지구물리와 물리탐사
Jigu-Mulli-wa-Mulli-Tamsa
Vol. 15, No. 4, 2012, p. 245~248

단 보

http://dx.doi.org/10.7582/GGE.2012.15.4.245

Upward Continuation of Potential Field on Spherical Patch Area

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구면부분지역에서의 퍼텐셜마당의 상향연속

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Abstract: Two dimensional Fourier transform can be used for the upward continuation of gravity or magnetic field data acquired at given altitude over a rectangular area. Earth's curvature is often neglected in most potential field continuations, however, it should be considered over several hundred kilometer field area. In this study, we developed a new method retaining terms of Earth's curvature to better perform the continuation of potential field on spherical patch area.

Keywords: gravity or magnetic field, upward continuation, Fourier transform

요 약: 일정고도에서 얻어진 중력 혹은 자력탐사자료에 대하여 보다 높은 고도에서의 값으로 연속하여 구하는 방법, 즉 상향연속의 경우 2차원 푸리에변환을 사용할 수 있다. 기존의 상향연속 방법에서는 지구의 곡률이 고려되지 않았으나, 지역이 수백킬로미터 이상으로 넓은 경우 곡률이 고려됨이 바람직하다. 본 연구에서는 지구의 곡률효과를 산정하는 중/자력마당의 새로운 상향연속방법을 계발하였다.

주요어: 중자력마당, 상향연속, 푸리에변환

Introduction

Gravity and magnetic fields are two potential fields, which have been widely used for important reconnaissance surveys in geophysical prospecting. Recently, the data acquisitions of these fields are often done by airborne or on-satellite measurements. These modern techniques enable us to cover larger area more easily than ever. With the measurement at given height, the upward/downward continuation of potential field is a powerful method to investigate the distribution and properties of sources at depth or other features at different heights. If the whole globe is under consideration, spherical harmonic series would be the relevant one to describe the global field. Upward continuation of gravity or magnetic field has generally been done by two dimensional Fourier transforms on rectangular and flat areas.

Although the flat Earth had been normally assumed in the potential field continuation, it is desirable to consider the curvature of the Earth, when data area exceeds a few degrees of latitude and longitude. In this study, we derive a new algorithm for upward continuation of gravity or magnetic field on a spherical patch area (Fig. 1). Both the field and its potential can be continued by similar ways with only different exponents for radial dependence.

Laplace Equation and Spherical Patch Transform

At field point $\vec{r}$ outside of Earth, the potential $U(\vec{r})$ of either gravity or magnetic field should satisfy Laplace equation; $\nabla^2 U(\vec{r}) = 0$. Spherical coordinates are the relevant coordinate system here, and Laplace equation for a potential $U(\vec{r})$ is given as the following (Arfken and Weber, 2005).

2012년 7월 26일 접수; 2012년 8월 28일 수정; 2012년 10월 23일 채택;
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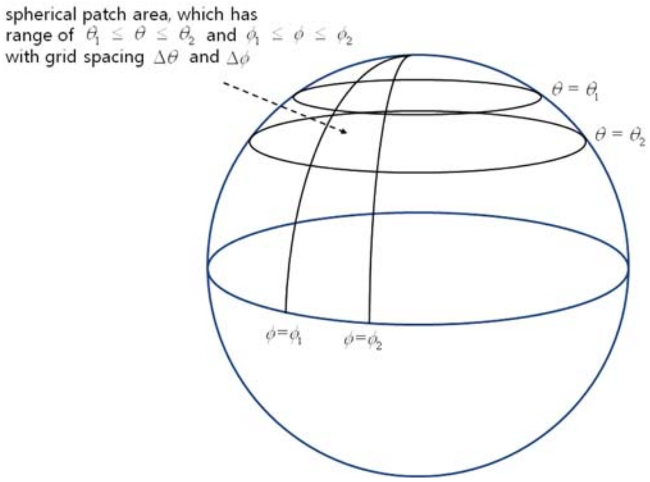

Fig. 1. Schematic figure of a spherical patch area dealt in this study.

$$\nabla^2 U(\vec{r}) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial U}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 U}{\partial\phi^2} = 0 \quad (1)$$

Suppose we know a set of values of potential field $U(\vec{r})$ on a spherical patch area, which is regularly gridded in ranges of latitude and longitude as follows.

$$\theta_1 < \theta < \theta_2 \text{ with grid spacing } \Delta\theta\text{, and}$$
$$\phi_1 < \phi < \phi_2 \text{ with grid spacing } \Delta\phi, \quad (2)$$

where total numbers of grids are K and L, *i.e.*, $\theta_2 - \theta_1 = (K - 1)\Delta\theta$ and $\phi_2 - \phi_1 = (L - 1)\Delta\phi$.

Let us denote $U(\vec{r}) = U(r,\theta_1+k\Delta\theta,\phi_1+l\Delta\phi)$ as $U(r,k,l)$ and its two dimensional discrete Fourier transform at the same radius (height) as $\tilde{U}(r,m,n)$. Then forward and inverse relations between $U(r,k,l)$ and $\tilde{U}(r,m,n)$ can be expressed as follows (Press *et al*., 1986).

$$\tilde{U}(r,m,n) = \frac{1}{K}\sum_{k=0}^{K-1}\frac{1}{L}\sum_{l=0}^{L-1} U(r,k,l)\exp\left(-2\pi i\frac{km}{K}\right)\exp\left(-2\pi i\frac{ln}{L}\right) \quad (3)$$

$$U(r,k,l) = \sum_{m=0}^{K-1}\sum_{n=0}^{L-1}\tilde{U}(r,m,n)\exp\left(+2\pi i\frac{km}{K}\right)\exp\left(+2\pi i\frac{ln}{L}\right) \quad (4)$$

Replacements of the differentiations $\partial/\partial\theta$ and $\partial/\partial\phi$ by $1/\Delta\theta\ \partial/\partial k$ and $1/\Delta\phi\ \partial/\partial l$ lead to discrete version of Eq. (1) as follows.

$$\nabla^2 U(r,k,l) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U(r,k,l)}{\partial r}\right) + \frac{1}{r^2\sin\theta}\left(\frac{1}{\Delta\theta}\frac{\partial}{\partial k}\right)\left(\frac{\sin\theta}{\Delta\theta}\frac{\partial U(r,k,l)}{\partial k}\right)$$
$$+ \frac{1}{r^2\sin^2\theta}\left(\frac{1}{\Delta\phi}\right)^2\frac{\partial^2 U(r,k,l)}{\partial l^2} = 0 \quad (5)$$

where $\sin\theta = \sin(\theta_1 + k\Delta\theta)$. Each terms of Eq. (5) without $1/r^2$ are re-written as the followings, where $U(r,k,l)$ is expressed as the inverse of $\tilde{U}(r,m,n)$.

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial U(r,k,l)}{\partial r}\right) = \left(r^2\frac{\partial^2}{\partial r^2} + 2r\frac{\partial}{\partial r}\right)U(r,k,l)$$
$$= \left(r^2\frac{\partial^2}{\partial r^2} + 2r\frac{\partial}{\partial r}\right)\sum_{m=0}^{K-1}\sum_{n=0}^{L-1}\tilde{U}(r,m,n)\exp\left(2\pi i\frac{km}{K}\right)\exp\left(2\pi i\frac{ln}{L}\right)$$
$$= \sum_{m=0}^{K-1}\sum_{n=0}^{L-1}\left(r^2\frac{\partial^2\tilde{U}(r,m,n)}{\partial r^2} + 2r\frac{\partial\tilde{U}(r,m,n)}{\partial r}\right)\exp\left(2\pi i\frac{km}{K}\right)\exp\left(2\pi i\frac{ln}{L}\right) \quad (6a)$$

$$\frac{1}{\sin\theta}\left(\frac{1}{\Delta\theta}\frac{\partial}{\partial k}\right)\left(\frac{\sin(\theta_1+(k-1)\Delta\theta)}{\Delta\theta}\frac{\partial U(r,k,l)}{\partial k}\right)$$
$$= \left(\frac{1}{(\Delta\theta)^2}\frac{\partial^2}{\partial k^2} + \frac{\cos\theta}{\Delta\theta\sin\theta}\frac{\partial}{\partial k}\right)U(r,k,l)$$
$$= \left(\frac{1}{(\Delta\theta)^2}\frac{\partial^2}{\partial k^2} + \frac{\cot\theta}{\Delta\theta}\frac{\partial}{\partial k}\right)\sum_{k=0}^{K-1}\sum_{n=0}^{L-1}\tilde{U}(r,m,n)\exp\left(2\pi i\frac{km}{K}\right)\exp\left(2\pi i\frac{ln}{L}\right)$$
$$= \sum_{k=0}^{K-1}\sum_{n=0}^{L-1}\left(\frac{1}{(\Delta\theta)^2}\frac{\partial^2}{\partial k^2} + \frac{\cot\theta}{\Delta\theta}\frac{\partial}{\partial k}\right)\tilde{U}(r,m,n)\exp\left(2\pi i\frac{km}{K}\right)\exp\left(2\pi i\frac{ln}{L}\right)$$
$$= \sum_{k=0}^{K-1}\sum_{n=0}^{L-1}\left(\frac{1}{(\Delta\theta)^2}\left(-4\pi^2\frac{m^2}{K^2}\right) + \frac{\cot\theta}{\Delta\theta}\left(2\pi i\frac{m}{K}\right)\right)\tilde{U}(r,m,n)\exp\left(2\pi i\frac{km}{K}\right)$$
$$\exp\left(2\pi i\frac{ln}{L}\right) \quad (6b)$$

$$\frac{1}{\sin^2\theta}\left(\frac{1}{\Delta\phi}\right)^2\frac{\partial^2 U(r,k,l)}{\partial l^2}$$
$$= \frac{1}{\sin^2\theta}\left(\frac{1}{\Delta\phi}\right)^2\frac{\partial^2}{\partial l^2}\sum_{m=0}^{K-1}\sum_{n=0}^{L-1}\tilde{U}(r,m,n)\exp\left(2\pi i\frac{km}{K}\right)\exp\left(2\pi i\frac{ln}{L}\right)$$
$$= \frac{1}{\sin^2\theta}\left(\frac{1}{\Delta\phi}\right)^2\sum_{m=0}^{K-1}\sum_{n=0}^{L-1}\left(-4\pi^2\frac{n^2}{L^2}\right)\tilde{U}(r,m,n)\exp\left(2\pi i\frac{km}{K}\right)\exp\left(2\pi i\frac{ln}{L}\right) \quad (6c)$$

Then the discrete version of Laplace equation in terms of $\tilde{U}(r,m,n)$ is found as follows.

$$\sum_{m=0}^{K-1}\sum_{n=0}^{L-1}\left[\begin{array}{c} r^2\frac{\partial^2\tilde{U}}{\partial r^2} + 2r\frac{\partial\tilde{U}}{\partial r} \\ +(\Delta\theta)^{-2}\left(-4\pi^2\frac{m^2}{K^2}\right)\tilde{U} + (\Delta\theta)^{-1}\left(2\pi i\frac{m}{K}\right)\cot\theta\tilde{U} \\ +(\Delta\phi)^{-2}\left(-4\pi^2\frac{n^2}{L^2}\right)\sin^{-2}\theta\tilde{U} \end{array}\right]$$
$$\times\ \exp\left(2\pi i\frac{km}{K}\right)\exp\left(2\pi i\frac{ln}{L}\right) = 0 \quad (7)$$

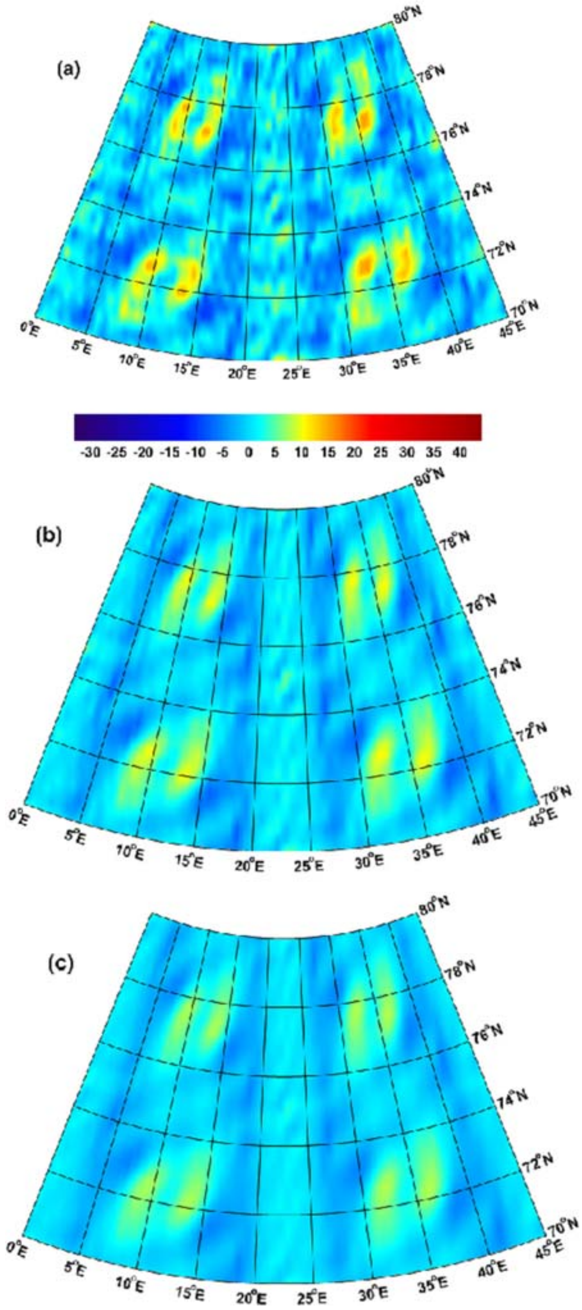

Fig. 2. Illustrations of gravity fields calculated by the algorithm of this study at different heights; (a) a gravity field at radius of r_1, (b) and (c) - two gravity fields acquired by upward continuation using the spherical patch transform of this study with radii $r_2 = 1.004r_1$ and $1.008r_1$. For example, if r_1 is specified as 6400 km, each target radii correspond to 6425.6 and 6451.2 km (b and c), under mass free assumption in the space between r_1 and r_2. The latitude range is between 70° and 80° N, and the longitude range is 45° (arbitrary unit).

Since Eq. (7) holds for arbitrary $\tilde{U}(r,m,n)$, it is claimed here that the summand of Eq. (7) vanishes. More specifically, the sum of the coefficients in the bracket of Eq. (7) should vanish. Then the resultant differential equation for $\tilde{U}(r,m,n)$ is written as follows.

$$r^2\frac{\partial^2\tilde{U}}{\partial r^2} + 2r\frac{\partial\tilde{U}}{\partial r} + A\tilde{U} = 0\,, \tag{8a}$$

where A is defined as

$$A = -\left(\frac{2\pi m}{K\Delta\theta}\right)^2 + \frac{2\pi i m}{K\Delta\theta}\cot(\theta_1 + k\Delta\theta) - \left(\frac{2\pi n}{L\Delta\phi\sin(\theta_1 + k\Delta\theta)}\right)^2 . \tag{8b}$$

Eq. (8a) belongs to Euler-Cauchy differential equation, and its solution is given as follows.

$$\tilde{U}(r,m,n) = Cr^{\lambda} \tag{9a}$$

Inserting Eq. (9a) into Eq. (8a) yields the condition for λ as follows.

$$\lambda(\lambda-1)r^{\lambda} + 2\lambda r^{\lambda} + Ar^{\lambda} = 0$$

The exponent λ is found as

$$\lambda = \frac{-1\pm\sqrt{1-4A}}{2}\,, \tag{9b}$$

where minus sign should be taken here. Unfortunately θ-dependence cannot be completely removed in the expression of A as in Eq. (8b).

The upward continuation of gravity potential U can be done by the following procedure.

$$U(r_1,k,l) \Rightarrow \tilde{U}(r_1,m,n) \Rightarrow \tilde{U}(r_2,m,n) \Rightarrow U(r_2,k,l)$$

The three successive steps are;

1) acquire Fourier transform $\tilde{U}(r_1,m,n)$ of the given potential $U(r_1,k,l)$,

2) perform the upward continuation in the wave number domain as $\tilde{U}(r_2,m,n) = \tilde{U}(r_1,m,n)\left(\frac{r_2}{r_1}\right)^{\lambda}$,

3) do the inverse Fourier transform to acquire $U(r_2,k,l)$.

As can be expected from the second step; $\tilde{U}(r_2) = \tilde{U}(r_1)\left(\frac{r_2}{r_1}\right)^{\lambda}$, the algorithm of this study can be used both for the upward and downward continuations.

In practice, usually the gravity field itself is processed. The continuation of gravity field g can be attained similarly.

$$g(r_1,k,l) \Rightarrow \tilde{g}(r_1,m,n) \Rightarrow \tilde{g}(r_2,m,n) \Rightarrow g(r_2,k,l)$$

It should be noted that the exponent in the second procedure is replaced by $\lambda - 1$ due to the difference in radial dependence. Our sample output figures, which were calculated according to the algorithm of this study, show typical feature of upward continuation (Fig. 2(a-c)).

Conclusions

A new method is developed to perform the upward continuation of potential field on spherical patch area by using two-dimensional Fourier transform. Our method includes the curvature of the Earth, which had been neglected in pre-existing continuation scheme. Thus, it yields more accurate estimation of gravity/magnetic field or their potentials than simple continuation done by flat Earth approximation, when the dimension of area exceeds one hundred kilometers.

Acknowledgments

This study was supported by the fund of CATER 2012-8010 provided by Center for Atmospheric Sciences and Earthquake Research (CATER) of Korea.

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