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POSITIVE SOLUTIONS FOR A CLASS OF TELEGRAPH SYSTEM WITH MULTIPARAMETERS

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Abstract. In this paper, we study the existence, non-existence, and multiplicity of positive solutions for a coupled telegraph system using the fixed-point theorem of cone expansion/compression type, the upper-lower solutions method, and fixed point index theory.

1. Introduction

In recent years, the study of semilinear elliptic problems in annular domains has received considerable attention. In [1, 2, 4, 6], the authors considered the existence of positive solutions of the following elliptic system:

$$\begin{cases} \Delta u + \lambda k_1(|x|)f(u,v) = 0, \\ \Delta v + \mu k_2(|x|)g(u,v) = 0 \text{ in } \Omega, \\ u = v = 0 \text{ on } \partial\Omega \end{cases}$$

either for $\lambda = \mu$ or $\lambda \neq \mu$, where $(\lambda,\mu) \in D_+ =: \mathbb{R}^2_+ \backslash \{(0,0)\}$, $k_i \in C([r_1,r_2], \mathbb{R}_+)$ $(i = 1,2)$, which does not vanishing identically on any subinterval of $[r_1,r_2]$ and $f,g \in C(\mathbb{R}^2_+, \mathbb{R}_+\backslash\{0\})$. In particular, we mention the works of Dunninger and Wang on homogeneous Dirichlet boundary conditions, as well as that of Lee on nonhomogeneous Dirichlet boundary conditions. On the basis of [2, 6], X. Yang studied the existence of positive solutions for $2m$-order nonlinear differential systems in [16]. And J. M. do Ó et al. studied the existence, non-existence, and multiplicity of positive solutions for a class of systems of second-order ordinary differential equations

$$\begin{cases} -u'' = g_1(t,u,v,a,b), \ \text{ in } (0,1), \\ -v'' = g_2(t,u,v,a,b), \ \text{ in } (0,1), \\ u(0) = v(0) = u(1) = v(1), \end{cases}$$

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using the fixed-point theorem of cone expansion/compression type, the upper-lower solutions method, and degree arguments in [9, 10].

Because of its important physical background, there are more people who have paid attention to the existence of time-periodic solutions of the telegraph equation or system with various boundary conditions for space variable x, see [5, 7, 8, 11, 12, 13, 14, 15] and the references therein. By using the fixed-point theorem of cone expansion/compression type, the upper-lower solutions method, and degree arguments, our study will be concerned with the existence of positive solutions for the following coupled telegraph system

$$(1)\quad \begin{cases} u_{tt} - u_{xx} + c_1 u_t + a_{11}(t,x)u + a_{12}(t,x)v = f_1(t,x,u,v,\lambda,\mu), \\ v_{tt} - v_{xx} + c_2 v_t + a_{21}(t,x)u + a_{22}(t,x)v = f_2(t,x,u,v,\lambda,\mu), \\ u(t+2\pi,x) = u(t,x+2\pi) = u(t,x), \quad (t,x), \\ v(t+2\pi,x) = v(t,x+2\pi) = v(t,x), \quad (t,x), \end{cases}$$

where $c_i > 0$ is constant, λ, μ are parameters, $a_{11},\ a_{22} \in C(\mathbb{R}^2, \mathbb{R}_+)$, a_{12}, $a_{21} \in C(\mathbb{R}^2, \mathbb{R}_-)$, $f_i \in C(\mathbb{R}^2 \times \mathbb{R}^4_+, \mathbb{R}_+)$, and $a_{ij},\ f_i$ are 2π-periodic in t and x. In particular, the method of upper and lower solutions will need the maximum principle of the coupled linear telegraph system, which was built in [15].

The paper is organized as follows: In Section 2, we make some preliminaries; Section 3 is devoted to proving the main results.

2. Preliminaries

Let $\top^2$ be the torus defined as $\top^2 = (\mathbb{R}/2\pi\mathbb{Z}) \times (\mathbb{R}/2\pi\mathbb{Z})$. Doubly 2π-periodic functions will be identified to be functions defined on $\top^2$. We use the notations

$$L^p(\top^2)\ ,\ C(\top^2)\ ,\ C^\alpha(\top^2)\ ,\ D(\top^2) = C^\infty(\top^2)\ , \ldots$$

to denote the spaces of doubly periodic functions with the indicated degree of regularity. The space $D^{'}(\top^2)$ denotes the space of distributions on $\top^2$.

Here and in the next, by a doubly periodic solution of (1) we mean that a $(u,v) \in L^1(\top^2) \times L^1(\top^2)$ satisfies (1) in the distribution sense, i.e.,

$$\begin{cases} \int_{\top_2} u(\varphi_{tt} - \varphi_{xx} - c_1\varphi_t + a_{11}\varphi) + a_{12}\int_{\top_2} v\varphi = \int_{\top^2} f_1\varphi, \\ \int_{\top_2} v(\phi_{tt} - \phi_{xx} - c_2\phi_t + a_{22}\phi) + a_{21}\int_{\top_2} u\phi = \int_{\top^2} f_2\phi, \\ \forall(\varphi,\phi) \in D'(\top^2) \times D'(\top^2). \end{cases}$$

For convenience, we rewritten this system as

$$\begin{cases} u_{tt} - u_{xx} + c_1 u_t + a_{11}(t,x)u + a_{12}(t,x)v = f_1(t,x,u,v,\lambda,\mu), \\ v_{tt} - v_{xx} + c_2 v_t + a_{21}(t,x)u + a_{22}(t,x)v = f_2(t,x,u,v,\lambda,\mu), \text{ in } D'(\top^2). \end{cases}$$

First, we consider the linear equation

$$(2)\qquad u_{tt} - u_{xx} + c_i u_t - \lambda_i u = h_i(t,x), \text{ in } D^{'}(\top^2),$$

where $c_i > 0,\ \lambda_i \in \mathbb{R},\ h_i(t,x) \in L^1(\top^2)\ \ (i = 1,2)$.

Let $\pounds_{\lambda_i}$ be the differential operator

$$\pounds_{\lambda_i} = u_{tt} - u_{xx} + c_i u_t - \lambda_i u,$$

acting on functions on $\top^2$. Following the discuss in [7, 13], we know that if $\lambda_i < 0$, $\pounds_{\lambda_i}$ has the resolvent R_{λ_i}

$$R_{\lambda_i} : L^1(\top^2) \to C(\top^2), \;\; h_i \mapsto u_i,$$

where u_i is the unique solution of (2), and the restriction of R_{λ_i} on $L^p(\top^2)(1 < p < \infty)$ or $C(\top^2)$ is compact. In particular, $R_{\lambda_i} : C(\top^2) \to C(\top^2)$ is a completely continuous operator.

For $\lambda_i = -c_i^2/4$, the Green function $G_i(t,x)$ of the differential operator $\pounds_{\lambda_i}$ is explicitly expressed, see Lemma 5.2 in [13]. From the definition of $G_i(t,x)$, we have

$$\begin{aligned} \underline{G_i} &:= \operatorname{ess\,inf}\; G_i(t,x) = e^{-3c_i\pi/2}/(1-e^{-c_i\pi})^2, \\ \overline{G_i} &:= \operatorname{ess\,sup}\; G_i(t,x) = (1+e^{-c_i\pi})/2(1-e^{-c_i\pi})^2. \end{aligned}$$

Let X denote the Banach space $C(\top^2)$. Then X is an ordered Banach space with cone

$$K_0 = \{u \in X \mid u(t,x) \geq 0\ ,\ \forall\ (t,x) \in \top^2\}.$$

Now, we consider the equation (2) when $-\lambda_i$ is replaced by $a_{ii}(t,x) \leq \frac{c_i^2}{4}$. In [7], the author has proved the following unique existence and positive estimate result.

Lemma 2.1. *Let $h_i(t,x) \in L^1(\top^2)$, X be the Banach space $C(\top^2)$. Then the equation* (2) *has a unique solution $u_i = P_i h_i$, $P_i : L^1(\top^2) \to X$ is a linear bounded operator with the following properties,*

(i) *$P_i : C(\top^2) \to C(\top^2)$ is a completely continuous operator*;

(ii) *If $h_i > 0$, a.e $(t,x) \in \top^2$, $P_i h_i$ has the positive estimate*

$$\underline{G_i}\|h_i\|_{L^1} \leq (P_i h_i) \leq \frac{\overline{G_i}}{\underline{G_i}\|a_{ii}\|_{L^1}}\|h_i\|_{L^1}. \tag{3}$$

To prove our main results, we need the following the fixed-point theorem of cone expansion/compression type, and fixed point index theory, the upper-lower solutions method. We refer to, for example, Guo and Lakshmikantham [3] for proofs and further results.

Lemma 2.2. *Let E be a Banach space, and let $K \subset E$ be a cone in E, Assume Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$, and let $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator such that either*

(i) *$\|Tu\| \leq \|u\|$, $u \in K \cap \partial\Omega_1$ and $\|Tu\| \geq \|u\|$, $u \in K \cap \partial\Omega_2$;*

(ii) *$\|Tu\| \geq \|u\|$, $u \in K \cap \partial\Omega_1$ and $\|Tu\| \leq \|u\|$, $u \in K \cap \partial\Omega_2$.*

Then T has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Lemma 2.3. *Let E be a Banach space with norm $\|\cdot\|$, and let K be a cone in E. For $r > 0$, define $K_r = \{u \in K : \|u\| < r\}$. Assume that $T : \overline{K_r} \to K$ is a compact map such that $Tu \neq u$, for $u \in \partial K_r = \{u \in K : \|u\| = r\}$.*

(i) *If $\|u\| \leq \|Tu\|$ for all $u \in \partial K_r$, then $i(T, K_r, K) = 1$.*

(ii) *If $\|u\| \geq \|Tu\|$ for all $u \in \partial K_r$, then $i(T, K_r, K) = 0$.*

Lemma 2.4. *Let E be a Banach space, K a cone in E and Ω bounded open in X. Let $0 \in \Omega$ and $T : K \cap \overline{\Omega} \to K$ be condensing. Suppose that $Tx \neq \lambda x$ for all $x \in K \cap \partial\Omega$ and all $\lambda \geq 1$. Then*

$$i(T, K \cap \Omega, K) = 1.$$

Lemma 2.5 ([15]). *Assume $a_{ii}(t,x) \in C(\top^2), 0 \leq a_{ii}(t,x) \leq \frac{c_i^2}{4}$ for $(t,x) \in \top^2$, and $\int_{\top^2} a_{ii}(t,x)dtdx > 0$; $a_{12}(t,x), a_{21}(t,x) \in C(\top^2, R_-)$. In addition, $\|a_{12}\|_{L^1}\overline{G_1} < \underline{G_1}\|a_{11}\|_{L^1}$, $\|a_{21}\|_{L^1}\overline{G_2} < \underline{G_2}\|a_{22}\|_{L^1}$. Then the linear telegraph system*

$$(4)\quad \begin{cases} u_{tt} - u_{xx} + c_1 u_t + a_{11}(t,x)u + a_{12}(t,x)v = g_1(t,x), \\ v_{tt} - v_{xx} + c_2 v_t + a_{21}(t,x)u + a_{22}(t,x)v = g_2(t,x), \quad \text{in } D'(\top^2) \\ u(t+2\pi, x) = u(t, x+2\pi) = u(t,x), \\ v(t+2\pi, x) = v(t, x+2\pi) = v(t,x), \end{cases}$$

where $g_i(t,x) \in L^1(\top^2)$, has at least one solution in $C(\top^2)\times C(\top^2)$ and satisfies the maximum principle.

Remark 2.6. If $\|a_{12}\|_{L^1}\overline{G_1} < \frac{1}{2}\underline{G_1}\|a_{11}\|_{L^1}$, $\|a_{21}\|_{L^1}\overline{G_2} < \frac{1}{2}\underline{G_2}\|a_{22}\|_{L^1}$, then the system (4) also satisfies the maximum principle.

Now, we consider the system

$$(5)\qquad \begin{cases} u_{tt} - u_{xx} + c_1 u_t + a_{11}(t,x)u + a_{12}(t,x)v = f_1(t,x,u,v), \\ v_{tt} - v_{xx} + c_2 v_t + a_{21}(t,x)u + a_{22}(t,x)v = f_2(t,x,u,v), \\ u(t+2\pi, x) = u(t, x+2\pi) = u(t,x), \quad (t,x) \in \mathbb{R}^2, \\ v(t+2\pi, x) = v(t, x+2\pi) = v(t,x), \quad (t,x) \in \mathbb{R}^2, \end{cases}$$

where a_{ij}, f_i satisfy assumptions (H_1) and (H_3).

Definition 2.7. Let $\alpha = (\alpha_1, \alpha_2) \in C(\top^2, \mathbb{R}) \times C(\top^2, \mathbb{R})$, we call (α_1, α_2) a lower solution of the problem (5) for all $(t,x) \in \top^2$ if

$$\begin{cases} \alpha_{1tt} - \alpha_{1xx} + c_1\alpha_{1t} + a_{11}(t,x)\alpha_1 + a_{12}(t,x)\alpha_2 \leq f_1(t,x,\alpha_1,\alpha_2), \\ \alpha_{2tt} - \alpha_{2xx} + c_2\alpha_{2t} + a_{21}(t,x)\alpha_1 + a_{22}(t,x)\alpha_2 \leq f_2(t,x,\alpha_1,\alpha_2), \\ \text{in } D'_+(\top^2) \times D'_+(\top^2) \\ \alpha_1(t+2\pi, x) = \alpha_1(t, x+2\pi) = \alpha_1(t,x), \quad (t,x) \in \mathbb{R}^2, \\ \alpha_2(t+2\pi, x) = \alpha_2(t, x+2\pi) = \alpha_2(t,x), \quad (t,x) \in \mathbb{R}^2. \end{cases}$$

Definition 2.8. Let $\beta = (\beta_1, \beta_2) \in C(\top^2, \mathbb{R}) \times C(\top^2, \mathbb{R})$, we call (β_1, β_2) a upper solution of the problem (5) for all $(t,x) \in \top^2$ if

$$\begin{cases} \beta_{1tt} - \beta_{1xx} + c_1\beta_{1t} + a_{11}(t,x)\beta_1 + a_{12}(t,x)\beta_2 \geq f_1(t,x,\beta_1,\beta_2), \\ \beta_{2tt} - \beta_{2xx} + c_2\beta_{2t} + a_{21}(t,x)\beta_1 + a_{22}(t,x)\beta_2 \geq f_2(t,x,\beta_1,\beta_2), \\ \text{in } D_+^{'}(\top^2) \times D_+^{'}(\top^2) \\ \beta_1(t+2\pi, x) = \beta_1(t, x+2\pi) = \beta_1(t,x), \quad (t,x) \in \mathbb{R}^2, \\ \beta_2(t+2\pi, x) = \beta_2(t, x+2\pi) = \beta_2(t,x), \quad (t,x) \in \mathbb{R}^2. \end{cases}$$

Remark 2.9. In the next, the inequalities related to upper and lower solutions are in the distribution sense, see [15].

Lemma 2.10 ([15])**.** *Let $(\alpha_1(t,x), \alpha_2(t,x))$ and $(\beta_1(t,x), \beta_2(t,x))$ be lower and upper solutions of* (8)*, respectively, such that*

(B_1) $0 \leq (\alpha_1(t,x), \alpha_2(t,x)) \leq (\beta_1(t,x), \beta_2(t,x)), \forall (t,x) \in \top^2$;

(B_2) *$f_i(t,x,u,v)$ for fixed $(t,x) \in \top^2$, is quasi-monotone nondecreasing with respect to u and v. Then the problem* (5) *has at least one solution $(u,v) \in D_\alpha^\beta$ such that*

$$(\alpha_1(t,x), \alpha_2(t,x)) \leq (u,v) \leq (\beta_1(t,x), \beta_2(t,x)), (t,x) \in \top^2.$$

3. Main result

For convenience, we now state our main result as following.

Theorem A. *Assume the following conditions hold*:

(H_1) $a_{ii} \in C(\top^2), 0 \leq a_{ii}(t,x) \leq \frac{c_i^2}{4}$ *for $(t,x) \in \top^2$, and* $\int_{\top^2} a_{ii}(t,x)dtdx > 0$, $a_{12}, a_{21} \in C(\top^2, \mathbb{R}_-)$, $a_{11}(t,x) + a_{12}(t,x) \geq 0$, $a_{21}(t,x) + a_{22}(t,x) > 0$, $\forall (t,x) \in \top^2$, $\|a_{12}\|_{L^1}\overline{G_1} < \frac{1}{2}\underline{G_1}\|a_{11}\|_{L^1}$, $\|a_{21}\|_{L^1}\overline{G_2} < \frac{1}{2}\underline{G_2}\|a_{22}\|_{L^1}$;

(H_2) *The functions $f_i : \top^2 \times [0,+\infty)^4 \to [0,+\infty)$ are continuous and nondecreasing in the last four variables. In other words,*

$$f_i(t,x,u_1,v_1,\lambda_1,\mu_1) \leq f_i(t,x,u_2,v_2,\lambda_2,\mu_2) \text{ for } i = 1,2$$

whenever $(u_1,v_1,\lambda_1,\mu_1) \leq (u_2,v_2,\lambda_2,\mu_2)$, where the inequality is understood inside every component;

(H_3) *Given $\lambda, \mu \geq 0$, for all $M > 0$, there exist $h_i(t,x) \in L^1(\top^2)$ such that*

$$0 \leq f_i(t,x,u,v,\lambda,\mu) \leq h_i(t,x) \text{ for all } (t,u,v) \in \top^2 \times [0,M]^2;$$

(H_4) *There exists a function $h(t,x) \in L^1(\top^2)$ such that*

$$\|h\|_{L^1} < \frac{1}{M^*}$$

and

$$\lim_{|(u,v,\lambda,\mu)| \to 0} \frac{f_1(t,x,u,v,\lambda,\mu) + f_2(t,x,u,v,\lambda,\mu)}{u+v+\lambda+\mu} < \frac{1}{2}h(t,x)$$

for each $(t,x) \in \top^2$, where $M^* = \max\{\frac{\overline{G_1}}{\underline{G_1}\|a_{11}\|_{L^1}}, \frac{\overline{G_2}}{\underline{G_2}\|a_{22}\|_{L^1}}\}$;

(H_5)

$$\lim_{u\to+\infty}\frac{f_1(t,x,u,v,0,0)+f_2(t,x,u,v,0,0)}{u}=+\infty$$

uniformly for $v\geq 0$ *and* $(t,x)\in\top^2$,

$$\lim_{v\to+\infty}\frac{f_1(t,x,u,v,0,0)+f_2(t,x,u,v,0,0)}{v}=+\infty$$

uniformly for $u\geq 0$ *and* $(t,x)\in\top^2$,

$$\lim_{|(\lambda,\mu)|\to+\infty}(f_1(t,x,0,0,\lambda,\mu)+f_2(t,x,0,0,\lambda,\mu))=+\infty$$

uniformly for $(t,x)\in\top^2$.
Then there exist a constant $\overline{\lambda}>0$ *and a non-increasing continuous function* $\Gamma:[0,\overline{\lambda}]\to[0,+\infty)$ *so that, for all* $\lambda\in[0,\overline{\lambda}]$, *the system* (1) *has*:
(i) *at least one positive solution for* $0\leq\mu\leq\Gamma(\lambda)$;
(ii) *no positive solutions for* $\mu>\Gamma(\lambda)$;
(iii) *at least two positive solutions for* $0<\mu<\Gamma(\lambda)$.

Let E denote the Banach space $C(\top^2)\times C(\top^2)$ with the norm $\|(u,v)\|=\|u\|_\infty+\|v\|_\infty$, $\|u\|_\infty=\max_{(t,x)\in\top^2}|u(t,x)|$. The cone K is defined as $K=K_1\times K_2$, where

$$K_1=\{u\in C(\top^2):u\geq\delta_1\|u\|_\infty\},\quad K_2=\{v\in C(\top^2):v\geq\delta_2\|v\|_\infty\},$$

and $\delta_1=\frac{{\underline{G_1}}^2\|a_{11}\|_{L^1}}{\overline{G_1}}$, $\delta_2=\frac{{\underline{G_2}}^2\|a_{22}\|_{L^1}}{\overline{G_2}}$.

By $P_i(i=1,2):L^1(\top^2)\to C(\top^2)$, we denote the solution operators as follows, respectively

$$\begin{aligned}&u_{tt}-u_{xx}+c_1u_t+a_{11}(t,x)u=h_1(t,x),\\&v_{tt}-v_{xx}+c_2v_t+a_{22}(t,x)v=h_2(t,x).\end{aligned}$$

Define mapping $T:K\to E$ by

$$T(u,v)=(Q_1(u,v),Q_2(u,v)),$$

where

$$\begin{aligned}Q_1(u,v)&:=P_1(-a_{12}v+f_1(t,x,u,v,\lambda,\mu)),\\Q_2(u,v)&:=P_2(-a_{21}u+f_2(t,x,u,v,\lambda,\mu)).\end{aligned}$$

First, we will show the existence of a positive solution for small parameters.

Lemma 3.1 ([14])**.** *Fix* $\lambda,\mu\geq 0$. *The operator* $T:E\to E$ *is well defined,* $T(K)\subseteq K$, *and* T *is completely continuous.*

Lemma 3.2. *Assume* (H_1), (H_3), (H_4) *hold. Then there exist* $R_0>0$ *and* ζ_0 *such that, for all* $(u,v)\in K_{R_0}$ *and all* (λ,μ) *with* $0<\lambda+\mu<\zeta_0$, *we have*

$$\|T(u,v)\|<\|(u,v)\|.$$

Proof. From condition (H_4), we choose $\sigma \in (0,1)$ such that

$$M^*\|h(t,x)\|_{L^1} < 1-\sigma.$$

In addition, there exists $R > 0$ such that, for all $0 \le u+v+\lambda+\mu \le R$ and $(t,x) \in \top^2$, we have

$$f_1(t,x,u,v,\lambda,\mu) + f_1(t,x,u,v,\lambda,\mu) \le \frac{1}{2}h(t,x)(u+v+\lambda+\mu).$$

Thus, it follows from condition (H_1), (H_4), for all $(u,v) \in K_{(1-\sigma)R}$, $\lambda+\mu \in (0,\sigma R)$ and $(t,x) \in \top^2$, we have

$$\begin{aligned}
&Q_1(u,v)(t,x) + Q_2(u,v)(t,x)\\
&= P_1(-a_{12}v + f_1(t,x,u,v,\lambda,\mu)) + P_2(-a_{21}v + f_2(t,x,u,v,\lambda,\mu))\\
&\le \frac{\overline{G_1}}{\underline{G_1}\|a_{11}\|_{L^1}}\|-a_{12}v + f_1(t,x,u,v,\lambda,\mu)\|_{L^1}\\
&\quad + \frac{\overline{G_2}}{\underline{G_2}\|a_{22}\|_{L^1}}\|-a_{21}u + f_2(t,x,u,v,\lambda,\mu)\|_{L^1}\\
&\le \frac{\overline{G_1}\|a_{12}\|_{L^1}}{\underline{G_1}\|a_{11}\|_{L^1}}\|v\|_\infty + \frac{\overline{G_2}\|a_{21}\|_{L^1}}{\underline{G_2}\|a_{22}\|_{L^1}}\|u\|_\infty\\
&\quad + M_1\|f_1(t,x,u,v,\lambda,\mu) + f_2(t,x,u,v,\lambda,\mu)\|_{L^1}\\
&\le \frac{1}{2}(\|u\|_\infty + \|v\|_\infty) + \frac{1}{2}M^*\|h(t,x)\|_{L^1}(\|u\|_\infty + \|v\|_\infty + \lambda + \mu)\\
&< (1-\sigma)R
\end{aligned}$$

by $(H1)$ and Lemma 2.1. Now taking $R_0 = (1-\sigma)R$ and $\zeta_0 = \sigma R_0$, for all $(u,v) \in K_{R_0}$, $0 < \lambda+\mu < \zeta_0$, by the above inequality, we have

$$\|T(u,v)\| = \|Q_1(u,v)\|_\infty + \|Q_2(u,v)\|_\infty < R_0 = \|(u,v)\|.$$

□

Lemma 3.3. *Assume (H_1)-(H_3), (H_5) hold. There exists $R_1 > 0$ such that, for all $(u,v) \in K_{R_1}$ and $(t,x) \in \top^2$, we have*

$$\|T(u,v)\| > \|(u,v)\|.$$

Proof. Inspired by some ideas of Lemma 3.2 in [9]. Now we show the proof.

For otherwise, there would exist an increasing sequence $R_n \to +\infty$, and a sequence $\{(u_n,v_n)\}$ in K so that the real sequence $\{R_n\}$ defined by $\|(u_n,v_n)\| = R_n$ would satisfy

$$\|T(u,v)\| \le \|(u,v)\|.$$

We consider two cases:

Case I : $\frac{\|u_n\|_\infty}{R_n} \to 0$ as $n \to +\infty$. Consequently, $\frac{\|v_n\|_\infty}{R_n} \to 1$ as $n \to +\infty$. By conditions (H_1)-(H_3), (H_5), we have

$$\begin{aligned}
T(u_n,v_n) &= Q_1(u_n,v_n) + Q_2(u_n,v_n)\\
&= P_1(-a_{12}v_n + f_1(t,x,u_n,v_n,\lambda,\mu))
\end{aligned}$$

$$
\begin{aligned}
&+ P_2(-a_{21}v_n + f_2(t,x,u_n,v_n,\lambda,\mu))\\
&\geq \underline{G_1}\|-a_{12}v_n + f_1(t,x,u_n,v_n,\lambda,\mu)\|_{L^1}\\
&\quad + \underline{G_2}\|-a_{21}u_n + f_2(t,x,u_n,v_n,\lambda,\mu)\|_{L^1}\\
&\geq G_*\|f_1(t,x,u_n,v_n,\lambda,\mu) + f_2(t,x,u_n,v_n,\lambda,\mu)\|_{L^1}\\
&\geq G_*\|f_1(t,x,\delta\|u_n\|_\infty,\delta\|v_n\|_\infty,0,0)\\
&\quad + f_2(t,x,\delta\|u_n\|_\infty,\delta\|v_n\|_\infty,0,0)\|_{L^1}\\
&\geq G_*\|\frac{J_n(t,x)}{\delta\|v_n\|_\infty}\delta\|v_n\|_\infty\|_{L^1}\\
&= G_*\delta M_n\frac{\|v_n\|_\infty}{R_n}R_n,
\end{aligned}
$$

where

$$
\begin{aligned}
\delta &= \min\left\{\frac{\underline{\underline{G_1}}^2\|a_{11}\|_{L^1}}{\overline{G_1}}\,,\ \frac{\underline{\underline{G_2}}^2\|a_{22}\|_{L^1}}{\overline{G_2}}\right\},\\
G_* &= \min\{\underline{G_1},\underline{G_2}\},\\
J_n(t,x) &= f_1(t,x,\delta\|u_n\|_\infty,\delta\|v_n\|_\infty,0,0) + f_2(t,x,\delta\|u_n\|_\infty,\delta\|v_n\|_\infty,0,0)
\end{aligned}
$$

and

$$M_n = \|\frac{J_n(t,x)}{\delta\|v_n\|_\infty}\|_{L^1} \to +\infty.$$

Therefore, we would have

$$1 \geq G_*\delta M_n\frac{\|v_n\|_\infty}{R_n},$$

which is impossible.

Case II : $\frac{\|u_n\|_\infty}{R_n} \to a > 0$ as $n \to +\infty$. Similarly, in this case we would have

$$
\begin{aligned}
T(u_n,v_n) &= Q_1(u_n,v_n) + Q_2(u_n,v_n)\\
&= P_1(-a_{12}v_n + f_1(t,x,u_n,v_n,\lambda,\mu))\\
&\quad + P_2(-a_{21}v_n + f_2(t,x,u_n,v_n,\lambda,\mu))\\
&\geq \underline{G_1}\|-a_{12}v_n + f_1(t,x,u_n,v_n,\lambda,\mu)\|_{L^1}\\
&\quad + \underline{G_2}\|-a_{21}u_n + f_2(t,x,u_n,v_n,\lambda,\mu)\|_{L^1}\\
&\geq G_*\|f_1(t,x,u_n,v_n,\lambda,\mu) + f_2(t,x,u_n,v_n,\lambda,\mu)\|_{L^1}\\
&\geq G_*\|f_1(t,x,\delta\|u_n\|_\infty,\delta\|v_n\|_\infty,0,0)\\
&\quad + f_2(t,x,\delta\|u_n\|_\infty,\delta\|v_n\|_\infty,0,0)\|_{L^1}\\
&\geq G_*\|\frac{J_n(t,x)}{\delta\|u_n\|_\infty}\delta\|u_n\|_\infty\|_{L^1}\\
&= G_*\delta M_n\frac{\|u_n\|_\infty}{R_n}R_n.
\end{aligned}
$$

From Case I, we also can have

$$1 \geq G_* \delta M_n \frac{\|u_n\|_\infty}{R_n},$$

which is impossible. □

Taking into account Lemma 3.2 and Lemma 3.3, the following is a direct consequence of Lemma 2.2.

Lemma 3.4. *There exists $\zeta_0 > 0$ so that, for all λ and μ with $0 < \lambda + \mu < \zeta_0$, the operator T has a fixed point $(u, v) \in K$ satisfying $R_0 < \|(u, v)\| < R$.*

Next, we will give a priori bounds and non-existence of solutions.

Lemma 3.5. *Assume that the system $(1)_{\lambda_2\mu_2}$ has a non-negative solution and that*

$$(0, 0) \leq (\lambda_1, \mu_1) \leq (\lambda_2, \mu_2).$$

Then the system $(1)_{\lambda_1\mu_1}$ has a non-negative solution.

Proof. Let the pair (u_2, v_2) be a non-negative solution of the system $(1)_{\lambda_2\mu_2}$. Since f_1 and f_2 are non-decreasing functions in the last two variables, we have that (u_2, v_2) is an upper solution of the system $(1)_{\lambda_1\mu_1}$. By the condition (H_1) and Lemma 2.5, it is to know that $(0, 0)$ is a lower solution of the system $(1)_{\lambda_1\mu_1}$. The conclusion results from Lemma 2.10. □

Lemma 3.6. *Assume (H_1)-(H_3), (H_5) hold. Then there exists a positive constant $C > 0$ such that, for all positive solution (u, v) of the system (1), we have*

$$\|(u, v)\| \leq C,$$

where C may be chosen independent of λ and μ.

Proof. The proof is analogous to that of Lemma 3.2 in [10]. □

Remark 3.7. Having the similar discussion as Remark 1 in [9], we also know that there exists $\xi > 0$ such that, for all $(\lambda, \mu) \in (0, +\infty) \times (0, +\infty)$ with $|(\lambda, \mu)| > \xi$, the system (1) has no positive solutions.

Define a set S by

$$S = \{\lambda > 0 : \text{the system (1) has a positive solution for some } \mu > 0\}.$$

From Lemma 3.4 and Remark 3.7, it implies that S is non-empty and bounded. Thus

$$0 < \overline{\lambda} = \sup S < +\infty.$$

Using the upper-lower solutions method, it is easy to see that for all $\lambda \in (0, \overline{\lambda})$, there exists $\mu > 0$ such that the system $(1)_{\lambda\mu}$ has a positive solution. We now define the function $\Gamma : [0, \overline{\lambda}] \to [0, +\infty)$ by

$$\Gamma(\lambda) = \sup\{\mu : \text{the system (1) has a positive solution}\}.$$

By Lemma 3.5, the function Γ is continuous and non-increasing. Moreover, $\Gamma(0) > 0$ as is easily verified. We claim that $\Gamma(\lambda)$ is attained. In fact, it suffices to use Lemma 3.6 and the compactness of the operator T. Finally, it follows from the definition of the function that the system (1) has at least one positive solution for $0 \leq \mu \leq \Gamma(\lambda)$, and furthermore that it has no positive solutions for $\mu > \Gamma(\lambda)$, which proves parts (i) and (ii) of Theorem 1.1, respectively.

Finally, we will establish existence of two positive solutions of the system (1), which corresponds to proving part (iii) of Theorem 1.1.

Fix $\lambda \in [0, \overline{\lambda}]$, and let (u^*, v^*) is the solution (1) at $(\lambda, \Gamma(\lambda))$ which is obtained using Lemma 3.5. Next we will establish another solution of the system $(1)_{\lambda,\mu}$ for $0 < \mu < \Gamma(\lambda)$.

Lemma 3.8. *For each* $0 < \mu < \Gamma(\lambda)$*, there exists* $\epsilon_0 > 0$ *so that, for all* $0 < \epsilon \leq \epsilon_0$ *and all* $(t,x) \in \top^2$*,* $(u^*_\epsilon, v^*_\epsilon)$ *is the upper solution of* (1) *at* (λ, μ)*, where* $u^*_\epsilon = u^* + \epsilon$*,* $v^*_\epsilon = v^* + \epsilon$*.*

Proof. Since f_i is increasing, we have that for each $0 < \mu < \Gamma(\lambda)$ we may find a positive constant $I = I(\mu)$ so that, for all $(t,x) \in \top^2$, we have

$$f_i(t,x,u^*,v^*,\lambda,\Gamma(\lambda)) - f_i(t,x,u^*,v^*,\lambda,\mu) \geq I > 0.$$

By the uniform continuity of f_i, there exists $0 < \epsilon_0$ so that, for all $(t,x) \in \top^2$ and all $0 < \epsilon \leq \epsilon_0$, we have

$$|f_i(t,x,u^*_\epsilon,v^*_\varepsilon,\lambda,\mu) - f_i(t,x,u^*,v^*,\lambda,\mu)| \leq \frac{I}{2}.$$

From the conditions, we also have $a_{11}(t,x) + a_{12}(t,x) \geq 0$. On the contrary, if $a_{11}(t,x) + a_{12}(t,x) < 0$, namely, $a_{11}(t,x) < -a_{12}(t,x)$, then $\|a_{11}\|_{L^1} \leq \|a_{12}\|_{L^1}$, which is contradict with the condition $\|a_{12}\|_{L^1}\overline{G_1} < \frac{1}{2}\underline{G_1}\|a_{11}\|_{L^1}$. Let $u^*_\epsilon = u^* + \epsilon$, $v^*_\epsilon = v^* + \epsilon$, from $(H1)$, $(H2)$, then

$$\begin{aligned}
&u^*_{\epsilon tt} - u^*_{\epsilon xx} + c_1 u^*_{\epsilon t} + a_{11}(t,x)u^*_\epsilon + a_{12}(t,x)v^*_\epsilon - f_1(t,x,u^*+\epsilon,v^*+\epsilon,\lambda,\mu)\\
=\ &u^*_{tt} - u^*_{xx} + c_1 u^*_t + a_{11}(t,x)u^* + a_{12}(t,x)v^* + a_{11}(t,x)\epsilon + a_{12}(t,x)\epsilon\\
&f_1(t,x,u^*+\epsilon,v^*+\epsilon,\lambda,\mu)\\
=\ &f_1(t,x,u^*,v^*,\lambda,\Gamma(\lambda)) - f_1(t,x,u^*+\epsilon,v^*+\epsilon,\lambda,\mu)\\
&+ a_{11}(t,x)\epsilon + a_{12}(t,x)\epsilon\\
>\ &f_1(t,x,u^*,v^*,\lambda,\Gamma(\lambda)) - f_1(t,x,u^*,v^*,\lambda,\mu)\\
&+ f_1(t,x,u^*,v^*,\lambda,\mu) - f_1(t,x,u^*+\epsilon,v^*+\epsilon)\\
\geq\ &I - \frac{I}{2} > 0
\end{aligned}$$

for all $(t,x) \in \top^2$. The inequality for v^*_ϵ can be shown similarly. Hence $(u^*_\epsilon, v^*_\epsilon)$ is an upper solution of (1) at (λ,μ) for all $0 < \epsilon \leq \epsilon_0$. □

Proof of (iii) *Theorem 1.1.* Define the set

$$D = \{(u,v) \in E : -\varepsilon < u < u^*_\epsilon, -\varepsilon < v < v^*_\epsilon\}.$$

Then D is bounded open set in E and $0 \in D$. The map T satisfies $K \cap \overline{D} \to K$ and is condensing, since it is completely continuous. Now let $(u,v) \in K \cap \partial D$, then there exists $(t_0, x_0) \in \top^2$ such that either $u(t_0,x_0) = u^*_\epsilon(t_0,x_0)$ or $v(t_0,x_0) = v^*_\epsilon(t_0,x_0)$. We assume $u(t_0,x_0) = u^*_\epsilon(t_0,x_0)$ without loss of generality, then by Lemma 3.8,

$$\begin{aligned} Q_1(u,v)(t_0,x_0) :&= P_1(-a_{12}v + f_1(t_0,x_0,u,v,\lambda,\mu)) \\ &\le P_1(-a_{12}v^*_\epsilon + f_1(t_0,x_0,u^*_\epsilon,u^*_\epsilon,\lambda,\mu)) \\ &\le \widetilde{u}^*(t_0,x_0) = u(t_0,x_0) \le \theta u(t_0,x_0) \end{aligned}$$

for all $\theta \ge 1$. Thus $T(u,v) \ne \theta(u,v)$ for all $(u,v) \in K \cap \partial D$ and $\theta \ge 1$, Lemma 2.4 now implies that

$$i(T, K \cap D, K) = 1.$$

On the other hand, a slight change in the proof of Lemma 3.6 shows the existence of an $r > 0$ sufficiently large, say $r > R_1$, where R_1 is as in Lemma 3.4, so that

$$\|T(u,v)\| > \|(u,v)\|$$

for every $\|(u,v)\| = r$ and $(u,v) \in K$.

Let $R = \max\{C+1, r, \|(u^*_\epsilon, v^*_\varepsilon)\|\}$, where C is as in Lemma 3.6. Set

$$K_R = \{(u,v) \in K : \|(u,v)\| < R\}.$$

Then Lemma 3.6 implies that $T(u,v) \ne (u,v)$ for $(u,v) \in K_R$. Consequently, part (i) of Lemma 2.3 implies $i(T, K_R, K) = 0$.

Now by the additivity property of the fixed point index we obtain

$$i(T, K \cap \Omega, K) + i(T, K_R \backslash \overline{K \cap \Omega}, K) = i(T, K_R, K) = 0.$$

Since $i(T, K \cap \Omega, K) = 1$, we conclude $i(T, K_R \; K \cap \Omega, K) = -1$. Therefore, T has another fixed point in $K_R \; \overline{K \cap \Omega}$, which was to be shown. □

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