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ON SUBMODULE TRANSFORMS T(N) AND S(N)

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Abstract. In this paper, we give some properties on submodule transforms.

0. Introduction

Let M be a module over commutative ring R with identity, S the set of nonzero divisors of R and R_S the total quotient ring of R. For a nonzero ideal I of R, let $I^{-1} = \{x \in R_S | xI \subseteq R\}$. I is said to be an *invertible ideal* of R if $II^{-1} = R$. Put $T = \{t \in S | tm = 0$ for some $m \in M$ implies $m = 0\}$. Then T is a multiplicatively closed subset of S and if M is torsion free, then $T = S$([9,Proposition 1.1]). In particular, if M is a faithful multiplication module then M is torsion free ([4,Lemma 4.1]) and so $T = S$. So in this case, $R_T = R_S$. Let N be a submodule of M. If $x = \frac{r}{t} \in R_T$ and $n \in N$, then we say that $xn \in M$ if there exists $m \in M$ such that $tm = rn$. Then this is a well defined operation([9,p399]). For a submodule N of M, $N^{-1} = \{x \in R_T | xN \subseteq M\} = [M :_{R_T} N]$. We say that N is *invertible* in M if $NN^{-1} = M$ and M is called a *Dedekind (resp. Prüfer)* module providing that every nonzero (resp. every nonzero finitely generated) submodule of M is invertible.

M is called a *multiplication module* if every submodule N of M has the form IM for some ideal I of R. An $R-$ module M is said to be *faithful* if Ann (M) $= [0 :_R M] = 0$.

Let R be an integral domain with quotient field $Q(R)$ and let I be an ideal of R. R.Gilmer and J.Huckaba ([7]) introduced the concept of ideal transform $T(I)$ of I; $T(I) = \cup_{n\geq 1}[R :_{Q(R)} I^n]$ and studied the problem of determining for which integral domain has the equality $T(IJ) = T(I)+T(J)$ for all ideals , or all finitely generated ideals, or all

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principal ideals I and J of R. Here $T(I)+T(J) = \{\alpha+\beta | \alpha \in T(I), \beta \in T(J)\}$, so that $T(I)+T(J)$ is not always a ring.

Ali([1]) generalized ideal transforms to submodules of modules over an integral domains as follows ; Let R be an integral domain and M a module over R. For a submodule N of M, $T(N) = \cup_{n\geq 0}[M :_{R_T} [N : M]^n N]$ where $[N : M]^0 = R$.

Consider the following conditions on M.

(T_1) $T([K : M]N) = T(K) + T(N)$ for all submodules K and N of M.

(T_2) $T([K : M]N) = T(K)+T(N)$ for all finitely generated submodules K and N of M.

We will say that M satisfies $T_1 - Property(resp. T_2 - Property)$ if $T([K : M]N) = T(K) + T(N)$ for all submodules (resp. all finitely generated submodules)K and N of M.

An $R-$ module M is called *cancellation* if for all ideals I and J of R, $IM = JM$ implies that $I = J$. In section 2 of this paper, we find new properties of submodule transforms of a faithful multiplication module over a domain (Theorem 2.4,Theorem 2.6 and Theorem 2.7).

In section 3, we define S-transform of submodules, S(N), for a submodule N of M and give some sufficient conditions for S(N) to be T(N)(Theorem 3.3 ,Theorem 3.4 and Theorem 3.5).

1. Ideal Transforms and Submodule transforms

In this section we give some properties to use in next sections.

Proposition 1.1. *Let I be an ideal of an integral domain R. Then $T(I) = T(I^n)$ for every positive integer n.*

Proof. Let $Q(R)$ be a quotient field of R and let $x \in T(I)$. Then $x \in Q(R)$ and $xI^r \subseteq R$ for some positive integer r. For any positive integer n, $I^n \subseteq I$ and hence $x(I^n)^r \subseteq xI^r \subseteq R$. So $x \in T(I^n)$. For the other inclusion, let $x \in T(I^n)$. Then $x \in Q(R)$ and $x(I^n)^s = xI^{ns} \subseteq R$ for some positive integer s. So $x \in T(I)$. □

Faithful multiplication module M over an integral domain R is torsion free ([4,Lemma 4.1]) and hence $T = S$([9,Proposition 1.1-(3)]). In this case $R_T = R_S = Q(R)$.

Proposition 1.2. *Let R be an integral domain,M a faithful multiplication module over R and N a submodule of M. Then $T(N) = T([N : M])$.*

Proof. $T(N) = \cup_{n\geq 0}[M :_{Q(R)} [N : M]^n N] = \cup_{n\geq 0}[M :_{Q(R)} [N : M]^{n+1}M] = \cup_{n\geq 0}[R :_{Q(R)} [N : M]^{n+1}] = T([N : M])$([1, p26]) □

Proposition 1.3. *Let R be an integral domain and M a faithful multiplication $R-$ module,then $[IN : M] = I[N : M]$ for all ideal I of R and any submodule N of M.*

Proof. Any faithful multiplication module M over an integral domain R is finitely generated ([6,Theorem 3.1]) and finitely generated faithful multiplication module is cancellation([10,Corollary to Theorem 9]). Since $IN = [IN : M]M$, $IN = I[N : M]M$ and M is cancellation, $[IN : M] = I[N : M]$. □

Proposition 1.4. *Let R be an integral domain , I an ideal of R and M a faithful multiplication R-module. Then $T(IM) = T(I^n M)$ for every positive integer n.*

Proof. $T(IM) = T([IM : M]) = T(I[M : M]) = T(IR) = T(I) = T(I^n) = T(I^n[M : M]) = T([I^n M : M]) = T(I^n M)$. □

2. Transforms T(N) of Submodules

In this section we consider some properties of submodule transforms of a faithful multiplication module over a domain.

Proposition 2.1. *Let R be an integral domain, M a faithful multiplication R-module and let N, K be submodules of M with $[N : M]N \subseteq K \subseteq N$. Then $T(K) = T(N)$.*

Proof. By [1,Theorem 1-(1)], $T(N) \subseteq T(K)$. $T([N : M]N) = T([N : M]N : M) = T([N : M]^2) = T([N : M]) = T(N)$ ([Proposition 1.1, 1.2 and 1.3]). Since $[N : M]N \subseteq K, T(K) \subseteq T([N : M]N) = T(N)$. □

Compare the following Proposition with [7,Corollary 3].

Proposition 2.2. *Let R be an integral domain, M a faithful multiplication R-module and let N, K be submodules of M with $T(N) \subseteq T(K)$.If K is finitely generated , then $T([K : M]N) = T(K) = T(K) + T(N)$.*

Proof. Since $T(N) \subseteq T(K)$, $T([N : M]) \subseteq T([K : M])$([Proposition 1.2]). We know that M is finitely generated ([6,Theorem 3.1]). Since K is finitely generated $[K : M]$ is also finitely generated ([3,Proposition 2.2-(2)]), $T([K : M][N : M]) = T([K : M]) = T([K : M]) + T([N : M])$([7, Corollary 3]). However, we know that $T([K : M][N : M]) =$

$T([K:M]N), T(K) = T([K:M])$ and $T(N) = T([N:M])$. Hence $T([K:M]N) = T(K) = T(K) + T(N)$. □

Compare the following Proposition with [7,Proposition 1-(f)].

Proposition 2.3. *Let R be an integral domain, M a faithful multiplication R-module and let N be a submodule of M. If $T(N) = R$ or $T(N) = Q(R)$ then $T([K:M]N) = T(K) + T(N)$.*

Proof. Suppose that $T(N) = R$. If $x \in T([K:M]N)$, then for some nonnegative integer n, $x[[K:M]N:M]^n[K:M]N \subseteq M$. Since $[[K:M]N:M] = [K:M][N:M]$(Proposition 1.4), we have that $x[K:M]^{n+1}[N:M]^nN \subseteq M$ and hence $x[K:M]^{n+1} \subseteq [M:[N:M]^nN] \subseteq T(N) = R$. Therefore $x \in [R:[K:M]^{n+1}] \subseteq T([K:M]) = T(K)$([Proposition 1.2]). Since $R \subseteq T(K)$, $T(N) \subseteq T(K)$ and so $T([K:M]N) \subseteq T(K) = T(K) + T(N)$. The other inclusion comes from [1, Theorem 1-(2)].

Now if $T(N) = Q(R)$ then $Q(R) = T(N) + T(K) \subseteq T([K:M]N)$ ([1,Theorem 1-(2)]) and since $T([K:M]N) \subseteq Q(R), T([K:M]N) = Q(R) = T(K) + T(N)$. □

Theorem 2.4. *Let R be an integral domain, M a faithful multiplication R-module and Λ the set of all submodule transforms of M. If M satisfies T_1- property , then $(\Lambda, +, \cup)$ is a distributive lattice.*

Proof. Let $T(K), T(N) \in \Lambda$. Since M satisfies T_1-property, $T([K:M]N) = T(K) + T(N)$. By [1,Theorem 1-(4)] we have $T(N) \cap T(K) = T(N+K)$. Hence Λ is closed under both "+" and "∩". It is then easy to show that Λ is a lattice. Now, to show that Λ is distributive, it is sufficient to prove that either of the distributive laws hold. We will prove that $(T(N)+T(K)) \cap T(L) = T(N) \cap T(L) + T(K) \cap T(L)$ for all $T(K), T(N), T(L) \in \Lambda$.

It is obvious that $T(N) \cap T(L) + T(K) \cap T(L) \subseteq (T(N)+T(K)) \cap T(L)$.

Note that $(T(N)+T(K)) \cap T(L) = T([K:M]N) \cap T(L) = T([K:M]N+L)$ and $[T(N) \cap T(L)] + [T(K) \cap T(L)] = T(N+L) + T(K+L) = T([(K+L):M](N+L))$.

Therefore we prove $T([K:M]N+L) \subseteq T([(K+L):M](N+L))$ for the other inclusion. However,

$[(K+L):M](N+L) = [(K+L):M]N + [(K+L):M]L$
$= [(K+L):M][N:M]M + [(K+L):M]L$
$= [N:M][(K+L):M]M + [(K+L):M]L$
$= [N:M](K+L) + [(K+L):M]L$
$= [N:M]K + [N:M]L + [(K+L):M]L.$

Since M is a multiplication module, $[N : M]K = [N : M][K : M]M = [K : M][N : M]M = [K : M]N$. Hence, $[N : M]K + [N : M]L + [(K + L) : M]L$
$\subseteq [N : M]K + L = [K : M]N + L$.

Again by [1,Theorem 1-(1)], $T([K : M]N+L) \subseteq T([(K+L) : M](N+L))$ and we complete our proof. □

We give a partial answer for the converse of above Theorem.

Corollary 2.5. *Let R be a Noetherian domain, M a faithful multiplication R-module and Λ the set of all submodule transforms of M.If $(\Lambda, +, \cap)$ is a distributive lattice, then M satisfies T_1- property.*

Proof. Let $\bar{\Lambda}$ be the set of all finitely generated submodule transforms of M. Since M is Noetherian ([5,Proposition 2.10]) $\Lambda = \bar{\Lambda}$.

Hence we know that M satisfies T_1-property if and only if M satisfies T_2-property. If $(\Lambda, +, \cap)$ is a distributive lattice then $(\bar{\Lambda}, +, \cap)$ is a distributive lattice. So M satisfies T_2-property ([1,Corollary 9]) and hence M satisfies T_1-property.

□

Theorem 2.6. *Let R be a Noetherian domain,M a faithful multiplication R-module and Λ the set of all submodule transforms of M. Then the following statements are equivalent.*

(1)M satisfies T_1-property.

(2)$(\Lambda, +, \cap)$ is a distributive lattice.

(3)$(\Lambda, +, \cap)$ is a lattice.

Proof. (1) $\Rightarrow$ (2) It follows from Theorem 2.4.

(2) $\Rightarrow$ (3) It is clear.

(3) $\Rightarrow$ (1) It follows from [1, Corollary 9] and $\Lambda = \bar{\Lambda}$ ([5,Proposition 2.10]). □

Theorem 2.7. *T_1-property holds in a faithful multiplication Dedekind module M over an integral domain R.*

Proof. As M is Noetherian ([2,Theorem 2.4]), M satisfies T_1-property if and only if M satisfies T_2-property. Furthermore M is Prüfer([2,Theorem 2.4]). The result comes from ([1,Proposition 4]). □

Proposition 2.8. *Let R be an integral domain, M a faithful multiplication R-module and Γ be the lattice of all submodules of M , Λ the set of all submodule transforms of M. If M satisfies T_1-property, then the map $\phi : (\Gamma, +, \cap) \rightarrow (\Lambda, +, \cap)$ defined by $\phi(N) = T(N)$ is an order* *reversing lattice homomorphism which interchanges the operations "+" and "∩".*

Proof. For any $N, K \in \Gamma$ with $N \subseteq K$, $T(K) \subseteq T(N)$ and hence $\phi(N) = T(N) \supseteq \phi(K) = T(K)$.

$\phi(N + K) = T(N + K) = T(N) \cap T(K)$([1,Theorem 1-(4)]).

$\phi(N \cap K) = T(N \cap K) = T([K : M]N) = T(K) + T(N)$(1,[Theorem 1-(3)]). □

3. Transforms $S(N)$ of submodules

Hays([8]) defined S-transform, $S(I)$, of an ideal I of an integral domain R with quotient field $Q(R)$; $S(I)$ is the set of elements $x \in Q(R)$ such that for each $a \in I, xa^{n_a} \in R$ for some positive integer n_a. Author gave some relations between $T(I)$ and $S(I)$. Now we generalize S-transform for ideals of a ring R to submodules of faithful multiplication modules over an integral domains.

Let R be an integral domain and M a faithful multiplication module over R.

We define *S- transform S(N)* for a submodule N of M to be the set of elements $x \in Q(R)$ such that for each $a \in [N : M]$ and for some positive integer n_a, $xa^{n_a}N \subseteq M$.

In this section we prove some properties about $S(N)$ and we give some sufficient conditions for $S(N)$ to be $T(N)$.

Proposition 3.1. *Let R be an integral domain and M a faithful multiplication module over R. For any submodule N of M, $T(N) \subseteq S(N)$.*

Proof. It is obvious. □

Proposition 3.2. *Let R be an integral domain and M a faithful multiplication module over R. For submodules N, K of M, if $N \subseteq K$ then $S(K) \subseteq S(N)$.*

Proof. Let a be any element in $[N : M]$ and let $x \in S(K)$. Then $a \in [K : M]$ and there exists some positive integer n_a such that $xa^{n_a}K \subseteq M$. Hence $xa^{n_a}N \subseteq xa^{n_a}K \subseteq M$ and $x \in S(N)$. □

Compare the following Theorem with [8,Theorem 1.3].

Theorem 3.3. *Let R be an integral domain and M a faithful multiplication module over R. If N is a finitely generated submodule of M then $T(N) = S(N)$.*

Proof. It is sufficient to show that $S(N) \subseteq T(N)$. Let $x \in S(N)$. Since N is a finitely generated submodule of M, [N:M] is also a finitely generated ideal of R([3,Proposition 2.2-(2)]). Now put $[N : M] = (a_1, \cdots, a_r)$ for some $a_i \in R$. Since $x \in S(N)$, there exist positive integers n_i such that $xa_i^{n_i}N \subseteq M$ for $1 \leq i \leq r$. Let $n = \sum_{i=1}^{r} n_i$. Then $[N : M]^n$ is generated by elements of the form $a_1^{m_1} \cdots a_r^{m_r}$ with $\sum_{i=1}^{r} m_i = n$. Thus $m_i \geq n_i$ for some $i(1 \leq i \leq r)$. Hence $x[N : M]^n N \subseteq M$ and $x \in T(N)$. □

Compare the following Propositions with [8, Lemma 1.11 and Lemma 1.12].

Theorem 3.4. *Let R be an integral domain and M a faithful multiplication module over R. If one of the following conditions hold, then $T(N) = S(N)$.*

(1) there exists finitely generated submodule $K \subseteq N$ such that $T(K) = T(N)$.

(2) there exists finitely generated submodule $K \subseteq N$ such that $[N : M]N \subseteq K \subseteq N$.

Proof. First, we assume that condition (1) holds. We show that $S(N) \subseteq T(N)$ because of $T(N) \subseteq S(N)$([Proposition 3.1]).

$S(N) \subseteq S(K)$ ([Proposition 3.2]) and $S(K) = T(K)$ ([Theorem 3.3]). Therefore $S(N) \subseteq T(K) = T(N)$.

Now we assume that condition (2) holds. $T([N : M]N) = T([N : M]N : M) = T([N : M][N : M]) = T([N : M]) = T(N)$. Since $[N : M]N \subseteq K$, $T(K) \subseteq T([N : M]N) = T(N) \subseteq T(K)$. By (1), $T(N) = S(N)$. □

An $R-$ module M is called *valuation module* if for all $m, n \in M$, $Rm \subseteq Rn$ or $Rn \subseteq Rm$. Equivalently, for all submodules N, K of M, either $N \subseteq K$ or $K \subseteq N$. ([2])

Theorem 3.5. *Let R be an integral domain and M a faithful multiplication valuation module over R. If $N \neq [N : M]N$ then $T(N) = S(N)$.*

Proof. Let $n \in N - [N : M]N$. Then $Rn \nsubseteq (N : M)N$. Since M is a valuation module, $(N : M)N \subseteq Rn(\subseteq N)$. Hence $T(N) \subseteq T(Rn) \subseteq T([N : M]N)$ and $T([N : M]N) = T([N : M][N : M]M) = T([N : M]^2M) = T([N : M]M)$([Proposition 1.3])$= T(N)$. Thus $T(N) = T(Rn)$.

Hence we obtain our result from Theorem 3.4-(1). □

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