \mathbf{V}\mathbf{q} = \mathbf{Q}\mathbf{v}\,, \qquad (9)$$

Substituting (9) into (7) yields the following form of equilibrium equation:

$$\mathbf{C}^T\mathbf{Q}\mathbf{C}\,\mathbf{x} = 0; \mathbf{C}^T\mathbf{Q}\mathbf{C}\,\mathbf{y} = 0\,. \qquad (10)$$

The above equation can be rewritten as

$$\mathbf{D}\,\mathbf{x} = 0; \mathbf{D}\,\mathbf{y} = 0 \qquad (11)$$

where $\mathbf{D} \in \mathbb{R}^{n \times n}$ is force density matrix.

4. RANK CONDITIONS

Two necessary rank conditions are required to produces a self-equilibrated structure:

A condition for force density matrix ($\mathbf{D}$):

$$r_D = \mathrm{rank}(\mathbf{D}) < n - 2\,. \qquad (12)$$

A condition for equilibrium matrix ($\mathbf{A}$):

$$r_A = \mathrm{rank}(\mathbf{A}) < m\,. \qquad (13)$$

More detailed explanations on these two rank conditions refer to the references (Connelly 1982; Motro 2003).

5. PRE-REQUISITES FOR FORM-FINDING

To clarify the basic concept and the procedure used in this study, Snelson's X-module is adopted to evaluate the important aspect of the form-finding process which is based on the force density method. The geometry of X-module is illustrated in Figure 2.
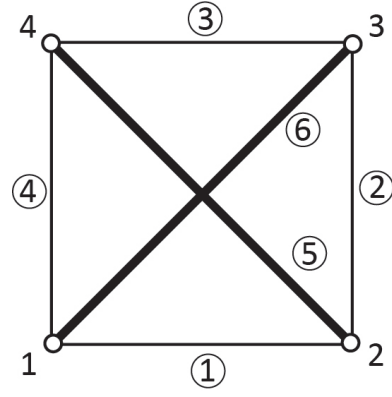

Figure 2. Snelson's X-module (d=2, n=4, m=6)

5.1 Initial force density

Since we already know that the self-equilibrated X-module requires a force density in the following values (Connelly 1998)

$$\mathbf{q}^0 = \{1,1,1,1,-1-1\}. \qquad (14)$$

We will use these values of force density $\mathbf{q}^0 \in \mathbb{R}^6$ in (14) for the evaluation of X-module in Section 5.

5.2 Force density matrix

In order to form the force density matrix $\mathbf{D} \in \mathbb{R}^{4 \times 4}$, we first evaluate the connectivity matrix $\mathbf{C} \in \mathbb{R}^{6 \times 4}$ and a diagonal matrix $\mathbf{Q} = \mathrm{diag}(\mathbf{q}) \in \mathbb{R}^{6 \times 6}$ which uses the values of force density $\mathbf{q}^0$ in (14).

$$\mathbf{C} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix};$$

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix};$$

$$\mathbf{D} = \mathbf{C}^T\mathbf{Q}\mathbf{C} = \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix}. \qquad (15)$$

Note that the force density matrix $\mathbf{D}$ coincides with the stress matrix $\mathbf{\Omega}$ produced by Connelly (1998).

5.3 Equilibrium matrix

We also can build the equilibrium matrix $\mathbf{A} \in \mathbb{R}^{8 \times 6}$ using connectivity matrix $\mathbf{C} \in \mathbb{R}^{6 \times 4}$ and the displacement matrices $(\mathbf{U}, \mathbf{V}) = \mathrm{diag}(\mathbf{C}\,\mathbf{x}, \mathbf{C}\,\mathbf{y}) \in \mathbb{R}^{6 \times 4}$:

$$\mathbf{A}_x = \mathbf{C}^T\mathbf{U} = \begin{bmatrix} -2 & 0 & 0 & 0 & -2 & 0 \\ -2 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & -2 & 0 & -2 & 0 \\ 0 & 0 & -2 & 0 & 0 & -2 \end{bmatrix};$$

$$\mathbf{A}_y = \mathbf{C}^T\mathbf{V} = \begin{bmatrix} 0 & 0 & 0 & -2 & -2 & 0 \\ 0 & -2 & 0 & 0 & 0 & -2 \\ 0 & 2 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 & 0 & 2 \end{bmatrix};$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_x \\ - \\ \mathbf{A}_y \end{bmatrix}. \tag{16}$$

Note that the above equilibrium matrix $\mathbf{A}$ can be simply extended into three-dimensional form with the addition of sub-matrix $\mathbf{A}_z$ which is associated with z-coordinate into (16).

5.4 Decomposition of force density matrix

Using a decomposition method, the characteristics of the force density matrix $\mathbf{D} \in \mathbb{R}^{n \times n}$ such as definiteness, nullity (N_D) and rank (r_D) can be identified.

Using eigenvalue decomposition (EVD), the force density matrix $\mathbf{D}$ is decomposed into the following form:

$$\mathbf{D} = \mathbf{\Phi}\mathbf{\Lambda}\mathbf{\Phi}^T \tag{17}$$

where $\mathbf{\Phi} \in \mathbb{R}^{n \times n}$ is eigenvector matrix and $\mathbf{\Lambda} \in \mathbb{R}^{n \times n}$ is a diagonal matrix containing eigenvalues of $\mathbf{D}$ in ascending order and it can be written in general form:

$$\mathbf{\Phi} = [\breve{\mathbf{\phi}}, \mathbf{\phi}] = [\breve{\mathbf{\phi}}_1, \cdots, \breve{\mathbf{\phi}}_{N_D}, \mathbf{\phi}_1, \cdots, \mathbf{\phi}_{r_D}];$$
$$\mathbf{\Lambda} = [\lambda_1, \lambda_2, \cdots, \lambda_n]. \tag{18}$$

For X-module, the matrix $\mathbf{D} \in \mathbb{R}^{4 \times 4}$ of (15) can be expressed as follows

$$\mathbf{\Phi} = [\breve{\mathbf{\phi}}, \mathbf{\phi}] = [\breve{\mathbf{\phi}}_1, \breve{\mathbf{\phi}}_2, \breve{\mathbf{\phi}}_3, \mathbf{\phi}_1];$$
$$\mathbf{\Lambda} = [\lambda_1, \lambda_2, \lambda_3, \lambda_4] \tag{19}$$

where the $\breve{\mathbf{\phi}} \in \mathbb{R}^{4 \times 3}, \mathbf{\phi} \in \mathbb{R}^4$ and $\mathbf{\Lambda} \in \mathbb{R}^{4 \times 4}$ are

$$\breve{\mathbf{\phi}} = \begin{bmatrix} -0.0846 & 0.4928 & 0.7071 \\ 0.0846 & -0.4928 & 0.7071 \\ 0.7815 & -0.3732 & 0 \\ 0.6124 & 0.6124 & 0 \end{bmatrix}; \mathbf{\phi} = \begin{bmatrix} -0.5000 \\ 0.5000 \\ -0.5000 \\ 0.5000 \end{bmatrix};$$

$$\mathbf{\Lambda} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}. \tag{20}$$

From the results of EVD, it turned out to be that the force density matrix $\mathbf{D}$ of the X-module with the force density values of (14) has the nullity $\mathcal{N} = 3$ which means three zero diagonal components exist in the matrix $\mathbf{\Lambda}$. Therefore this X-module satisfies the condition of (12).

It should be noted that the column vectors $\breve{\mathbf{\phi}}$ of (18), which is associated with the zero diagonal entries in the $\mathbf{\Lambda}$ matrix, will be the candidate of the position vector for the nodes of tensegrity structures. More specific explanation on these criteria will be described in Section 6.

5.5 Decomposition of equilibrium matrix

In order to satisfy a state of self-stress of target tensegrity structure, the rank conditions in Section 4 should be checked. For the non-trivial solution of (6), the condition of (13) should be satisfied. As described in Reference (Motro 2003), the rank deficiency provides the number of independent state of self-stress and total number of infinitesimal mechanism.

Since the equilibrium matrix $\mathbf{A} \in \mathbb{R}^{nd \times m}$ can be decomposed by using the SVD into the following form:

$$\mathbf{A} = \mathbf{\Upsilon}\mathbf{\Sigma}\mathbf{W}^T \tag{21}$$

where $\mathbf{\Upsilon} \in \mathbb{R}^{nd \times nd}, \mathbf{\Sigma} \in \mathbb{R}^{nd \times m}$ and $\mathbf{W} \in \mathbb{R}^{m \times m}$ can be written in general form

$$\mathbf{\Upsilon} = [\mathbf{u}, \mathbf{\mu}] = [\mathbf{u}_1, \cdots, \mathbf{u}_{r_A}, \mathbf{\mu}_1, \cdots, \mathbf{\mu}_{nd-r_A}];$$
$$\mathbf{\Sigma} = [\sigma_1, \sigma_2, \cdots, \sigma_n];$$
$$\mathbf{W} = [\mathbf{w}, \breve{\mathbf{q}}] = [\mathbf{w}_1, \cdots, \mathbf{w}_{r_A}, \breve{\mathbf{q}}_1, \cdots, \breve{\mathbf{q}}_{m-r_A}]. \tag{22}$$

For the X-module, the equilibrium matrix $\mathbf{A} \in \mathbb{R}^{8 \times 6}$ of (21) can be expressed as

$$\mathbf{\Upsilon} = [\mathbf{u}, \mathbf{\mu}] = [\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5, \mathbf{\mu}_1, \mathbf{\mu}_2, \mathbf{\mu}_3]$$
$$\mathbf{\Sigma} = [\sigma_1, \sigma_2, \cdots, \sigma_8]$$
$$\mathbf{W} = [\mathbf{w}, \breve{\mathbf{q}}] = [\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4, \mathbf{w}_5, \breve{\mathbf{q}}_1] \tag{23}$$

in which $\mathbf{u}, \mathbf{\mu}, \mathbf{\Sigma}, \mathbf{w}$ and $\breve{\mathbf{q}}$ are

$$\mathbf{u} = \begin{bmatrix} -0.3536 & 0.3536 & 0.0000 & -0.6124 & 0.0000 \\ 0.3536 & 0.3536 & 0.0000 & 0.6124 & 0.0000 \\ 0.3536 & -0.3536 & 0.5584 & -0.2041 & 0.1466 \\ -0.3536 & -0.3536 & -0.5584 & 0.2041 & -0.1466 \\ -0.3536 & 0.3536 & 0.4061 & 0.2041 & -0.4103 \\ -0.3536 & -0.3536 & 0.1523 & 0.2041 & 0.5569 \\ 0.3536 & -0.3536 & -0.1523 & -0.2041 & -0.5569 \\ 0.3536 & 0.3536 & -0.4061 & -0.2041 & 0.4103 \end{bmatrix};$$

$$\mathbf{\mu} = \begin{bmatrix} -0.4764 & -0.3339 & -0.3785 \\ -0.4764 & -0.0690 & -0.3785 \\ -0.1839 & -0.2851 & -0.5098 \\ -0.1839 & -0.2851 & -0.5098 \\ -0.4330 & -0.3339 & -0.2757 \\ -0.2274 & -0.5500 & -0.1444 \\ -0.2274 & -0.5500 & -0.1444 \\ -0.4330 & -0.3339 & -0.2757 \end{bmatrix};$$

$$\mathbf{\Sigma} = \begin{bmatrix} 4.899 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4.0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2.8284 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2.8284 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2.8284 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix};$$

$$\mathbf{w} = \begin{bmatrix} 0.2887 & 0.0000 & 0.0000 & 0.8660 & 0.0000 \\ 0.2887 & 0.0000 & -0.2154 & -0.2887 & -0.7876 \\ 0.2887 & 0.0000 & 0.7898 & -0.2887 & 0.2073 \\ 0.2887 & 0.0000 & -0.5744 & -0.2887 & 0.5803 \\ 0.5774 & -0.7071 & 0.0000 & 0.0000 & 0.0000 \\ 0.5774 & 0.7071 & -0.0000 & 0.0000 & 0.0000 \end{bmatrix};$$

$$\breve{\mathbf{q}} = [0.4082, 0.4082, 0.4082, 0.4082, -0.4082, -0.4082]^T. \tag{24}$$

The rank of diagonal matrix Σ is $r_A = 5$; the vector $\mu \in \mathbb{R}^{(2n-r_A=3)}$ denotes the $(2n-r_A = 3)$ infinitesimal mechanisms; and the vector $\breve{q} \in \mathbb{R}^{m-r_A}$ is related to the states of self-stress. Let $\mu \in \mathbb{R}^{2n \times (2n-r_A)=8 \times 3}$ be a matrix of mechanism. Note that mechanism does not elongate any member so that $A^T \mu = 0$.

5.6 Final geometry confirmation

The geometry of tensegrity can be also calculated by using the null space basis of the force density matrix $D \in \mathbb{R}^{n \times n}$. The reduced echelon form of D will be evaluated by using Gauss-Jordan elimination with partial pivoting. Note that the Matlab command z=rref(D) can easily produce the pivot variables and free variable.

For X-module with the force density of (14), the basis of null space for force density matrix $D \in \mathbb{R}^{4 \times 4}$ of (15) can be obtained as

$$ z = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \qquad (25) $$

In this case, we can choose three free nodes since the force density matrix D has the nullity $\mathcal{N} = 3$. If we assume the x and y coordinates of free nodes are α, β, γ and $\delta, \varepsilon, \zeta$ for the nodes 2, 3 and 4. Then, the geometry (or configuration) of target tensegrity will be defined in the following form:

$$ \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \alpha \begin{Bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{Bmatrix} + \beta \begin{Bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{Bmatrix} + \gamma \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{Bmatrix} $$

$$ \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix} = \delta \begin{Bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{Bmatrix} + \varepsilon \begin{Bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{Bmatrix} + \zeta \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{Bmatrix}. \qquad (26) $$

For an example, if we use certain coordinates of free nodes 2, 3, 4 such as $(\alpha = 1, \delta = -1)$ $(\beta = 1, \varepsilon = 1)$ $(\gamma = -1, \zeta = 1)$, we can calculate the coordinate of all nodes of tensegrity structure

$$ \begin{bmatrix} x_1, y_1 \\ x_2, y_2 \\ x_3, y_3 \\ x_4, y_4 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & -1 \\ 1 & 1 \\ -1 & 1 \end{bmatrix}. \qquad (27) $$

Free nodes will be always the same coordinate values with the given values such as (α, δ); (β, ε); (γ, ζ). The coordinate pivot node will be then calculated. Using this analytical method, we can therefore confirm the nodal coordinate values of the final tensegrity structure produced by the present form-finding process.

6. FORM-FINDING PROCESS

In this section, the proposed form-finding process is described. The overall process is illustrated in Figure 3 and it is mainly inspired by the research works done by Estrada (2006), Tibert ad Pellegrino (2003) and Motro (2003). A successful form-finding process for the planar tensegrity structures highly depends on the decision of position vector from the result of decomposition of force density matrix $D \in \mathbb{R}^{n \times n}$. A simple and precise decision criteria how to choose new position vector for the geometry of tensegrity structure

is described.

Two decision criterions are used simultaneously for the form-finding process for tensegrity structures:

C1. The candidate position vector should have shortest projected length of tensegrity structure. Note that Estrada (2006) used and adopted QR decomposition to factorize the projected length.

C2. The projected lengths of each member calculated by using candidate position vector may have no zero-length. In particular, the components of the vector $(L_i = C\breve{\phi}_i)$ is desirable to have non-zero value during the iteration. In addition, the column vector having the smallest projected length would be one of the most promising candidate position vectors.
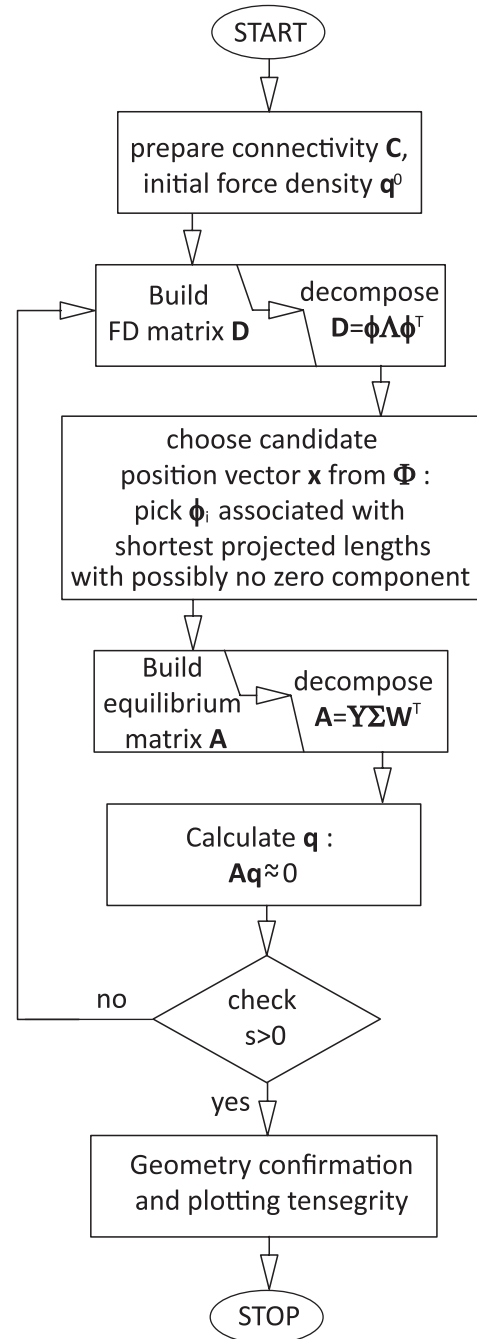
Figure 3. Flowchart of the proposed form-finding process

For example, if we use the same X-module in Section 5, we can choose a candidate matrix $\breve{\Phi}$ having (d+1) column vectors as follows

$$\breve{\Phi} = \begin{bmatrix} -0.0846 & 0.4928 & 0.7071 \\ 0.0846 & -0.4928 & 0.7071 \\ 0.7815 & -0.3732 & 0 \\ 0.6124 & 0.6124 & 0 \end{bmatrix}. \quad (28)$$

From the above candidate column vectors $\breve{\Phi}$, the length can be calculated as follows

$$\mathbf{L} = \mathbf{C}\breve{\Phi} = \begin{bmatrix} -0.1691 & 0.9856 & 0.0000 \\ -0.6969 & -0.1196 & 0.7071 \\ 0.1691 & -0.9856 & 0.0000 \\ -0.6969 & -0.1196 & 0.7071 \\ -0.8660 & 0.8660 & 0.7071 \\ -0.5278 & -1.1052 & 0.7071 \end{bmatrix}. \quad (29)$$

From the matrix of (29), two zero components are found in the third column vector. In this case, we therefore choose the first and second column as the candidate position vector for the next iteration as follows

$$[\breve{\Phi}_1 \breve{\Phi}_2] = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix} = \begin{bmatrix} -0.0846 & 0.4928 \\ 0.0846 & -0.4928 \\ 0.7815 & -0.3732 \\ 0.6124 & 0.6124 \end{bmatrix}. \quad (30)$$

Finally, the candidate position vector of (30) will be used for the calculation of equilibrium matrix $\mathbf{A}$.

Note that if there is no enough nullity (d+1) in the matrix $\mathbf{D}$, we have to use the criteria C2 for selection of the candidate position vector which should be at least (d+1) columns vector with the consideration of the C1 criteria.

The Pseudo-code deduced by using the present decision criteria is described in Table 2.

Table 2. Pseudo-code for form-finding procedure

Set the initial force density to $\mathbf{q}^0 = \{1,1,..,1,1,-1,...,-1\}$
Set $\mathbf{q}^k = \mathbf{q}^0$
While s=0
    Decompose $\mathbf{D}$ into $\mathbf{\Phi}\mathbf{\Lambda}\mathbf{\Phi}^T$
    Check the nullity ($^N D$) of $\mathbf{D}$
    Set the number of position vector (N > d + 1) to N_D
    IF $N_D > 1$
      Calculate the projected length $\mathbf{L}_i = |\lambda_i| \, \mathbf{C}\boldsymbol{\phi}_i, i = 1, N$
    ELSE
      Calculate the projected length $\mathbf{L}_i = \mathbf{C}\boldsymbol{\phi}_i, i = 1, N$
    ENDIF
    Find two shortest projected lengths L_1 and L_2 with possibly no zero projected length component
    Pick the $\boldsymbol{\phi}_1$, $\boldsymbol{\phi}_2$ in $\mathbf{L}_1$, $\mathbf{L}_2$ for x- and y-coordinate
    Form the equilibrium matrix $\mathbf{A}$
    Decompose $\mathbf{A}$ into $\mathbf{Y}\mathbf{\Sigma}\mathbf{W}^T$
    Get new force density $\mathbf{q}^{k+1}$ using $\mathbf{q}^k$ and $\mathbf{W}$
    Check the state of self-stress s using $\mathbf{\Sigma}$
EndWhile

In the present form-finding process, the geometry confirmation is also carried out by using analytical procedure described in Section 5.5.

7. NUMERICAL EXAMPLES

Several planar tensegrity structures are considered to test the capability of proposed form-finding procedure. The numerical results are then compared to the existing reference solutions if it is available in the open literatures.

7.1 Snelson's X-module

This module is frequently used to explain basic behaviour of tensegrity structure. As a simplest tensegrity, this module was often used to explain many important aspects of tensegrity structures including super stability explained by Connelly (1982, 1998). He pointed out that this module produces the positive semi-definite stress matrix $\mathbf{\Omega}$ with rank $r_\Omega=1$ as we described in Section 5. Note that the topology of X-module is illustrated in Figure 2 of Section 5. It has four nodes (n=4), six members (m=6) which is consist of two struts and four cables (N4-S2-C4[†]). In this example, we use three different initial force density vectors $\mathbf{q}^0$ which are denoted as Cases I, II and III. All three cases require only one iteration to achieve its convergence. The initial force density $\mathbf{q}^0$ and the final force density $\mathbf{q}^f$, the normalized final force density $\mathbf{q}_n^f$ and the lengths of each member are described in Table 3. Note that the normalized values of force density $\mathbf{q}_n^f$ is calculated by division of all terms in $\mathbf{q}^f$ with the force density value of the first member.

Table 3. Force densities and member lengths of X-module

Var.	Element					
	1	2	3	4	5	6
Case I:						
$\mathbf{q}^0$	1.0	1.0	1.0	1.0	-0.5	-0.5
$\mathbf{q}^f$	0.8333	0.8333	0.8333	0.8333	-0.8333	-0.8333
$\mathbf{q}_n^f$	1.0	1.0	1.0	1.0	-1.0	-1.0
ℓ	1.0	1.0	1.0	1.0	1.4142	1.4142
Case II:						
$\mathbf{q}^0$	1.0	1.0	1.0	1.0	-1.0	-1.0
$\mathbf{q}^f$	1.0	1.0	1.0	1.0	-1.0	-1.0
$\mathbf{q}_n^f$	1.0	1.0	1.0	1.0	-1.0	-1.0
ℓ	1.0	0.7071	1.0	0.7071	1.2247	1.2247
Case III:						
$\mathbf{q}^0$	1.0	1.0	1.0	1.0	-1.5	-1.5
$\mathbf{q}^f$	1.5	1.5	1.5	1.5	-1.5	-1.5
$\mathbf{q}_n^f$	1.0	1.0	1.0	1.0	-1.0	-1.0
ℓ	1.0	1.0	1.0	1.0	1.4142	1.4142

The geometries of X-module in self-equilibrium produced by the final force densities $\mathbf{q}^f$ are described in Table 4. All the geometries of X-module are confirmed by the analytical method described in Section 5.5. It is found to be that the initial force density matrices

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[†] We follow the notation provided by Motro (2003): N=No. of nodes, S= No. of struts, C=No. of cables

$\mathbf{D}(\mathbf{q}^0)$ of Cases I, II and III have nullity $N_D =$1, 3 and 1 respectively and all three cases have nullity $N_D = 3$ and a single state of self-stress s=1 after form-finding process.

Table 4. Final geometries of X-module

Coordinates	Node (a)			
	1	2	3	4
Case I:				
x_a	-0.4390	0.5543	0.4390	-0.5543
y_a	0.5543	0.4390	-0.5543	-0.4390
Case II:				
x_a	-0.0846	0.0846	0.7815	0.6124
y_a	0.4928	-0.4928	-0.3732	0.6124
Case III:				
x_a	-0.4703	-0.5281	0.4703	0.5281
y_a	-0.5281	0.4703	0.5281	-0.4703

7.2 Hexagonal tensegrity (HT)

The HT is used to find a form in self-equilibrium. The geometry of HT is illustrated in Figure 4. It has six nodes, three struts and six cables (N6-S3-C6) and the detailed explanation on this tensegrity is provided in the previous works (Tibert and Pellegrino, 2002). The initial force density values $\mathbf{q}^0$ could be any combination as demonstrated in Section 7.1 but we decided to use the positive unit force density value (q=1) for cables and the negative unit force density value (q=-1) for struts. After one iteration, the convergence is achieved and produced the final force density $\mathbf{q}^f$ as described in Table 5.
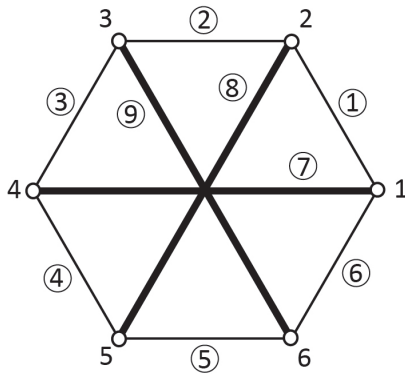
Figure 4. Hexagonal tensegrity (d=2, n=6, m=9)

From the form-finding process, the initial force density matrix $\mathbf{D}(\mathbf{q}^0)$ has nullity $N_D = 1$ and after one iteration but the final force density matrix has nullity $N_D = 3$ and a single state of self-stress s=1. It should be noted that the initial force density matrix $\mathbf{D}(\mathbf{q}^0)$ produces two negative eigenvalues from the EVD. It means that the HT with the initial force density $\mathbf{q}^0$ will not remain at rest even though it could be an equilibrium configuration. In particular, the final force density matrix $\mathbf{D}(\mathbf{q}^f)$ produces three same eigenvalues $\lambda_1 = \lambda_2 = \lambda_3 = 3.3333$. The final geometry of HT is described in Table 6.
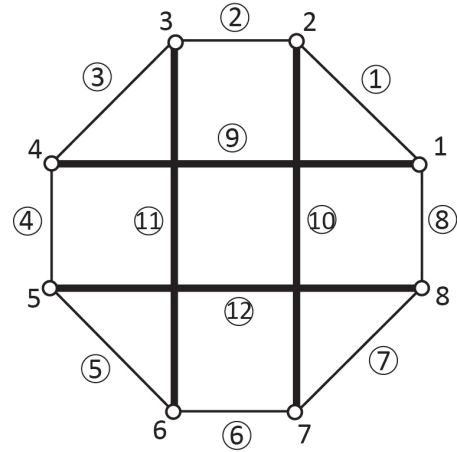
Table 5. Force densities and member lengths of HT

Element	Force densities and member lengths			
	$\mathbf{q}^0$	$\mathbf{q}^f$	$\mathbf{q}_n^f$	ℓ
1~6	1.0	1.1111	1.0	0.5774
7~9	-1.0	-0.5556	-0.5	1.1547

Table 6. Final geometry of HT

Node (a)	Present	
	x_a	y_a
1	0.5197	0.2514
2	0.4776	-0.3244
3	-0.0422	-0.5758
4	-0.5197	-0.2514
5	-0.4776	0.3244
6	0.0422	0.5758

7.3 Octagonal tensegrity I (OT-I)

The OT-I has eight nodes, four struts and eight cables (N8-S4-C8). Its geometry is illustrated in Figure 5. The OT-I was described by Connelly (1998) as super stable planer tensegrity.

Figure 5. OT-I (d=2, n=8, m=12)

The initial force density $\mathbf{q}^0$ is provided in Table 7. From the EVD, the initial force density matrix $\mathbf{D}(\mathbf{q}^0)$ has nullity $N_D =2$ and two negative eigenvalues. After one iteration, the final force density matrix $\mathbf{D}(\mathbf{q}^f)$ has nullity $N_D = 3$ without negative eigenvalue and produces a single state of self-stress (s = 1). The force densities and final lengths of members are described in Table 7 and the final nodal coordinates is also provided in Table 8.

Table 7. Force densities and member lengths of OT-I

Element	Force densities and member lengths			
	$\mathbf{q}^0$	$\mathbf{q}^f$	$\mathbf{q}_n^f$	ℓ
1,3,5,7	1.0000	1.2896	1.0000	0.2298
2,4,6,8	1.0000	0.7970	0.6180	0.5257
9~12	-1.0000	-0.3044	-0.2361	0.9732

Table 8. Final geometry of OT-I

Node (a)	Present	
	x_a	y_a
1	0.4472	-0.2236
2	0.5000	0.0000
3	0.2236	0.4472
4	0.0000	0.5000
5	0.4472	0.2236
6	-0.5000	0.0000
7	0.2236	-0.4472
8	0.0000	-0.5000

Table 10. Final geometry of OT-II

Node (a)	Present	
	x_a	y_a
1	-0.4866	0.1913
2	-0.4866	-0.1913
3	-0.1149	-0.4619
4	0.1149	-0.4619
5	0.4866	-0.1913
6	0.4866	0.1913
7	0.1149	0.4619
8	-0.1149	0.4619

7.4 Octagonal tensegrity I (OT-II)

This module has eight nodes, four struts and eight cables (N8-S4-C8). The geometry of OT-II is illustrated in Figure 6.
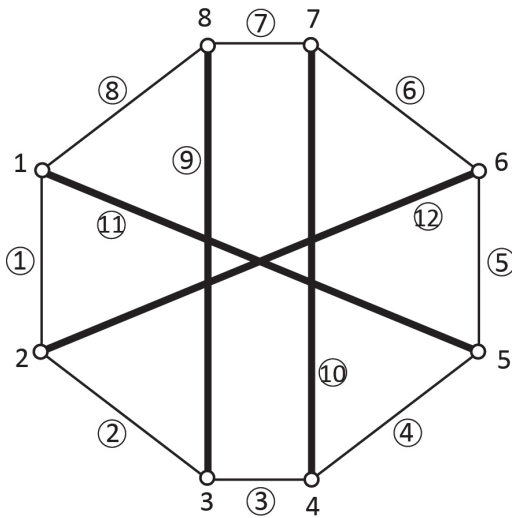

Figure 6. OT-II (d=2, n=8, m=12)

The initial force density $\mathbf{q}^0$ is provided in Table 9. From the form-finding process, it is found to be that the initial force density matrix $\mathbf{D}(\mathbf{q}^0)$ has nullity $N_D =1$ with two negative eigenvalues $\lambda_1 = -1.4142, \lambda_2 = -1.2361$.

After one iteration, the form-finding process is converged and the final force density matrix $\mathbf{D}(\mathbf{q}^f)$ has nullity $N_D = 3$ and final equilibrium matrix $\mathbf{A}$ produces the single state of self-stress $(s = 1)$. The force densities and final lengths of members are described in Table 9 and the final nodal coordinates is also provided in Table 10.

Table 9. Force densities and member lengths of OT-II

Element	Force densities and member lengths			
	$\mathbf{q}^0$	$\mathbf{q}^f$	$\mathbf{q}_n^f$	ℓ
1,5	1.0000	0.9711	1.0000	0.3827
2,4,6,8	1.0000	0.8917	0.9182	0.4598
3,7	1.0000	1.4428	1.4857	0.2298
9,10	-1.0000	-0.2612	-0.2689	0.9239
11,12	-1.0000	-0.3406	-0.3507	1.0458

7.5 Octet module (OM)

This module has six nodes, three struts and nine cables (N6-S3-C9). The geometry of OM is illustrated in Figure 7. Connelly (1998) pointed out that this module is infinitesimally rigid in the plane but are not infinitesimally rigid in three-dimensional space.
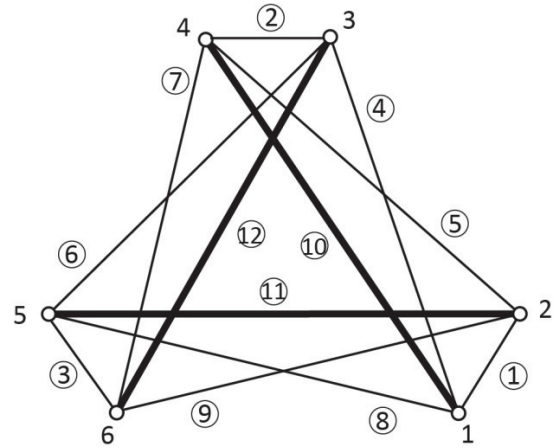

Figure 7. Octet Module (d=2, n=6, m=12)

The initial force density $\mathbf{q}^0$ is provided in Table 11. From the form-finding process, it is found to be that the initial force density matrix $\mathbf{D}(\mathbf{q}^0)$ has nullity $N_D =2$.

After one iteration, the form-finding process is converged and the final force density matrix $\mathbf{D}(\mathbf{q}^f)$ has nullity $N_D = 4$ and final equilibrium matrix $\mathbf{A}$ produces multiple state of self-stress $(s = 2)$. Note that the final force density matrix of this module has the nullity $(d+1=4)$ and therefore, this module has a possibility to be expanded into a spatial tensegrity. Presumably, the simplex tensegrity could be the final spatial form of this module and it will be dealt with in the further study. The force densities and final lengths of members are described in Table 11 and the final nodal coordinates is also provided in Table 12.

Table 11. Force densities and member lengths of OM

Element	Force densities and member lengths			
	$\mathbf{q}^0$	$\mathbf{q}^f$	$\mathbf{q}_n^f$	ℓ
1~3	1.0000	1.183	1.0000	0.2989
4~9	1.0000	0.683	0.5774	1.0000
10~12	-1.0000	-1.183	-1.0000	1.1154

Table 12. Final geometry of OM

Node (a)	Present	
	x_a	y_a
1	0.5138	-0.2634
2	0.5766	0.0288
3	-0.0288	0.5766
4	-0.3133	0.4850
5	-0.4850	-0.3133
6	-0.2634	-0.5138

7.6 Square prism tensegrity (SPT)

This module has eight nodes, four struts and twelve cables (N8-S4-C12). The geometry of SPT is illustrated in Figure 8.
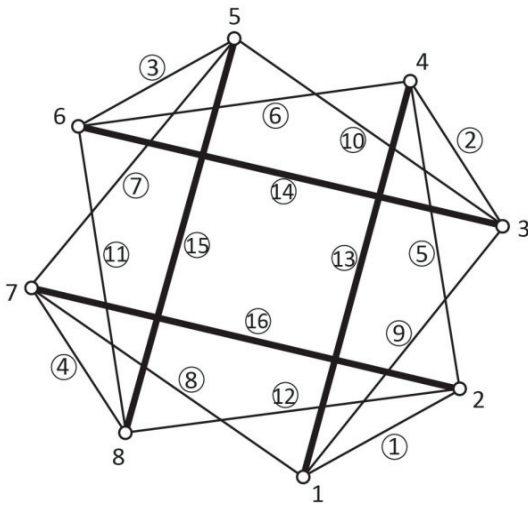

Figure 8. SPT (d=2, n=8, m=16)

Table 13. Force densities and member lengths of SPT

Element	Force densities and member lengths			
	q^0	q^f	q_n^f	ℓ
1~4	1.000	1.1381	1.0000	0.3827
5~12	1.000	0.8047	0.7071	0.7071
13~16	-1.000	-1.1381	-1.0000	0.9239

Table 14. Final geometry of SPT

Node (a)	Present	
	x_a	y_a
1	-0.4003	0.2996
2	-0.0712	0.4949
3	0.2996	0.4003
4	0.4949	0.0712
5	0.4003	-0.2996
6	0.0712	-0.4949
7	-0.2996	-0.4003
8	-0.4949	-0.0712

The initial force density $\mathbf{q}^0$ is provided in Table 13. From the form-finding process, it is found to be that the initial force density matrix $\mathbf{D}(\mathbf{q}^0)$ has nullity $N_D = 2$. After one iteration, the form-finding process is converged and the final force density matrix $\mathbf{D}(\mathbf{q}^f)$ has nullity $N_D = 4$ and final equilibrium matrix $\mathbf{A}$ produces the multiple state of self-stress ($s = 3$). The force densities and final lengths of members are described in Table 13 and the final nodal coordinates is also provided in Table 14. This module has also a possibility to be expanded into a spatial tensegrity since the final force density matrix of this module $\mathbf{D}(\mathbf{q}^f)$ has the nullity (d+1=4).

7.7 Super stable tensegrity (SST)

This module has ten nodes, five struts and fifteen cables (N10-S5-C15). The geometry of OM is illustrated in Figure 9. It is generated by the dihedral groups in joint work of Connelly and Terrell as described by Connelly and Back (1998).
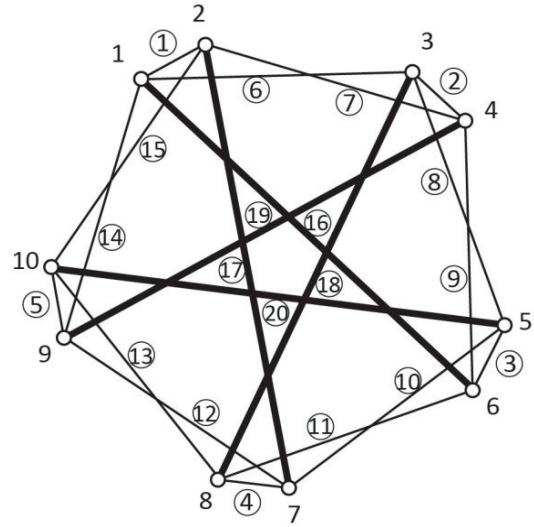
Figure 9. SST (d=2, n=10, m=20)

The initial force density $\mathbf{q}^0$ is provided in Table 15. From the form-finding process, it is found to be that the initial force density matrix $\mathbf{D}(\mathbf{q}^0)$ has nullity $N_D = 2$. After one iteration, the form-finding process is converged and the final force density matrix $\mathbf{D}(\mathbf{q}^f)$ has nullity $N_D = 4$ and final equilibrium matrix $\mathbf{A}$ produces the multiple state of self-stress ($s = 3$). This module has a possibility to be expanded into a spatial tensegrity since the final force density matrix of this module $\mathbf{D}(\mathbf{q}^f)$ has the nullity (d+1=4). The force densities and final lengths of members are described in Table 15 and the final nodal coordinates is also provided in Table 16.

Table 15. Force densities and member lengths of SST

Element	Force densities and member lengths			
	q^0	q^f	q_n^f	ℓ
1~5	1.000	0.821	1.0000	0.1399
6~15	1.000	1.130	1.3764	0.5257
16~20	-1.000	-0.821	-1.0000	0.8834

Table 16. Final geometry of SST

Node (a)	Present	
	x_a	y_a
1	0.3690	-0.2526
2	0.4290	-0.1262
3	0.3543	0.2729
4	0.2526	0.3690
5	-0.1501	0.4213
6	-0.2729	0.3543
7	-0.4470	-0.0126
8	-0.4213	-0.1501
9	-0.1262	-0.4290
10	0.0126	-0.4470

8. CONCULSIONS

Form-finding process is provided for planar tensegrity structures. Decision criteria for selecting the candidate position vector of tensegrity vertex (or node) and final geometry confirmation process based on analytical method are incorporated in the present form-finding process. From benchmark tests, the proposed form-finding process is efficient to produce the planar tensegrity structures in equilibrium for a single or multiple state of self-stress. The numerical results obtained by using the proposed form-finding process are provided as a benchmark test suite for the future study on the form-finding of planar tensegrity structures. Further investigation on the extension of the proposed form finding process of planar tensegrity into the spatial tensegrity could be the next research work.

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