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VAGUE q-IDEALS IN BCI-ALGEBRAS

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Abstract. The notion of vague q-ideals of BCI-algebras is introduced, and several properties of them are investigated. Relations between a vague ideal and a vague q-ideal are discussed. Characterizations of a vague q-ideal are considered.

1. Introduction

Several authors from time to time have made a number of generalizations of Zadeh's fuzzy set theory [12]. Of these, the notion of vague set theory introduced by Gau and Buehrer [3] is of interest to us. Using the vague set in the sense of Gau and Buehrer, Biswas [2] studied vague groups. Jun and Park [6,10] studied vague ideals and vague deductive systems in subtraction algebras. In [8], the concept of vague BCK/BCI-algebras is discussed. S. S. Ahn, Y. U. Cho and C. H. Park [1] studied vague quick ideals of BCK/BCI-algebras. Y. B. Jun and K. J. Lee ([7]) introduced the notion of positive implicative vague ideals in BCK-algebras. They established relations between a vague ideal and a positive implicative ideals.

In this paper, we also use the notion of vague set in the sense of Gau and Buehrer to discuss the vague theory in BCI-algebras. We introduce the notion of vague q-ideal of BCI-algebras and investigate several properties of them. We study a relation between a vague ideal and a vague q-ideal. We establish characterizations of a vague q-ideal.

2. Preliminaries

We review some definitions and properties that will be useful in our results.

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By a *BCI-algebra* we mean an algebra $(X, *, 0)$ of type (2,0) satisfying the following conditions:

(a1) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,
(a2) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,
(a3) $(\forall x \in X)\ (x * x = 0)$,
(a4) $(\forall x, y \in X)\ (x * y = 0,\ y * x = 0 \Rightarrow\ x = y)$.

In any BCI-algebra X one can define a partial order "$\leq$" by putting $x \leq y$ if and only if $x * y = 0$.

A BCI-algebra X has the following properties:

(b1) $(\forall x \in X)\ (x * 0 = x)$.
(b2) $(\forall x, y, z \in X)\ ((x * y) * z = (x * z) * y)$.
(b3) $(\forall x, y \in X)\ (0 * (x * y) = (0 * x) * (0 * y))$.
(b4) $(\forall x, y \in X)\ (x * (x * (x * y)) = x * y)$.
(b5) $(\forall x, y, z \in X)\ (x \leq y\ \Rightarrow\ x * z \leq y * z,\ z * y \leq z * x)$.
(b6) $(\forall x, y, z \in X)\ ((x * z) * (y * z) \leq x * y)$.

A non-empty subset S of a BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ whenever $x, y \in S$. A non-empty subset A of a BCI-algebra X is called an *ideal* of X if it satisfies:

(c1) $0 \in A$,
(c2) $(\forall x \in A)\ (\forall y \in X)\ (y * x \in A\ \Rightarrow\ y \in A)$.

Note that every ideal A of a BCI-algebra X satisfies:

$$(\forall x \in A)\,(\forall y \in X)\,(y \leq x\ \Rightarrow\ y \in A).$$

A non-empty subset A of a BCI-algebra X is called a *q-ideal* of X if it satisfies (c1) and

(c3) $(\forall x, y, z \in A)(x * (y * z) \in A, y \in A \Rightarrow x * z \in A)$.

Note that any q-ideal is an ideal, but the converse is not true in general.

We refer the reader to the book [4] for further information regarding BCI-algebras.

Definition 2.1.([2]) A *vague set* A in the universe of discourse U is characterized by two membership functions given by:

1. A true membership function
$$t_A : U \to [0, 1],$$
and
2. A false membership function
$$f_A : U \to [0, 1],$$

where $t_A(u)$ is a lower bound on the grade of membership of u derived from the "evidence for u", $f_A(u)$ is a lower bound on the negation of u derived from the "evidence against u", and

$$t_A(u) + f_A(u) \leq 1.$$

Thus the grade of membership of u in the vague set A is bounded by a subinterval $[t_A(u), 1 - f_A(u)]$ of $[0, 1]$. This indicates that if the actual grade of membership of u is $\mu(u)$, then

$$t_A(u) \leq \mu(u) \leq 1 - f_A(u).$$

The vague set A is written as

$$A = \{\langle u, [t_A(u), f_A(u)]\rangle \mid u \in U\},$$

where the interval $[t_A(u), 1 - f_A(u)]$ is called the *vague value* of u in A, denoted by $V_A(u)$.

For $\alpha, \beta \in [0, 1]$ we now define (α, β)-cut and α-cut of a vague set. Recall that if $I_1 = [a_1, b_1]$ and $I_2 = [a_2, b_2]$ are two subintervals of $[0, 1]$, we can define a relation by $I_1 \succeq I_2$ if and only if $a_1 \geq a_2$ and $b_1 \geq b_2$.

Definition 2.2.([2]) Let A be a vague set of a universe X with the true-membership function t_A and the false-membership function f_A. The (α, β)-*cut* of the vague set A is a crisp subset $A_{(\alpha,\beta)}$ of the set X given by

$$A_{(\alpha,\beta)} = \{x \in X \mid V_A(x) \succeq [\alpha, \beta]\}.$$

Clearly $A_{(0,0)} = X$. The (α, β)-cuts of the vague set A are also called *vague-cuts* of A.

Definition 2.3.([2]) The α-*cut* of the vague set A is a crisp subset A_α of the set X given by $A_\alpha = A_{(\alpha,\alpha)}$.

Note that $A_0 = X$, and if $\alpha \geq \beta$ then $A_\alpha \subseteq A_\beta$ and $A_{(\alpha,\beta)} = A_\alpha$.

Equivalently, we can define the α-cut as

$$A_\alpha = \{x \in X \mid t_A(x) \geq \alpha\}.$$

3. Vague q-ideals

For our discussion, we shall use the following notations on interval arithmetic:

Let $I[0,1]$ denote the family of all closed subintervals of $[0,1]$. We define the term "imax" to mean the maximum of two intervals as

$$\text{imax}(I_1, I_2) := [\max(a_1, a_2), \max(b_1, b_2)],$$

where $I_1 = [a_1, b_1]$, $I_2 = [a_2, b_2] \in I[0,1]$. Similarly we define "imin". The concepts of "imax" and "imin" could be extended to define "isup" and "iinf" of infinite number of elements of $I[0,1]$.

It is obvious that $L = \{I[0,1], \text{isup}, \text{iinf}, \succeq\}$ is a lattice with universal bounds $[0,0]$ and $[1,1]$ (see [2]).

In what follows let X be a BCI-algebra unless specified otherwise.

Definition 3.1.([8]) A vague set A of a BCI-algebra X is called a *vague BCI-algebra* of X if the following condition is true:

(d0) $(\forall x \in X)(V_A(x * y) \succeq \text{imin}\{V_A(x), V_A(y)\})$.

that is,

$$t_A(x * y) \geq \min\{t_A(x), t_A(y)\},$$
$$1 - f_A(x * y) \geq \min\{1 - f_A(x), 1 - f_A(y)\}$$

for all $x, y \in X$.

Definition 3.2.([8]) A vague set A of X is called a *vague ideal* of X if the following conditions are true:

(d1) $(\forall x \in X)(V_A(0) \succeq V_A(x))$,
(d2) $(\forall x, y \in X)(V_A(x) \succeq \text{imin}\{V_A(x * y), V_A(y)\})$.

that is,

$$t_A(0) \geq t_A(x), 1 - f_A(0) \geq 1 - f_A(x),$$
$$\text{and } t_A(x) \geq \min\{t_A(x * y), t_A(y)\}$$
$$1 - f_A(x) \geq \min\{1 - f_A(x * y), 1 - f_A(y)\}$$

for all $x, y \in X$.

Proposition 3.3.([8]) *Every vague ideal of a BCI-algebra X satisfies the following properties:*

(i) $(\forall x, y \in X)(x \leq y \Rightarrow V_A(x) \succeq V_A(y))$,
(ii) $(\forall x, y, z \in X)(V_A(x * z) \succeq \textit{imin}\{V_A((x * y) * z), V_A(y)\})$.

Definition 3.4. A vague set A of X is called a *vague q-ideal* of X if it satisfies (d1) and

(d3) $(\forall x, y, z \in X)(V_A(x * z) \succeq \text{imin}\{V_A(x * (y * z)), V_A(y)\})$.

that is,

$$t_A(x*z) \geq \min\{t_A(x*(y*z)), t_A(y)\},$$
$$1-f_A(x*z) \geq \min\{1-f_A(x*(y*z)), 1-f_A(y)\}$$

for all $x, y, z \in X$.

Example 3.5. Let $X := \{0, a, b\}$ be a BCI-algebra([9]) in which the $*$-operation is given by the following table:

$*$	0	a	b
0	0	0	b
a	a	0	b
b	b	b	0

Let A be the vague set in X defined as follows:

$$A = \{\langle 0, [0.8, 0.1]\rangle, \langle a, [0.8, 0.1]\rangle, \langle b, [0.5, 0.3]\rangle\}.$$

It is routine to verify that A is a vague q-ideal of X.

Theorem 3.6. *Every vague q-ideal of a BCI-algebra X is both a vague ideal of X and a vague BCI-algebra of X.*

Proof. Let A be a vague q-ideal of X. Put $z := 0$ in (d3). Use (b1), we have (d2). Hence A is a vague ideal of X.
Putting $y := z$ in (d3), for any $x, y, z \in X$ we have

$$\begin{aligned} V_A(x*z) &\succeq \text{imin}\{V_A(x*(z*z)), V_A(z)\} \\ &= \text{imin}\{V_A(x*0), V_A(z)\} \\ &= \text{imin}\{V_A(x), V_A(z)\}. \end{aligned}$$

It means that A is a vague BCI-algebra of X □

The converse of Theorem 3.6 is not true in general as the following example.

Example 3.7. Let $X := \{0, a, b, c\}$ be a BCI-algebra([9]) in which the $*$-operation is given by the following table:

$*$	0	a	b	c
0	0	c	b	a
a	a	0	c	b
b	b	a	0	c
c	c	b	a	0

Let A be the vague set in X defined as follows:

$$A = \{\langle 0, [0.7, 0.2]\rangle, \langle a, [0.5, 0.4]\rangle, \langle b, [0.5, 0.4]\rangle, \langle c, [0.5, 0.4]\rangle\}.$$

It is routine to verify that A is both a vague ideal of X and a vague BCI-algebra of X. But it is not a vague q-ideal of X since $V_A(c * a) = V_A(b) \not\succeq \text{imin}\{V_A(c * (0 * a)), V_A(0)\}$.

Theorem 3.8. *Let A be a vague ideal of a BCI-algebra. Then the following are equivalent:*

(1) *A is a vague q-ideal of X.*
(2) $(\forall x, y \in X)(V_A(x * y) \succeq V_A(x * (0 * y))$.
(3) $(\forall x, y, z \in X)(V_A((x * y) * z) \succeq V_A(x * (y * z))$.

Proof. (1)⇒(2) Put $y = 0$ and $z = y$ in (d3). Hence for any $x, y \in X$, we have

$$\begin{aligned} V_A(x * y) \succeq& \text{imin}\{V_A(x * (0 * y)), V_A(0)\} \\ =& V_A(x * (0 * y)). \end{aligned}$$

(2)⇒(3) Since for any $x, y, z \in X$

$$\begin{aligned} ((x * y) * (0 * z)) * (x * (y * z)) =& ((x * y) * (x * (y * z))) * (0 * z) \\ \leq& ((y * z) * y) * (0 * z) \\ =& (0 * z) * (0 * z) = 0, \end{aligned}$$

we have $((x * y) * (0 * z)) * (x * (y * z)) = 0$, it follows from Proposition 3.3(i) that $V_A(x * (y * z)) \preceq V_A((x * y) * (0 * z)) \preceq V_A((x * y) * z)$. Thus (3) holds.

(3)⇒(1) Using Proposition 3.3(ii) and (3), we have

$$\begin{aligned} V_A(x * z) \succeq& \text{imin}\{V_A((x * y) * z), V_A(y)\} \\ \succeq& \text{imin}\{V_A(x * (y * z)), V_A(y)\}, \end{aligned}$$

for all $x, y, z \in X$. Thus (d3) holds. Thus A is a vague q-ideal of X. □

Theorem 3.9. *Let A be a vague ideal of a BCI-algebra X such that*

$$(\forall x, y \in X)(V_A(x * y) \succeq V_A(x)).$$

Then it is a vague q-ideal of X.

Proof. Using (d2) and assumption, we have

$$\begin{aligned} V_A(x * z) \succeq& \text{imin}\{V_A((x * z) * (y * z)), V_A(y * z)\} \\ =& \text{imin}\{V_A((x * (y * z)) * z), V_A(y * z)\} \\ \succeq& \text{imin}\{V_A(x * (y * z)), V_A(y * z)\} \\ \succeq& \text{imin}\{V_A(x * (y * z)), V_A(y)\} \end{aligned}$$

for all $x, y, z \in X$. Hence (d3) holds. Thus A is a vague q-ideal of X. □

The converse of Theorem 3.9 is not true in general as seen the following example.

Example 3.10. Consider a BCI-algebra $X := \{0, a, b\}$ and a vague set A as in Example 3.5. Then A is a vague q-ideal of X, but it does not satisfy $V_A(x * y) \succeq V_A(x)$ since $V_A(0 * b) = V_A(b) \not\succeq V_A(0)$.

Definition 3.11 A BCI-algebra X is said to be *associative* ([4]) if $(x * y) * z = x * (y * z)$ for any $x, y, z \in X$. A BCI-algebra X is said to be *quasi-associative* ([11]) if $(x * y) * z \leq x * (y * z)$ for any $x, y, z \in X$.

Every associative BCI-algebra X is quasi-associative, but the converse is not true in general (see [11]).

Proposition 3.12. *Let X be a quasi-associative BCI-algebra. Every vague ideal of X is a vague q-ideal of X.*

Proof. Let A be a vague ideal of X. Since X is a quasi-associative BCI-algebra, we have $(x * y) * z \leq x * (y * z)$ for any $x, y, z \in X$. It follows from Proposition 3.3(i) that $V_A((x * y) * z) \succeq V_A(x * (y * z))$. By Theorem 3.8, A is a vague q-ideal of X. □

Proposition 3.12 is not true in general if X is not a quasi-associative BCI-algebra as seen in the following example.

Example 3.13. Consider a BCI-algebra $X = \{0, a, b, c\}$ and a vague set A of X as in Example 3.7. Since $(a * b) * c \not\leq a * (b * c)$, X is not a quasi-associative BCI-algebra. Then A is a vague ideal of X but not a vague q-ideal of X.

Corollary 3.14. *Let X be an associative BCI-algebra. Every vague ideal of X is a vague q-ideal of X.*

Proof. Straightforward. □

Theorem 3.15. *Let A be a vague q-ideal of a BCI-algebra X. Then for any $\alpha, \beta \in [0, 1]$, the vague-cut $A_{(\alpha,\beta)}$ of A is a crisp q-ideal of X.*

Proof. Obviously, $0 \in A_{(\alpha,\beta)}$. Let $x * (y * z) \in A_{(\alpha,\beta)}$ and $y \in A_{(\alpha,\beta)}$. Then $V_A(x * (y * z)) \succeq [\alpha, \beta]$ and $V_A(y) \succeq [\alpha, \beta]$, i.e., $t_A(x * (y * z)) \geq \alpha, t_A(y) \geq \alpha$ and $1 - f_A(x * (y * z)) \geq \beta, 1 - f_A(y) \geq \beta$. It follows that

$$t_A(x * z) \geq \min\{t_A(x * (y * z)), t_A(y)\} \geq \alpha$$

and

$$1 - f_A(x * z) \geq \min\{1 - f_A(x * (y * z)), 1 - f_A(y)\} \geq \beta.$$

Hence $x * z \in A_{(\alpha,\beta)}$ and so $A_{(\alpha,\beta)}$ is a crisp q-ideal of X. □

The ideals like $A_{(\alpha,\beta)}$ are also called *vague cut q-ideals* of X.

Theorem 3.16. *Any q-ideal I of a BCI-algebra X is a vague-cut ideal of some vague q-ideal of X.*

Proof. Proof. Consider the vague set A of X given by

$$V_A(x) = \begin{cases} [\alpha,\alpha] & \text{if } x \in I \\ [0,0] & \text{if } x \notin I \end{cases}$$

where $\alpha \in (0,1)$. Since $0 \in I$, we have $V_A(0) = [\alpha,\alpha] \succeq V_A(x)$ for all $x \in X$. Let $x, y, z \in X$ be such that $x*(y*z) \in I$ and $y \in I$. If $x*z \notin I$, then

$$t_A(x*z) = 0 \leq \min\{t_A(x*(y*z)), t_A(y)\}$$

$$\text{and } 1 - f_A(x*z) = 0 \leq \min\{1 - f_A(x*(y*z)), 1 - f_A(y)\}.$$

If $x*z \in I$, then

$$t_A(x*z) = \alpha = \min\{t_A(x*(y*z)), t_A(y)\}$$

$$\text{and } 1 - f_A(x*z) = \alpha = \min\{1 - f_A(x*(y*z)), 1 - f_A(y)\}.$$

Thus A is a vague q-ideal of X. Clearly, $I = A_{(\alpha,\alpha)}$. □

Theorem 3.17. *Let A be a vague q-ideal of a BCI-algebra X. Then the set*

$$I := \{x \in X | V_A(x) = V_A(0)\}$$

is a crisp q-ideal of X.

Proof. Clearly, $0 \in I$. Let $x, y, z \in X$ be such that $x*(y*z) \in I$ and $y \in I$. Then $V_A(x*(y*z)) = V_A(0)$ and $V_A(y) = V_A(0)$ and so

$$V_A(x*z) \succeq \text{imin}\{V_A(x*(y*z)), V_A(y)\} = V_A(0).$$

Since $V_A(0) \succeq V_A(x)$ for all $x \in X$, it follows that $V_A(x*z) = V_A(0)$. Hence $x*z \in I$. Therefore I is a crisp q-ideal of X. □

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