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CROSS COMMUTATORS ON BACKWARD SHIFT INVARIANT SUBSPACES OVER THE BIDISK II

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Abstract. In the previous paper, we gave a characterization of backward shift invariant subspaces of the Hardy space over the bidisk on which $[S_{z^n}, S_w^*] = 0$ for a positive integer $n \geq 2$. In this case, it holds that $S_{z^n} = cI$ for some $c \in \mathbb{C}$. In this paper, it is proved that if $[S_\varphi, S_w^*] = 0$ and $\varphi \in H^\infty(\Gamma_z)$, then $S_\varphi = cI$ for some $c \in \mathbb{C}$.

1. Introduction

Let Γ^2 be the 2-dimensional unit torus. We write $(z, w) = (e^{is}, e^{it})$ for variables in $\Gamma^2 = \Gamma_z \times \Gamma_w$. Let $L^2 = L^2(\Gamma^2)$ be the usual Lebesgue space on Γ^2 with the norm

$$\|f\|_2 = \Big(\int_0^{2\pi} \int_0^{2\pi} |f(e^{is}, e^{it})|^2 \frac{dsdt}{(2\pi)^2} \Big)^{1/2}.$$

With the usual inner product, L^2 is a Hilbert space. Let $H^2 = H^2(\Gamma^2)$ be the Hardy space over Γ^2. We denote by $H^2(\Gamma_z)$ and $H^2(\Gamma_w)$ the Hardy spaces on the unit circle Γ in variables z and w, respectively. We think of $H^2(\Gamma_z)$ and $H^2(\Gamma_w)$ as closed subspaces H^2. For each $f \in H^2$, we can write f as

$$f = \sum_{i=0}^{\infty} \oplus f_i(w) z^i, \qquad f_i(w) \in H^2(\Gamma_w).$$

Let P be the orthogonal projection from L^2 onto H^2. For a closed subspace M of L^2, we denote by P_M the orthogonal projection from L^2 onto M. For a function $\psi \in L^\infty$, the Toeplitz operator T_ψ on H^2 is defined by $T_\psi f = P(\psi f)$ for $f \in H^2$. It is well known that $T_\psi^* = T_{\overline{\psi}}$, and $T_{\varphi(z)}^* T_{\psi(w)} = T_{\psi(w)} T_{\varphi(z)}^*$ for every $\varphi(z) \in H^\infty(\Gamma_z)$ and $\psi(w) \in H^\infty(\Gamma_w)$. A function $f \in H^2$ is called inner if $|f| = 1$ on Γ^2 almost everywhere. A nonzero closed subspace M of H^2 is

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called invariant if $zM \subset M$ and $wM \subset M$. In one variable case, the well known Beurling theorem [2] says that an invariant subspace M of $H^2(\Gamma_z)$ has a form $M = q(z)H^2(\Gamma_z)$, where $q(z)$ is an inner function. In two variable case, the structure of invariant subspaces of H^2 is extremely complicated, see [3, 10].

Let M be an invariant subspace of H^2 with $M \neq \{0\}$ and $M \neq H^2$. Then $T_z^*(H^2 \ominus M) \subset H^2 \ominus M$ and $T_w^*(H^2 \ominus M) \subset H^2 \ominus M$. In this paper, we write

$$N = H^2 \ominus M.$$

Usually, N is called a backward shift invariant subspace of H^2. See [1, 9] for studies of backward shift invariant subspaces over the unit circle Γ.

For a function $\psi \in L^\infty$, we denote by R_ψ the operator on M defined by $R_\psi f = P_M(\psi f)$ for $f \in M$. It holds $R_\psi^* = R_{\overline{\psi}}$ and $R_z = T_z|_M$. We denote by $[R_z, R_w^*]$ the cross commutator of R_z and R_w, that is, $[R_z, R_w^*] = R_z R_w^* - R_w^* R_z$. In [8], Mandrekar proved that $[R_z, R_w^*] = 0$ if and only if M is Beurling type, that is, $M = qH^2$ for some inner function q on Γ^2. This is a nice characterization of Beurling type invariant subspaces of H^2. More generally, in [4] the authors proved that $[R_z, R_w^*] = 0$ if and only if $[R_{\psi_1(z)}, R_{\psi_2(w)}^*] = 0$ for nonconstant functions $\psi_1(z), \psi_2(w) \in H^\infty(\Gamma)$.

We define the operator S_ψ on N by $S_\psi f = P_N(\psi f)$ for $f \in N$. Then we have $S_\psi^* = S_{\overline{\psi}}$ and $S_z^* = T_z^*|_N$. In [6], it is proved that $[S_z, S_w^*] = 0$ if and only if N has one of the following forms;

- $N = H^2 \ominus q_1(z)H^2$,
- $N = H^2 \ominus q_2(w)H^2$,
- $N = (H^2 \ominus q_1(z)H^2) \cap (H^2 \ominus q_2(w)H^2)$

for nonconstant one variable inner functions $q_1(z)$ and $q_2(w)$. In [7], it is shown that the condition $[S_{z^2}, S_w^*] = 0$ does not imply $[S_z, S_w^*] = 0$. In [5], the authors proved that for $n \geq 2$, $[S_{z^n}, S_w^*] = 0$ if and only if one of the following conditions holds;

(i) $[S_z, S_w^*] = 0$,

(ii) $S_{z^n} S_w^* = 0$,

(iii) there exists a Blaschke product $b(z)$ with

$$b(z) = \prod_{j=1}^{n} \frac{z - \alpha_j}{1 - \overline{\alpha}_j z}, \quad 0 < |\alpha_j| < 1,$$

where $\alpha_i \neq \alpha_j$ for every i, j with $i \neq j$ and $\alpha_1^n = \alpha_2^n = \cdots = \alpha_n^n$ such that $N \subset H^2 \ominus b(z)H^2$.

In [7, Theorem 2.2], it is proved that (ii) holds if and only if either $N \subset H^2(\Gamma_z)$ or $N \subset H^2 \ominus z^n H^2$. If $N \subset H^2(\Gamma_z)$, then we have $[S_z, S_w^*] = 0$. Moreover, in [5] it is proved that if $[S_{z^n}, S_w^*] = 0$ and $[S_z, S_w^*] \neq 0$, then $M \cap H^\infty(\Gamma_z) = \theta(z)H^\infty(\Gamma_z)$ for an inner function $\theta(z)$, and $z^n \in \mathbb{C} + \theta(z)H^\infty(\Gamma_z)$. In this case, we have $S_{z^n} = cI$ for some $c \in \mathbb{C}$.

The purpose of this paper is to generalize the above phenomeron. Let $\varphi(z) \in H^\infty(\Gamma_z)$ be a nonconstant function. Suppose that $[S_{\varphi(z)}, S_w^*] = 0$ and

$[S_z, S_w^*] \neq 0$. In Section 2, we prove that $M \cap H^\infty(\Gamma_z) \neq \{0\}$ and $M \cap H^2(\Gamma_z) \neq H^2(\Gamma_z)$. Hence by the Beurling theorem, $M \cap H^2(\Gamma_z) = \theta(z)H^2(\Gamma_z)$ for a nonconstant inner function $\theta(z)$. Thus we get $\theta(z)H^2 \subset M$. Write

$$M_\theta = M \ominus \theta(z)H^2.$$

We prove that $M_\theta \neq \{0\}$ and $T^*_{\varphi(z)} M_\theta \subset M_\theta$. In another word, $\varphi(z)N \subset N \oplus \theta(z)H^2$ holds. In Section 3, we study on the one variable Hardy space $H^2(\Gamma_z)$. Let N_1, N_2 be backward shift invariant subspaces of $H^2(\Gamma_z)$ satisfying $\{0\} \neq N_2 \subsetneqq N_1 \neq H^2(\Gamma_z)$. It is proved that $\varphi(z)N_2 \subset N_2 \oplus (H^2(\Gamma_z) \ominus N_1)$ if and only if $\varphi(z) \in \mathbb{C} + (H^2(\Gamma_z) \ominus N_1)$. As applications of these facts, in Section 4 we prove that $\varphi(z) \in \mathbb{C} + \theta(z)H^\infty(\Gamma_z)$ and $S_\varphi = cI$ for some $c \in \mathbb{C}$.

2. Equivalent conditions for $[S_{\varphi(z)}, S_w^*] = 0$

Let N be a backward shift invariant subspace of H^2 with $N \neq \{0\}$ and $N \neq H^2$, and let $\varphi(z) \in H^\infty(\Gamma_z)$ be a nonconstant function. We write operators T_φ and T_w^* on $H^2 = M \oplus N$ in the matrix forms as

$$T_\varphi = \begin{pmatrix} * & P_M T_\varphi|_N \\ 0 & S_\varphi \end{pmatrix}, \quad T_w^* = \begin{pmatrix} * & 0 \\ P_N T_w^*|_M & S_w^* \end{pmatrix} \quad \text{on } H^2 = \begin{pmatrix} M \\ \oplus \\ N \end{pmatrix}.$$

Let

$$A = P_M T_\varphi|_N \qquad \text{and} \qquad B = P_N T_w^*|_M.$$

Since $T_\varphi T_w^* = T_w^* T_\varphi$ on H^2, we have

$$S_\varphi S_w^* = BA + S_w^* S_\varphi.$$

Hence we get the following.

Lemma 2.1. $[S_\varphi, S_w^*] = 0$ *if and only if* $BA = 0$.

It is not difficult to see that

$$\begin{aligned} \ker B &= \{f \in M : T_w^* f \in M\} \\ &= \{f \in M \ominus wM : T_w^* f = 0\} \oplus wM \\ &= \big(M \cap H^2(\Gamma_z)\big) \oplus wM \end{aligned}$$

and

$$\overline{\text{range}\, A} = M \ominus \ker A^* = M \ominus \{f \in M : T_\varphi^* f \in M\}.$$

Then by Lemma 2.1, we have the following.

Lemma 2.2. $[S_\varphi, S_w^*] = 0$ *if and only if*

$$M \ominus \{f \in M : T_\varphi^* f \in M\} \subset \big(M \cap H^2(\Gamma_z)\big) \oplus wM.$$

Lemma 2.3. *If* $[S_\varphi, S_w^*] = 0$ *and* $[S_z, S_w^*] \neq 0$, *then* $M \cap H^2(\Gamma_z)$ *is a nontrivial invariant subspace of* $H^2(\Gamma_z)$.

Proof. Since $M \neq H^2$, trivially $M \cap H^2(\Gamma_z) \neq H^2(\Gamma_z)$ holds. Suppose that $M \cap H^2(\Gamma_z) = \{0\}$. By Lemma 2.2,

$$M \ominus \{f \in M : T_\varphi^* f \in M\} \subset wM.$$

Hence

$$M \ominus wM \subset \{f \in M : T_\varphi^* f \in M\}.$$

Since $T_w T_\varphi^* = T_\varphi^* T_w$ on H^2, if $f \in M$ and $T_\varphi^* f \in M$, then $T_\varphi^*(w^n f) = w^n T_\varphi^* f \in M$ for every $n \geq 0$, so that by the above we get

$$w^n(M \ominus wM) \subset \{f \in M : T_\varphi^* f \in M\}.$$

Therefore

$$M = \sum_{n=0}^{\infty} \oplus w^n (M \ominus wM) \subset \{f \in M : T_\varphi^* f \in M\}.$$

Thus we get $T_\varphi^* M \subset M$. This shows that $\varphi(z) N \subset N$.

Let

$$\mathcal{A} = \{\psi(z) \in H^\infty(\Gamma_z) : \psi N \subset N\}.$$

Then both functions 1 and $\varphi(z)$ are contained in $\mathcal{A}$. For $\psi \in \mathcal{A}$ and $h \in N$, we have

$$N \ni T_z^*(\psi h) = (T_z^* \psi) h + \psi(0) T_z^* h.$$

Hence $(T_z^* \psi) N \subset N$, so that $T_z^* \mathcal{A} \subset \mathcal{A}$. It is easy to see that $\mathcal{A}$ is a weak-* closed subalgebra of $H^\infty(\Gamma_z)$. Let

$$L = \Big\{ f(z) \in H^1(\Gamma_z) : \int_0^{2\pi} f(e^{i\theta}) \overline{\psi(e^{i\theta})} \, \frac{d\theta}{2\pi} = 0 \ \text{ for every } \psi(z) \in \mathcal{A} \Big\}.$$

Then L is a closed subspace of $H^1(\Gamma_z)$. Since $T_z^* \mathcal{A} \subset \mathcal{A}$ and $1 \in \mathcal{A}$, we have $zL \subset L$.

Suppose that $L \neq \{0\}$. By the Beurling theorem, $L = q(z) H^1(\Gamma_z)$ for an inner function $q(z)$. Since $1 \in \mathcal{A}$, $q(0) = 0$. Hence $\overline{z} q(z) \in H^\infty(\Gamma_z)$. Since $\varphi(z)^n \in \mathcal{A}$ for $n \geq 1$,

$$\int_0^{2\pi} e^{-i\theta} q(e^{i\theta}) \overline{\varphi(e^{i\theta})}^{\,n} e^{i\theta} h(e^{i\theta}) \, \frac{d\theta}{2\pi} = \int_0^{2\pi} q(e^{i\theta}) h(e^{i\theta}) \overline{\varphi(e^{i\theta})}^{\,n} \, \frac{d\theta}{2\pi} = 0$$

for every $h(z) \in H^1(\Gamma_z)$. Hence $\overline{z} q(z) \overline{\varphi(z)}^{\,n} \in H^\infty(\Gamma_z)$ for every $n \geq 1$. By the Schneider theorem [11], we have $\overline{\varphi(z)} \in H^\infty(\Gamma_z)$. This shows that $\varphi(z)$ is constant. Since we assumed that $\varphi(z)$ is nonconstant, this is a contradiction. Therefore $L = \{0\}$. Hence $\mathcal{A} = H^\infty(\Gamma_z)$. Especially, we have $z \in \mathcal{A}$ and $zN \subset N$. Then $T_z|_N = S_z$. Since $T_w^*|_N = S_w^*$ and $T_z T_w^* = T_w^* T_z$ on H^2, we have $S_z S_w^* = S_w^* S_z$. This is a desired contradiction. □

In the rest of this section, we assume that $M \cap H^2(\Gamma_z) \neq \{0\}$. Since $M \neq H^2$, $M \cap H^2(\Gamma_z) \neq H^2(\Gamma_z)$. By the Beurling theorem,

$$M \cap H^2(\Gamma_z) = \theta(z) H^2(\Gamma_z)$$

for some nonconstant inner function $\theta(z)$. Hence $\theta(z)H^2 \subset M$. Write

$$M_\theta = M \ominus \theta(z)H^2.$$

Then

$$M = M_\theta \oplus \theta(z)H^2 \quad \text{and} \quad H^2 \ominus \theta(z)H^2 = M_\theta \oplus N.$$

By the definition of M_θ, we have $wM_\theta \subset M_\theta$ and $M_\theta \cap H^2(\Gamma_z) = \{0\}$. Note that if $[S_\varphi, S_w^*] = 0$ and $[S_z, S_w^*] \neq 0$, then $M_\theta \neq \{0\}$. For, if $M_\theta = \{0\}$, then $M = \theta(z)H^2$ and $N = H^2 \ominus \theta(z)H^2$. Then we have $[S_z, S_w^*] = 0$, see [6], and this is a contradiction.

Lemma 2.4. *Let $f \in M_\theta$. Then $T_w^* f \in M_\theta$ if and only if $f \in wM_\theta$.*

Proof. Suppose that $T_w^* f \in M_\theta$. Then

$$f - f(z,0) \in wM_\theta \subset M_\theta.$$

Since $f \in M_\theta$, $f(z,0) \in M_\theta$. Since $M_\theta \cap H^2(\Gamma_z) = \{0\}$, $f(z,0) = 0$. Hence $f \in wM_\theta$. The converse is trivial. □

Let P_θ be the orthogonal projection from H^2 onto $H^2 \ominus \theta(z)H^2$, and Q_φ be the operator on $H^2 \ominus \theta(z)H^2$ defined by $Q_\varphi f = P_\theta(\varphi f)$ for $f \in H^2 \ominus \theta(z)H^2$. We can write both operators Q_φ and $T_w^*|_{(H^2 \ominus \theta H^2)}$ as

$$Q_\varphi = \begin{pmatrix} * & P_{M_\theta} T_\varphi|_N \\ 0 & S_\varphi \end{pmatrix} \quad \text{on} \quad H^2 \ominus \theta(z)H^2 = \begin{pmatrix} M_\theta \\ \oplus \\ N \end{pmatrix}$$

and

$$T_w^*|_{(H^2 \ominus \theta H^2)} = \begin{pmatrix} * & 0 \\ P_N T_w^*|_{M_\theta} & S_w^* \end{pmatrix} \quad \text{on} \quad H^2 \ominus \theta(z)H^2 = \begin{pmatrix} M_\theta \\ \oplus \\ N \end{pmatrix}.$$

Let

$$A_\theta = P_{M_\theta} T_\varphi|_N \quad \text{and} \quad B_\theta = P_N T_w^*|_{M_\theta}.$$

Lemma 2.5. $[S_\varphi, S_w^*] = 0$ *if and only if* $B_\theta A_\theta = 0$.

Proof. Let $f \in H^2 \ominus \theta(z)H^2 = M_\theta \oplus N$. We have $T_w^*(\varphi(z)f) = \varphi(z)T_w^* f$. Write

$$\varphi(z)f = Q_\varphi f \oplus f_1 \in (M_\theta \oplus N) \oplus \theta(z)H^2.$$

Since $T_w^* f_1 \in \theta(z)H^2$ and $T_w^*(Q_\varphi f) \perp \theta(z)H^2$, we get $T_w^* Q_\varphi f = Q_\varphi T_w^* f$. Thus $Q_\varphi T_w^* = T_w^* Q_\varphi$ on $M_\theta \oplus N$. Similarly as Lemma 2.1, we can prove the assertion. □

The following is a slight generalization of [7, Theorem 4.4].

Theorem 2.6. *The following conditions are equivalent*;

(i) $[S_\varphi, S_w^*] = 0$,
(ii) $M_\theta \ominus \{f \in M_\theta : T_\varphi^* f \in M_\theta\} \subset wM_\theta$,
(iii) $T_\varphi^* M_\theta \subset M_\theta$,

(iv) $\varphi(z)N \subset N \oplus \theta(z)H^2$.

Proof. By Lemma 2.4,

$$\ker B_\theta = \{f \in M_\theta : T_w^* f \in M_\theta\} = wM_\theta.$$

Also we have

$$\overline{\text{range}\, A_\theta} = M_\theta \ominus \ker A_\theta^* = M_\theta \ominus \{f \in M_\theta : T_\varphi^* f \in M_\theta\}.$$

Hence by Lemma 2.5, we get (i) $\Leftrightarrow$ (ii).

If (ii) holds, then

$$M_\theta \ominus wM_\theta \subset \{f \in M_\theta : T_\varphi^* f \in M_\theta\}.$$

Hence for each $n \geq 0$, we have

$$T_{\varphi(z)}^* w^n (M_\theta \ominus wM_\theta) = w^n T_{\varphi(z)}^* (M_\theta \ominus wM_\theta) \subset w^n M_\theta \subset M_\theta.$$

Since

$$M_\theta = \sum_{n=0}^{\infty} \oplus w^n (M_\theta \ominus wM_\theta),$$

we have $T_\varphi^* M_\theta \subset M_\theta$. Thus we get (iii).

(iii) $\Rightarrow$ (ii) is trivial.

It is not difficult to see that (iii) $\Leftrightarrow$ (iv). □

Suppose that $[S_\varphi, S_w^*] = 0$ and $[S_z, S_w^*] \neq 0$. Then we proved that

$$\theta(z)H^2 \subsetneqq M \quad \text{and} \quad \varphi(z)(H^2 \ominus M) \subset (H^2 \ominus M) \oplus \theta(z)H^2.$$

Note that $\theta(z)H^2$ and M are invariant subspaces of H^2. Now we fix an inner function $\theta(z)$. Here we have a question for which $\varphi(z) \in H^\infty(\Gamma_z)$ satisfies the above condition. In the next section, we study a similar question in the one variable Hardy space $H^2(\Gamma_z)$. In Section 4, we revisit on this question.

3. A theorem on the unit circle

In this section, we prove the following theorem.

Theorem 3.1. *Let N_1, N_2 be backward shift invariant subspaces of $H^2(\Gamma_z)$ with $0 \neq N_2 \subsetneqq N_1 \neq H^2(\Gamma_z)$, and $\varphi(z) \in N_1$. Then*

$$\varphi(N_2 \cap H^\infty(\Gamma_z)) \subset N_2 \oplus (H^2(\Gamma_z) \ominus N_1)$$

if and only if $\varphi(z) = cP_{N_1}1$ for some $c \in \mathbb{C}$. In this case, if we define the operator S_φ on N_1 by $S_\varphi f = P_{N_1}(\varphi f)$ for $f \in N_1$, then $S_\varphi = cI$.

To prove the theorem, we need two lemmas which are not difficult to show.

Lemma 3.2. *Let N be a backward shift invariant subspace of $H^2(\Gamma_z)$. Then $N \cap H^\infty(\Gamma_z)$ is dense in N.*

Lemma 3.3. *Let N be a backward shift invariant subspace of $H^2(\Gamma_z)$ with $N \neq \{0\}$ and $N \neq H^2(\Gamma_z)$. If $\varphi \in H^2(\Gamma_z)$ is a nonconstant function, then $\varphi\big(N \cap H^\infty(\Gamma_z)\big) \not\subset N$.*

Proof of Theorem 3.1. By the Beurling theorem,

$$H^2(\Gamma_z) \ominus N_1 = \theta H^2(\Gamma_z)$$

for some nonconstant inner function θ.

First, suppose that

$$\varphi(N_2 \cap H^\infty(\Gamma_z)) \subset N_2 \oplus (H^2(\Gamma_z) \ominus N_1).$$

Since $N_2 \neq \{0\}$, by Lemma 3.2 there exists $h_1 \in N_2 \cap H^\infty(\Gamma_z)$ with $h_1(0) = 1$. Write

$$\varphi h_1 = f_1 \oplus \theta g_1 \in N_2 \oplus \big(H^2(\Gamma_z) \ominus N_1\big) = N_2 \oplus \theta H^2(\Gamma_z). \tag{3.1}$$

Also for each $h \in N_2 \cap H^\infty(\Gamma_z)$, we can write

$$\varphi h = f \oplus \theta g \in N_2 \oplus \theta H^2(\Gamma_z). \tag{3.2}$$

When $h(0) = 0$, we shall prove that

$$g(0) = 0. \tag{3.3}$$

Since

$$T_z^*(\varphi h) = \varphi T_z^* h + h(0) T_z^* \varphi = \varphi T_z^* h,$$

by (3.2) we have

$$\begin{aligned} \varphi T_z^* h &= T_z^*(\varphi h) = T_z^*(f + \theta g) \\ &= T_z^* f + \theta T_z^* g + g(0) T_z^* \theta \\ &= (T_z^* f + g(0) T_z^* \theta) + \theta T_z^* g. \end{aligned}$$

Note that $T_z^* h \in N_2 \cap H^\infty(\Gamma_z)$ and $T_z^* f + g(0) T_z^* \theta \perp \theta H^2(\Gamma_z)$. By the assumption, $\varphi T_z^* h \in N_2 \oplus \theta H^2(\Gamma_z)$. Hence

$$T_z^* f + g(0) T_z^* \theta \in N_2.$$

Since $T_z^* f \in N_2$, $g(0) T_z^* \theta \in N_2$. To prove (3.3), suppose that $g(0) \neq 0$. Then $T_z^* \theta \in N_2$. Let N be a backward shift invariant subspace generated by $T_z^* \theta$. Since $N_1 = H^2(\Gamma_z) \ominus \theta H^2(\Gamma_z)$, we have $N = N_1$. Since $T_z^* \theta \in N_2$, $N \subset N_2$. This contradicts $N_2 \subsetneqq N_1$. Therefore $g(0) = 0$. Thus we get (3.3).

By (3.1) and (3.2),

$$\varphi(h - h(0) h_1) = (f - h(0) f_1) \oplus \theta(g - h(0) g_1) \in N_2 \oplus \theta H^2(\Gamma_z).$$

Since $(h - h(0) h_1)(0) = 0$, by (3.3) we get

$$g(0) = h(0) g_1(0). \tag{3.4}$$

By (3.2) again,

$$\varphi T_z^* h + h(0) T_z^* \varphi = T_z^*(\varphi h) = (T_z^* f + g(0) T_z^* \theta) + \theta T_z^* g,$$

so that

$$\varphi T_z^* h = \big(- h(0) T_z^* \varphi + T_z^* f + g(0) T_z^* \theta\big) \oplus \theta T_z^* g.$$

Since $T_z^* h \in N_2 \cap H^\infty(\Gamma_z)$ and $\varphi \perp \theta H^2(\Gamma_z)$, by the assumption we have

$$-h(0) T_z^* \varphi + T_z^* f + g(0) T_z^* \theta \in N_2.$$

Similarly we have

$$\begin{aligned}\varphi T_z^{*2} h &= \Big(-(T_z^* h)(0) T_z^* \varphi - h(0) T_z^{*2} \varphi + T_z^{*2} f + g(0) T_z^{*2} \theta \\ &\qquad + (T_z^* g)(0) T_z^* \theta \Big) \oplus \theta T_z^{*2} g.\end{aligned}$$

Repeating the same argument, we get

$$\begin{aligned}\varphi T_z^{*n} h &= \Big[-\Big(\sum_{j=0}^{n-1} \big(T_z^{*(n-j-1)} h\big)(0) T_z^{*(j+1)} \varphi \Big) + T_z^{*n} f \\ &\qquad + \Big(\sum_{j=0}^{n-1} (T_z^{*j} g)(0) T_z^{*(n-j)} \theta \Big) \Big] \oplus \theta T_z^{*n} g.\end{aligned}$$

Since $h \in N_2 \cap H^\infty(\Gamma_z)$, $T_z^{*n} h \in N_2 \cap H^\infty(\Gamma_z)$. Hence by (3.2) and (3.4),

$$(T_z^{*n} g)(0) = (T_z^{*n} h)(0) g_1(0)$$

for every $n \geq 0$. This shows that $g = g_1(0)h$. By (3.2), we obtain

$$(\varphi - g_1(0)\theta)h = f \in N_2$$

for every $h \in N_2 \cap H^\infty(\Gamma_z)$. By Lemma 3.3, $\varphi - g_1(0)\theta$ is constant. Write $\varphi - g_1(0)\theta = c$. Since $\varphi \in N_1$, we have $\varphi = cP_{N_1} 1$.

Next, suppose that $\varphi = cP_{N_1} 1$. Then

$$\varphi = cP_{N_1} 1 = c(1 - \overline{\theta(0)}\theta).$$

Hence for $f \in N_2 \cap H^\infty(\Gamma_z)$, we have

$$\varphi f = cf - c\overline{\theta(0)}\theta f \in N_2 \oplus \theta H^2(\Gamma_z).$$

Thus we get $\varphi\big(N_2 \cap H^\infty(\Gamma_z)\big) \subset N_2 \oplus (H^2(\Gamma_z) \ominus N_1)$. □

Corollary 3.4. *Let N_1, N_2 be backward shift invariant subspaces of $H^2(\Gamma_z)$ with $\{0\} \neq N_2 \subsetneqq N_1 \neq H^2(\Gamma_z)$, and $\varphi(z) \in L^\infty(\Gamma_z)$. Define the operator S_φ on N_1 by $S_\varphi h = P_{N_1}(\varphi h)$ for $h \in N_1$. Then $S_\varphi N_2 \subset N_2$ if and only if*

$$\varphi \in \mathbb{C} + H^2(\Gamma_z)^\perp + (H^2(\Gamma_z) \ominus N_1) = \overline{H^2(\Gamma_z)} + (H^2(\Gamma_z) \ominus N_1).$$

Proof. Write $H^2(\Gamma_z) \ominus N_1 = \theta H^2(\Gamma_z)$ for some inner function θ. Let

$$\varphi = \varphi_1 \oplus \varphi_2 \oplus \theta\varphi_3 \in H^2(\Gamma_z)^\perp \oplus N_1 \oplus \theta H^2(\Gamma_z).$$

It is easy to see that

$$P_{N_1}\big(\varphi_1(N_2 \cap H^\infty(\Gamma_z))\big) \subset N_2$$

and

$$P_{N_1}\big(\theta\varphi_3(N_2 \cap H^\infty(\Gamma_z))\big) = \{0\}.$$

Hence $S_\varphi N_2 \subset N_2$ if and only if $P_{N_1}\big(\varphi_2(N_2 \cap H^\infty(\Gamma_z))\big) \subset N_2$. By Theorem 3.1, $S_\varphi N_2 \subset N_2$ if and only if

$$\begin{aligned}\varphi &= \varphi_1 + cP_{N_1}1 + \theta\varphi_3 \\ &= \varphi_1 + c(1 - \overline{\theta(0)}\theta) + \theta\varphi_3 \\ &= \varphi_1 + c + \theta(\varphi_3 - c\overline{\theta(0)}).\end{aligned}$$

This completes the proof. □

The following corollaries follow from Corollary 3.4 directly.

Corollary 3.5. *Let N_1, N_2 be backward shift invariant subspaces of $H^2(\Gamma_z)$ with $\{0\} \neq N_2 \subsetneqq N_1 \neq H^2(\Gamma_z)$, and $\varphi(z) \in H^\infty(\Gamma_z)$. Then $\varphi N_2 \subset N_2 \oplus (H^2(\Gamma_z) \ominus N_1)$ if and only if $\varphi \in \mathbb{C} + (H^2(\Gamma_z) \ominus N_1)$.*

Corollary 3.6. *Let N_1, N_2 be backward shift invariant subspaces of $H^2(\Gamma_z)$ with $\{0\} \neq N_2 \subset N_1 \neq H^2(\Gamma_z)$, and $\varphi(z) \in H^\infty(\Gamma_z)$. If $\varphi N_2 \subset N_2 \oplus (H^2(\Gamma_z) \ominus N_1)$, then $N_1 = N_2$ if and only if $\varphi \notin \mathbb{C} + (H^2(\Gamma_z) \ominus N_1)$.*

Corollary 3.7. *Let M_1, M_2 be invariant subspaces of $H^2(\Gamma_z)$ with $\{0\} \neq M_1 \subsetneqq M_2 \neq H^2(\Gamma_z)$, and $\varphi(z) \in H^\infty(\Gamma_z)$. Then $T_\varphi^*(M_2 \ominus M_1) \subset M_2 \ominus M_1$ if and only if $\varphi \in \mathbb{C} + M_1$.*

Corollary 3.8. *Let M_1, M_2 be invariant subspaces of $H^2(\Gamma_z)$ with $\{0\} \neq M_1 \subsetneqq M_2 \subset H^2(\Gamma_z)$, and $\varphi(z) \in H^\infty(\Gamma_z)$. If $T_\varphi^*(M_2 \ominus M_1) \subset M_2 \ominus M_1$, then $\varphi \notin \mathbb{C} + M_1$ if and only if $M_2 = H^2(\Gamma_z)$.*

4. The main theorem

As applications of the results in Sections 2 and 3, we prove the following.

Theorem 4.1. *Let N be a backward shift invariant subspace of H^2 with $N \neq \{0\}$ and $N \neq H^2$. Let $\varphi(z) \in H^\infty(\Gamma_z)$ be a nonconstant function. If $[S_\varphi, S_w^*] = 0$ and $[S_z, S_w^*] \neq 0$, then $\varphi(z) - c \in M \cap H^\infty(\Gamma_z)$ for some $c \in \mathbb{C}$ and $S_\varphi = cI$.*

Proof. By Lemma 2.3, $M \cap H^2(\Gamma_z) = \theta(z)H^2(\Gamma_z)$ for a nonconstant inner function $\theta(z)$. Since $\theta(z)H^2 \subset M$, as in Section 2 we write

$$M_\theta = M \ominus \theta(z)H^2. \tag{4.1}$$

Since $[S_z, S_w^*] \neq 0$, we have $M_\theta \neq \{0\}$. By Theorem 2.6,

$$\varphi(z)N \subset N \oplus \theta(z)H^2 \tag{4.2}$$

and

$$T_\varphi^* M_\theta \subset M_\theta. \tag{4.3}$$

To prove the assertion, we assume that

$$\varphi(z) - c \notin \theta(z)H^\infty(\Gamma_z) \tag{4.4}$$

for every $c \in \mathbb{C}$. We shall prove that $[S_z, S_w^*] = 0$. This will be a desired contradiction. We consider two cases $\theta(0) = 0$ and $\theta(0) \neq 0$ separately.

Case 1. Suppose that $\theta(0)=0$. If $\theta(z)=cz$ for some constant c with $|c|=1$, then it is easy to see that

$$M=\theta(z)H^2+q(w)H^2$$

for either a nonconstant inner function $q(w)$ or $q(w)=0$. In this case, by [6] we have $[S_z,S_w^*]=0$. So, we may assume that $\theta(z)=z\theta_1(z)$ for a nonconstant inner function $\theta_1(z)$. Then

$$H^2\ominus\theta(z)H^2=H^2(\Gamma_w)\oplus z\big(H^2\ominus\theta_1(z)H^2\big). \tag{4.5}$$

We divide the proof into two subcases.

Subcase 1.1. Assume that $\theta_1(z)M_\theta\subset\theta(z)H^2$. Then $M_\theta\subset zH^2$. Hence $H^2(\Gamma_w)\subset N$. For each nonnegative integer n, let

$$L_n=\big\{f(z)\in H^2(\Gamma_z)\ominus\theta(z)H^2(\Gamma_z):w^nf(z)\in N\big\}.$$

Then $1\in L_n$, L_n is a nonzero closed subspace of $H^2(\Gamma_z)\ominus\theta(z)H^2(\Gamma_z)$, and $T_z^*L_n\subset L_n$. By (4.2),

$$w^n\varphi(z)L_n\subset\varphi(z)N\subset N\oplus\theta(z)H^2,$$

so we have

$$\varphi(z)L_n\subset L_n\oplus\theta(z)H^2(\Gamma_z).$$

By (4.4) and Corollary 3.6, $L_n=H^2(\Gamma_z)\ominus\theta(z)H^2(\Gamma_z)$. Hence

$$w^n\big(H^2(\Gamma_z)\ominus\theta(z)H^2(\Gamma_z)\big)\subset N$$

for every $n\geq 0$. Therefore

$$H^2\ominus\theta(z)H^2=\sum_{n=0}^{\infty}\oplus w^n\big(H^2(\Gamma_z)\ominus\theta(z)H^2(\Gamma_z)\big)\subset N.$$

By (4.1), $H^2\ominus\theta(z)H^2=M_\theta\oplus N$, so that $M_\theta=\{0\}$. This contradicts $[S_z,S_w^*]\neq 0$.

Subcase 1.2. Assume that $\theta_1(z)M_\theta\not\subset\theta(z)H^2$. By (4.5), for every $g\in M_\theta$ we can write

$$g=f_g(w)\oplus zh_g(z,w), \tag{4.6}$$

where $f_g\in H^2(\Gamma_w)$ and $h_g\in H^2\ominus\theta_1(z)H^2$. Since $\theta_1(z)M_\theta\subset M$, we have

$$\theta_1(z)g=\theta_1(z)f_g(w)\oplus z\theta_1(z)h_g(z,w)\in M=M_\theta\oplus\theta(z)H^2,$$

so that $\theta_1(z)f_g(w)\in M_\theta$. Since $\theta_1(z)M_\theta\not\subset\theta(z)H^2$, $f_g(w)\neq 0$ for some $g\in M_\theta$. Then $\{f_g(w):g\in M_\theta\}\neq\{0\}$. Since $wM_\theta\subset M_\theta$, by (4.6) $\overline{\{f_g(w):g\in M_\theta\}}$ is a nonzero T_w-invariant subspace of $H^2(\Gamma_w)$. Hence there is a one variable inner function $q(w)$ such that

$$q(w)H^2(\Gamma_w)=\overline{\{f_g(w):g\in M_\theta\}}. \tag{4.7}$$

Since $\theta_1(z)\{f_g(w) : g \in M_\theta\} \subset M_\theta$, we have

$$\theta_1(z)q(w)H^2(\Gamma_w) \subset M_\theta. \tag{4.8}$$

If $q(w)$ is constant, then $\theta_1(z) \in M_\theta$ and

$$\theta(z)H^2(\Gamma_z) \subsetneqq \mathbb{C} \cdot \theta_1(z) + \theta(z)H^2(\Gamma_z) \subset M \cap H^2(\Gamma_z),$$

so that $\theta(z)H^2(\Gamma_z) \neq M \cap H^2(\Gamma_z)$. This is a contradiction. Hence $q(w)$ is nonconstant. By (4.6) and (4.7), we get

$$\big(H^2(\Gamma_w) \ominus q(w)H^2(\Gamma_w)\big) \perp M_\theta. \tag{4.9}$$

For each nonnegative integer n, let

$$L_n = \big\{f(z) \in H^2(\Gamma_z) \ominus \theta(z)H^2(\Gamma_z) : f(z)w^n q(w) \in M_\theta\big\}.$$

By (4.8), $\theta_1(z) \in L_n$. Since $zM_\theta \subset M_\theta \oplus \theta(z)H^2$, $L_n \oplus \theta(z)H^2(\Gamma_z)$ is an invariant subspace of $H^2(\Gamma_z)$. By (4.3), we have $T^*_\varphi L_n \subset L_n$. By (4.4) and Corollary 3.8, $L_n = H^2(\Gamma_z) \ominus \theta(z)H^2(\Gamma_z)$. Hence

$$w^n q(w)\big(H^2(\Gamma_z) \ominus \theta(z)H^2(\Gamma_z)\big) \subset M_\theta$$

for every $n \geq 0$. Thus we get

$$q(w)\big(H^2 \ominus \theta(z)H^2\big) \subset M_\theta. \tag{4.10}$$

By (4.9), $H^2(\Gamma_w) \ominus q(w)H^2(\Gamma_w) \subset N$. For each $\psi(w) \in H^2(\Gamma_w) \ominus q(w)H^2(\Gamma_w)$, let

$$L_\psi = \big\{f(z) \in H^2(\Gamma_z) \ominus \theta(z)H^2(\Gamma_z) : f(z)\psi(w) \in N\big\}.$$

Then $1 \in L_\psi$, and in the same way as Subcase 1.1, L_ψ is a nonzero closed subspace of $H^2(\Gamma_z) \ominus \theta(z)H^2(\Gamma_z)$ such that $T^*_z L_\psi \subset L_\psi$ and $\varphi(z)L_\psi \subset L_\psi \oplus \theta(z)H^2(\Gamma_z)$. Hence by (4.4) and Corollary 3.6, $L_\psi = H^2(\Gamma_z) \ominus \theta(z)H^2(\Gamma_z)$. Therefore

$$\psi(w)\big(H^2(\Gamma_z) \ominus \theta(z)H^2(\Gamma_z)\big) \subset N$$

for every $\psi(w) \in H^2(\Gamma_w) \ominus q(w)H^2(\Gamma_w)$, and hence

$$\big(H^2 \ominus \theta(z)H^2\big) \ominus q(w)\big(H^2 \ominus \theta(z)H^2\big) \subset N. \tag{4.11}$$

Since $H^2 \ominus \theta(z)H^2 = M_\theta \oplus N$, by (4.10) and (4.11) we get

$$N = \big(H^2 \ominus \theta(z)H^2\big) \ominus q(w)\big(H^2 \ominus \theta(z)H^2\big).$$

By [6], this shows that $[S_z, S^*_w] = 0$.

Case 2. Suppose that $\theta(0) \neq 0$. Let $\varphi'(z) = \varphi(z) - \langle\varphi, \theta\rangle\theta(z)$. Then $S_\varphi = S_{\varphi'}$, so that we may assume that $\varphi \perp \theta$. Write

$$\varphi(z) = \varphi_1(z) + \theta(z)z\varphi_2(z), \tag{4.12}$$

where $\varphi_1 \in H^2(\Gamma_z) \ominus \theta H^2(\Gamma_z)$ and $\varphi_2 \in H^2(\Gamma_z)$. By (4.4), $\varphi_1(z) \neq 0$. Since $\theta(0) \neq 0$, $T^*_z\varphi_1(z) \neq 0$. For each $h \in N$, by (4.2) we can write

$$\varphi h = f_h + \theta g_h \in N \oplus \theta H^2.$$

Applying T_z^* for the both side of the above, we have

$$\varphi T_z^* h + h(0,w) T_z^* \varphi = T_z^* f_h + g(0,w) T_z^* \theta + \theta T_z^* g_h.$$

Hence by (4.12),

$$\begin{aligned} \varphi T_z^* h &= -h(0,w)T_z^*\varphi + T_z^* f_h + g_h(0,w)T_z^*\theta + \theta T_z^* g_h \\ &= -h(0,w)T_z^*\varphi_1 + T_z^* f_h + g_h(0,w)T_z^*\theta + \theta(T_z^* g_h - h(0,w)\varphi_2). \end{aligned}$$

Note that

$$-h(0,w)T_z^*\varphi_1 + T_z^* f_h + g_h(0,w)T_z^*\theta \perp \theta H^2.$$

Since $h \in N$, we have $T_z^* h \in N$, so that by (4.2) we have

$$-h(0,w)T_z^*\varphi_1 + T_z^* f_h + g_h(0,w)T_z^*\theta \in N.$$

Since $f_h \in N$, also we have $T_z^* f_h \in N$ and

$$-h(0,w)T_z^*\varphi_1 + g_h(0,w)T_z^*\theta \in N. \tag{4.13}$$

Write

$$\Theta(z) = \theta^2(z) - \theta(0)\theta(z).$$

We have

$$T^*_{\theta\frac{\theta-\theta(0)}{z}} T_z^* = T^*_{\theta^2-\theta(0)\theta} = T^*_\Theta.$$

Since

$$T^*_{\theta\frac{\theta-\theta(0)}{z}} N \subset N,$$

$$-h(0,w)\big(T_\Theta^*\varphi_1 + aT_\Theta^*\theta\big) + g(0,w)T_\Theta^*\theta \in N.$$

Since $\varphi_1 \in N \subset H^2 \ominus \theta H^2$, we have $T_\Theta^*\varphi_1 = 0$. Since $T_\Theta^*\theta = -\overline{\theta(0)}$, we get

$$a\overline{\theta(0)}h(0,w) - \overline{\theta(0)}g(0,w) \in N.$$

Since $\theta(0) \neq 0$,

$$ah(0,w) - g(0,w) \in N.$$

Thus we get

$$ah(0,w) - g(0,w) \perp \theta(z)H^2.$$

Because $\theta(0) \neq 0$, we have $ah(0,w) - g(0,w) = 0$. Hence by (4.9),

$$-h(0,w)T_z^*\varphi_1(z) \in N.$$

Note that $T_z^*\varphi_1(z) \neq 0$. In the same way as Subcase 1.2,

$$h(0,w)\big(H^2(\Gamma_z) \ominus \theta(z)H^2(\Gamma_z)\big) \subset N \subset H^2 \ominus \theta(z)H^2$$

for every $h \in N$. Since $T_w^* N \subset N$ and $N \neq \{0\}$, $\overline{\{h(0,w) : h \in N\}}$ is a nontrivial T_w^*-invariant subspace of $H^2(\Gamma_w)$, so that

$$\overline{\{h(0,w) : h \in N\}} = H^2(\Gamma_w) \ominus q(w)H^2(\Gamma_w)$$

for either nontrivial inner function $q(w)$ or $q(w) = 0$. Hence

$$(H^2 \ominus \theta(z)H^2) \ominus q(w)(H^2 \ominus \theta(z)H^2) \subset N.$$

For every $f \in N$, write

$$f = \sum_{n=0}^{\infty} \oplus f_n(w) z^n.$$

Since $T_z^* N \subset N$, $f_n(w) \in H^2(\Gamma_w) \ominus q(w) H^2(\Gamma_w)$ for every $n \geq 0$. Hence

$$N \subset (H^2 \ominus \theta(z) H^2) \ominus q(w)(H^2 \ominus \theta(z) H^2).$$

Therefore

$$N = (H^2 \ominus \theta(z) H^2) \ominus q(w)(H^2 \ominus \theta(z) H^2).$$

This shows that $[S_z, S_w^*] = 0$. This completes the proof. □

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