Pricing Outside Lookback Options with Guaranteed Floating Strike

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Abstract

A floating-strike lookback call (or put) option gives the holder the right to buy (or sell) at some percentage of the lowest (or highest) price of the underlying asset. This paper will propose an outside lookback call (or put) option that gives the holder the right to buy (or sell) one underlying asset at its guaranteed floating-strike price that is some percentage times the smaller (or the greater) of a specific guaranteed amount and the lowest (or highest) price of the other underlying asset. In addition, this paper derives explicit pricing formulas for these outside lookback options. Section 3 and Section 4 assume that the underlying assets pay no dividends. In contrast, Section 5 derives explicit pricing formulas for these options when their underlying assets pay dividends continuously at a rate proportional to their prices. Some numerical examples are also discussed.

Keywords: Floating strike, outside lookback option, Brownian motion.

1. Introduction

A floating-strike lookback call (or put) option gives the holder the right to buy (or sell) at some percentage of the lowest (highest) price of the underlying asset. Goldman *et al.* (1979) derived explicit pricing formulas for floating-strike lookback options where the highest (or lowest) price of the underlying asset is attained during the whole life of the options. Conze and Viswanathan (1991) derived explicit pricing formulas for partial floating-strike lookback options that give the holder the right to buy (or sell) at some percentage times the lowest (or highest) price. Heynen and Kat (1994b, 1997) suggested a way to reduce the price of these partial floating-strike lookback options while preserving some of their good qualities and derives explicit pricing formulas for the proposed options. Lee (2008) derived explicit pricing formulas for floating-strike lookback options whose monitoring period starts at an arbitrary date and ends at another arbitrary date before maturity.

However, researches listed above concern lookback options whose payoff depends on one underlying asset. Lee (2009) discussed outside floating-strike lookback options whose payoffs depend on prices of two underlying assets: the terminal value of one asset is used to determine the payoff, and the maximum (or minimum) value of the other asset to determine the floating strike. This paper proposes an outside floating-strike lookback call (or put) option that gives the holder the right to buy (or sell) one underlying asset at its guaranteed floating-strike price that is some percentage times the smaller (or the greater) of a specific guaranteed amount and the lowest (or highest) price of the other underlying asset. These proposed options will be a generalization of the corresponding floating-strike lookback options. In addition, this paper will present explicit pricing formulas for these proposed options.

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This paper is organized as follows. Section 2 discusses some basics for pricing contingent claims and will derive some useful expectations and probabilities for pricing the proposed options. Section 3 and Section 4 present explicit pricing formulas for the outside floating-strike lookback put and call options, respectively. In addition, Section 5 derives explicit pricing formulas for these options when their underlying assets pay dividends continuously at a rate proportional to their prices. These pricing formulas are generalization of the pricing formulas in Section 3 and Section 4. Some numerical examples are discussed.

2. Esscher Transforms and Some Useful Formulas

This section discusses some basics for pricing contingent claims and calculates some useful expectations and probabilities for pricing the proposed options. If we assume the Black-Scholes framework, then according to the fundamental theorem of asset pricing, the prices of contingent claims such as options can be calculated as the discounted expectations of the corresponding payoffs with respect to the equivalent martingale measure. Gerber and Shiu (1994, 1996) showed that Esscher transforms are an efficient tool to find the equivalent martingale measure if the logarithms of the prices of the underlying assets are stochastic processes with stationary and independent increments. This section briefly summarizes a special case of the method of Esscher transforms and demonstrates that the factorization formula is a main feature of this method and that it can simplify many calculations. For general methods of option pricing, see Baxter and Rennie (1998), Nelken (1996) and Zhang (1998).

Let $S_1(t)$ and $S_2(t)$ denote the time-t prices of two underlying assets. Assume that these assets pay no dividends. Assume that for $t \geq 0$, $i = 1$ and 2,

$$S_i(t) = S_i(0) \exp(X_i(t)), \tag{2.1}$$

where $\{\boldsymbol{X}(t) = (X_1(t), X_2(t))'\}$ is a 2-dimensional Brownian motion with drift vector $\boldsymbol{\mu} = (\mu_1, \mu_2)'$, $X_i(0) = 0$ and diffusion matrix $\boldsymbol{V}$ equal to

$$\begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}. \tag{2.2}$$

Thus the 2-dimensional Brownian motion is a stochastic process with independent and stationary increments and $\boldsymbol{X}(t) = (X_1(t), X_2(t))'$ has a bivariate normal distribution with mean vector $\boldsymbol{\mu}t$ and covariance matrix $\boldsymbol{V}t$.

For a nonzero real vector $\boldsymbol{h} = (h_1, h_2)'$, the moment generating function of $\boldsymbol{X}(t)$, $E[e^{\boldsymbol{h}'\boldsymbol{X}(t)}]$, exists for all $t \geq 0$, because $\{\boldsymbol{X}(t)\}$ is the Brownian motion as described above. The stochastic process

$$\left\{ e^{\boldsymbol{h}'\boldsymbol{X}(t)} E\left[e^{\boldsymbol{h}'\boldsymbol{X}(1)} \right]^{-t} \right\}$$

is a positive martingale that can be used to define a new probability measure Q. In technical terms, this process is used to define the Radon-Nikodym derivative dQ/dP, where P is the original probability measure. We call Q the Esscher measure of parameter vector $\boldsymbol{h}$.

For a random variable Y that is a real-valued function of $\{\boldsymbol{X}(t), 0 \leq t \leq T\}$, the expectation of Y under the new probability measure Q is calculated as

$$E\left[Y \frac{e^{\boldsymbol{h}'\boldsymbol{X}(\boldsymbol{T})}}{E\left[e^{\boldsymbol{h}'\boldsymbol{X}(1)} \right]^{\boldsymbol{T}}} \right], \tag{2.3}$$

which will be denoted by $E[Y; \boldsymbol{h}]$. The risk-neutral measure is the Esscher measure of parameter vector $\boldsymbol{h} = \boldsymbol{h}^*$ with respect to which the process $\{e^{-rt} S_i(t)\}$ is a martingale. Here, r is a risk-free rate. Thus

$$E\left[e^{-rt} S_i(t); \boldsymbol{h}^*\right] = S_i(0). \tag{2.4}$$

Therefore, $\boldsymbol{h}^*$ is the solution of

$$\boldsymbol{\mu} + \boldsymbol{V} \boldsymbol{h}^* = \left(r - \frac{\sigma_1^2}{2}, r - \frac{\sigma_2^2}{2}\right)'. \tag{2.5}$$

For $t \geq 0$, the moment generating function of $\boldsymbol{X}(t)$ under Esscher measure of parameter vector $\boldsymbol{h}$ is

$$E\left[e^{z'X(t)}; \boldsymbol{h}\right] = \exp\left\{(\boldsymbol{\mu}' + \boldsymbol{h}'\boldsymbol{V})zt + \frac{z'\boldsymbol{V}zt}{2}\right\}, \tag{2.6}$$

which implies that $\boldsymbol{X}(t)$ has a bivariate normal distribution with mean vector $(\boldsymbol{\mu} + \boldsymbol{V}\boldsymbol{h})t$ and variance $\boldsymbol{V}t$ under the Esscher measure. It can be shown that the process $\{\boldsymbol{X}(t)\}$ under the Esscher measure has independent and stationary increments. Thus, this process is a two-dimensional Brownian motion with drift vector

$$\boldsymbol{\mu} + \boldsymbol{V}\boldsymbol{h} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} + \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} \mu_1 + \sigma_1^2 h_1 + \rho\sigma_1\sigma_2 h_2 \\ \mu_2 + \rho\sigma_1\sigma_2 h_1 + \sigma_2^2 h_2 \end{pmatrix} \tag{2.7}$$

and diffusion matrix $\boldsymbol{V}$ under the Esscher measure of parameter vector $\boldsymbol{h}$.

   Let us consider a special case of the factorization formula (Gerber and Shiu, 1994, 1996). For a random variable Y that is a real-valued function of $\{\boldsymbol{X}(t), 0 \leq t \leq T\}$,

$$E\left[e^{g'X(T)}Y; \boldsymbol{h}\right] = E\left[e^{g'X(T)}; \boldsymbol{h}\right] E[Y; \boldsymbol{h} + \boldsymbol{g}]. \tag{2.8}$$

In particular, for an event B whose condition is determined by $\{\boldsymbol{X}(t), 0 \leq t \leq T\}$, formula (2.8) can be expressed as follows:

$$E\left[e^{g'X(T)}I(B); \boldsymbol{h}\right] = E\left[e^{g'X(T)}; \boldsymbol{h}\right] \Pr(B; \boldsymbol{h} + \boldsymbol{g}), \tag{2.9}$$

where $I(\cdot)$ denotes the indicator function and $\Pr(B; \boldsymbol{h})$ denotes the probability of the event B under the Esscher measure of parameter vector $\boldsymbol{h}$.

   Now, let

$$M_2(T) = \max\{X_2(\tau), 0 \leq \tau \leq T\} \tag{2.10}$$

and

$$m_2(T) = \min\{X_2(\tau), 0 \leq \tau \leq T\}, \tag{2.11}$$

for $T > 0$. In Heynen and Kat (1994a), it can be shown that the joint distribution function of $M_2(T)$ and $X_1(T)$ is

$$\begin{aligned} &\Pr(X_1(T) \leq x, M_2(T) \leq m) \\ &= \Phi_2\left(\frac{x - \mu_1 T}{\sigma_1 \sqrt{T}}, \frac{m - \mu_2 T}{\sigma_2 \sqrt{T}}; \rho\right) - e^{\frac{2\mu_2}{\sigma_2^2} m} \Phi_2\left(\frac{x - \mu_1 T}{\sigma_1 \sqrt{T}} - \frac{2\rho m}{\sigma_2 \sqrt{T}}, \frac{-m - \mu_2 T}{\sigma_2 \sqrt{T}}; \rho\right), \end{aligned} \tag{2.12}$$

where $\Phi_2(a, b; \rho)$ denotes the bivariate standard normal distribution function with correlation coefficient ρ. The joint distribution (2.12) will be used for calculating the following expectation (2.13).

Next, consider some useful expectations for pricing the proposed options. Assume that $\xi = 2\mu_2/\sigma_2^2$, $\eta = 1 - 2\rho(\sigma_1/\sigma_2)$ and $c + \xi \neq 0$. The proof of (2.13) will be given in the Appendix. Here, $a \vee b$ denotes $\max(a, b)$. For $d \geq 0$,

$$E\left[e^{c \cdot (M_2(T) \vee d)} I((M_2(T) \vee d) > X_1(T) + k)\right]$$

$$= e^{c\mu_2 T + \frac{1}{2}c^2\sigma_2^2 T} \Phi_2\left(\frac{-k - \left(\mu_1 + c\rho\sigma_1\sigma_2 - \mu_2 - c\sigma_2^2\right)T}{\sqrt{\left(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2\right)T}}, \frac{-d + \left(\mu_2 + c\sigma_2^2\right)T}{\sigma_2\sqrt{T}}; \frac{-\rho\sigma_1\sigma_2 + \sigma_2^2}{\sqrt{\left(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2\right)\sigma_2^2}}\right)$$

$$- \frac{c}{c + \xi}e^{(c+\xi)d} \Phi_2\left(\frac{\eta d - k - \mu_1 T}{\sigma_1\sqrt{T}}, \frac{-d - \mu_2 T}{\sigma_2\sqrt{T}}; \rho\right) + e^{c \cdot d} \Phi_2\left(\frac{d - k - \mu_1 T}{\sigma_1\sqrt{T}}, \frac{d - \mu_2 T}{\sigma_2\sqrt{T}}; \rho\right)$$

$$+ \left(\frac{\xi}{c + \xi}\frac{\eta}{|\eta|} + 2\rho\frac{\sigma_1}{\sigma_2}\frac{1}{|\eta|}\right)e^{\frac{c+\xi}{\eta}k}e^{\frac{c+\xi}{\eta}\mu_1 T + \frac{1}{2}\left(\frac{c+\xi}{\eta}\right)^2\sigma_1^2 T}$$

$$\times \Phi_2\left(\frac{k/\eta - d + \left(\left(\mu_1 + \frac{c+\xi}{\eta}\sigma_1^2\right)/\eta\right)T}{\sqrt{\left(\sigma_1^2/\eta^2\right)T}}, \frac{-k/\eta - \left(\left(\mu_1 + \frac{c+\xi}{\eta}\sigma_1^2\right)/\eta + \mu_2 + \frac{c+\xi}{\eta}\rho\sigma_1\sigma_2\right)T}{\sqrt{\left(\sigma_1^2/\eta^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2/\eta\right)T}};\right.$$

$$\left.\frac{-\sigma_1^2/\eta^2 - \rho\sigma_1\sigma_2/\eta}{\sqrt{\left(\sigma_1^2/\eta^2\right)\left(\sigma_1^2/\eta^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2/\eta\right)}}\right) + \frac{c}{c + \xi}e^{-(c+\xi)\mu_2 T + \frac{1}{2}(c+\xi)^2\sigma_2^2 T}$$

$$\times \Phi_2\left(\frac{-k - \left(\mu_1 - (c + \xi)\rho\sigma_1\sigma_2 + \eta\mu_2 - \eta(c + \xi)\sigma_2^2\right)T}{\sqrt{\left(\sigma_1^2 + \eta^2\sigma_2^2 + 2\eta\rho\sigma_1\sigma_2\right)T}}, \frac{-d - \left(\mu_2 - (c + \xi)\sigma_2^2\right)T}{\sigma_2\sqrt{T}};\right.$$

$$\left.\frac{\rho\sigma_1\sigma_2 + \eta\sigma_2^2}{\sqrt{\left(\sigma_1^2 + \eta^2\sigma_2^2 + 2\eta\rho\sigma_1\sigma_2\right)\sigma_2^2}}\right)$$

$$=: H\left(d, \begin{pmatrix} c \\ k \end{pmatrix}, \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}, \rho, T\right), \tag{2.13}$$

which will be useful for pricing the proposed put option. In addition, applying (2.13), for $d \leq 0$,

$$E\left[e^{c \cdot (m_2(T) \wedge d)} I((m_2(T) \wedge d) < X_1(T) + k)\right]$$

$$= E\left[e^{-c \cdot [\text{Max}\{-X_2(\tau), 0 \leq \tau \leq T\} \vee (-d)]} I\left([\text{Max}\{-X_2(\tau), 0 \leq \tau \leq T\} \vee (-d)] > -X_1(T) + (-k)\right)\right]$$

$$=: H\left(-d, -\begin{pmatrix} c \\ k \end{pmatrix}, -\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}, \rho, T\right), \tag{2.14}$$

which will be useful for pricing the proposed call option. Here, $a \wedge b$ denotes $\min(a, b)$. Note that the stochastic process $\{(-X_1(t), -X_2(t))'\}$ is a 2-dimensional Brownian motion with drift vector $(-\mu_1, -\mu_2)'$ and diffusion matrix $\boldsymbol{V}$.

Finally, let us discuss some useful probability formulas for pricing the proposed options. For

$d \geq 0$, applying (2.13) with $c = 0$, we have

$$\Pr\left((M_2(T) \vee d) > X_1(T) + k\right) = E\left[e^{0 \cdot (M_2(T) \vee d)} I((M_2(T) \vee d) > X_1(T) + k)\right]$$

$$=: H\left(d, \binom{0}{k}, \binom{\mu_1}{\mu_2}, \binom{\sigma_1}{\sigma_2}, \rho, T\right). \tag{2.15}$$

Similarly, for $d \leq 0$, applying (2.14) with $c = 0$, we have

$$\Pr\left((m_2(T) \wedge d) < X_1(T) + k\right) = E\left[e^{0 \cdot (m_2(T) \wedge d)} I((m_2(T) \wedge d) < X_1(T) + k)\right]$$

$$=: H\left(-d, -\binom{0}{k}, -\binom{\mu_1}{\mu_2}, \binom{\sigma_1}{\sigma_2}, \rho, T\right). \tag{2.16}$$

3. Outside Floating-Strike Lookback Put Option

The proposed outside floating-strike lookback put option gives the holder the right to sell one underlying asset at its guaranteed floating-strike price that is some percentage times the greater of a specific guaranteed amount and the highest price of the other underlying asset. This section will derive an explicit pricing formula for the outside floating-strike lookback put option.

Let us take a close look at the payoff of the outside floating-strike lookback put option. Assume that $\lambda (> 0)$ is the percentage over the greater of the highest price and $L \geq S_2(0)$. The payoff of this put option can be is written as follows:

$$(\lambda \cdot \{\max(S_2(\tau), 0 \leq \tau \leq T) \vee L\} - S_1(T))_+. \tag{3.1}$$

To simplify writing, we define all expectations in this and next sections as taken with respect to the risk-neutral measure. Under this measure, the underlying stochastic processes $\{X_i(\tau), \tau \geq 0\}$ is a Brownian motion with drift vector $(r - \sigma_1^2/2, r - \sigma_2^2/2)'$ and diffusion matrix $\boldsymbol{V}$. By the fundamental theorem of asset pricing, the time-0 value of the payoff (3.1) is

$$e^{-rT} E\left[\left(\lambda \cdot S_2(0)e^{M_2(T) \vee d} - S_1(0)e^{X_1(T)}\right)_+\right], \tag{3.2}$$

where $d = \ln(L/S_2(0)) \geq 0$. Calculating this discounted expectation (3.2) seems to require significant complicated and tedious integration; however, formulas (2.13) and (2.15) can simplify and reduce many of the calculations.

Therefore, the time-0 value of the outside floating-strike lookback put option can be rewritten and decomposed into the sum of two expectations,

$$e^{-rT} E\left[\left(\lambda \cdot S_2(0)e^{M_2(T) \vee d} - S_1(0)e^{X_1(T)}\right) I\left((M_2(T) \vee d) > X_1(T) + \ln\left(\frac{S_1(0)}{\lambda \cdot S_2(0)}\right)\right)\right]$$

$$= \lambda e^{-rT} S_2(0) E\left[e^{M_2(T) \vee d} I\left((M_2(T) \vee d) > X_1(T) + \ln\left(\frac{S_1(0)}{\lambda \cdot S_2(0)}\right)\right)\right]$$

$$- e^{-rT} S_1(0) E\left[e^{X_1(T)} I\left((M_2(T) \vee d) > X_1(T) + \ln\left(\frac{S_1(0)}{\lambda \cdot S_2(0)}\right)\right)\right]. \tag{3.3}$$

Applying (2.13), the first expectation in the RHS (right hand side) of (3.3) can be

$$H\left(d, \binom{1}{\ln\left(\frac{S_1(0)}{\lambda \cdot S_2(0)}\right)}, \binom{r - \frac{1}{2}\sigma_1^2}{r - \frac{1}{2}\sigma_2^2}, \binom{\sigma_1}{\sigma_2}, \rho, T\right). \tag{3.4}$$

Applying the factorization formula (2.9), (2.15) and the fact that $\{e^{-rt}S_i(t)\}$ is a martingale under the risk-neutral measure, the second term in the RHS of (3.3) will be

$$e^{-rT}S_1(0)E\left[e^{X_1(T)}\right]\Pr\left((M_2(T) \vee d) > X_1(T) + \ln\left(\frac{S_1(0)}{\lambda \cdot S_2(0)}\right); (1,0)'\right)$$

$$= S_1(0)H\left(d, \begin{pmatrix} 0 \\ \ln\left(\frac{S_1(0)}{\lambda \cdot S_2(0)}\right) \end{pmatrix}, \begin{pmatrix} r + \frac{1}{2}\sigma_1^2 \\ r - \frac{1}{2}\sigma_2^2 + \rho\sigma_1\sigma_2 \end{pmatrix}, \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}, \rho, T\right). \tag{3.5}$$

Note that the drift vector is shifted because of

$$\begin{pmatrix} r - \frac{1}{2}\sigma_1^2 \\ r - \frac{1}{2}\sigma_2^2 \end{pmatrix} + \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} r + \frac{1}{2}\sigma_1^2 \\ r - \frac{1}{2}\sigma_2^2 + \rho\sigma_1\sigma_2 \end{pmatrix}. \tag{3.6}$$

Hence, placing (3.4) and (3.5) into (3.3), we have the time-0 value of the outside floating-strike put option

$$\lambda e^{-rT}S_2(0)H\left(d, \begin{pmatrix} 1 \\ \ln\left(\frac{S_1(0)}{\lambda \cdot S_2(0)}\right) \end{pmatrix}, \begin{pmatrix} r - \frac{1}{2}\sigma_1^2 \\ r - \frac{1}{2}\sigma_2^2 \end{pmatrix}, \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}, \rho, T\right)$$

$$- S_1(0)H\left(d, \begin{pmatrix} 0 \\ \ln\left(\frac{S_1(0)}{\lambda \cdot S_2(0)}\right) \end{pmatrix}, \begin{pmatrix} r + \frac{1}{2}\sigma_1^2 \\ r - \frac{1}{2}\sigma_2^2 + \rho\sigma_1\sigma_2 \end{pmatrix}, \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}, \rho, T\right). \tag{3.7}$$

For numerical results of pricing formula (3.7), see Table 1. It is observed that $L/S_2(0)$, T and σ_1/σ_2 increase formula (3.7), but r and ρ decrease it. Thus, the guaranteed strike price L, the maturity and the volatility of the first asset S_1 increase the put option price. Meanwhile, the interest rate and the correlation coefficient decrease the price.

Let us derive the time-t value of this option. By the fundamental theorem of asset pricing, the time-t value of the payoff (3.1) will be expressed as discounted conditional expectation

$$e^{-r(T-t)}E\left[(\lambda \cdot \{\max(S_2(\tau), 0 \le \tau \le T) \vee L\} - S_1(T))_+ | (S_1(\tau), S_2(\tau)), 0 \le \tau \le t\right]. \tag{3.8}$$

According to the model (2.1), for $i = 1, 2$,

$$S_i(T) = S_i(0)e^{X_i(t)+X_i(T)-X_i(t)} = S_i(t)e^{X_i(T)-X_i(t)}. \tag{3.9}$$

The floating-strike price of (3.8) can be rewritten as follows:

$$\lambda \cdot \{\max(S_2(\tau), 0 \le \tau \le t) \vee \max(S_2(\tau), t \le \tau \le T) \vee L\}$$

$$= \lambda \cdot S_2(t)e^{\max(X_2(\tau)-X_2(t), t \le \tau \le T) \vee \ln\left[\frac{\{\max(S_2(\tau), 0 \le \tau \le t) \vee L\}}{S_2(t)}\right]}. \tag{3.10}$$

Here, note that

$$(\max(X_2(\tau) - X_2(t), t \le \tau \le T), X_1(T) - X_1(t)) \stackrel{d}{=} (M_2(T-t), X_1(T-t)), \tag{3.11}$$

where notation $\stackrel{d}{=}$ implies that the two random vectors follow the same distribution. In addition, the vector in the LHS (left hand side) of (3.11) is independent of $(S_1(\tau), S_2(\tau))$ for $0 \le \tau \le t$. Thus,

Table 1: Put option prices ($S_1(0) = S_2(0) = 100$, $\lambda = 1$, $\sigma_2 = 0.2$)

T	σ_1/σ_2	r	$L/S_2(0)$						
			1	1.05	1.1		1	1.05	1.1
			$\rho = 0$				$\rho = 0.3$		
0.25	0.5	0.04	8.32	8.95	11.21		8.12	8.77	11.17
		0.06	8.14	8.71	10.83		7.92	8.51	10.78
		0.08	7.96	8.48	10.48		7.72	8.25	10.41
	1.0	0.04	9.60	10.16	12.06		9.18	9.76	11.84
		0.06	9.43	9.96	11.75		9.00	9.52	11.49
		0.08	9.27	9.76	11.46		8.82	9.30	11.16
	1.5	0.04	11.24	11.78	13.53		10.70	11.32	13.24
		0.06	11.08	11.58	13.24		10.52	11.10	12.93
		0.08	10.92	11.40	12.98		10.35	10.88	12.63
0.50	0.5	0.04	11.69	12.07	13.49		11.38	11.73	13.30
		0.06	11.33	11.66	12.92		10.98	11.26	12.68
		0.08	11.00	11.29	12.41		10.61	10.83	12.10
	1.0	0.04	13.55	13.91	15.14		12.92	13.25	14.62
		0.06	13.22	13.54	14.66		12.56	12.83	14.07
		0.08	12.91	13.21	14.23		12.21	12.44	13.55
	1.5	0.04	15.90	16.26	17.43		15.12	15.52	16.83
		0.06	15.58	15.91	16.98		14.76	15.12	16.32
		0.08	15.28	15.58	16.57		14.69	14.95	16.00

according to (3.9), (3.10), (3.11) and this independence, the time-t value (3.8) is the same as the time-0 value (3.2) except that $T = T - t$, $S_i(0) = S_i(t)$, and $L = \max(S_2(\tau), 0 \leq \tau \leq t) \vee L$. Therefore, we have the time-t value of this put option,

$$\lambda e^{-r(T-t)} S_2(t) H \left(d_t^{\max}, \left(\frac{1}{\ln\left(\frac{S_1(t)}{\lambda \cdot S_2(t)}\right)} \right), \left(\begin{matrix} r - \frac{1}{2}\sigma_1^2 \\ r - \frac{1}{2}\sigma_2^2 \end{matrix} \right), \left(\begin{matrix} \sigma_1 \\ \sigma_2 \end{matrix} \right), \rho, T - t \right)$$

$$- S_1(t) H \left(d_t^{\max}, \left(\frac{0}{\ln\left(\frac{S_1(t)}{\lambda \cdot S_2(t)}\right)} \right), \left(\begin{matrix} r + \frac{1}{2}\sigma_1^2 \\ r - \frac{1}{2}\sigma_2^2 + \rho\sigma_1\sigma_2 \end{matrix} \right), \left(\begin{matrix} \sigma_1 \\ \sigma_2 \end{matrix} \right), \rho, T - t \right). \tag{3.12}$$

where $d_t^{\max} = \ln[\{\max(S_2(\tau), 0 \leq \tau \leq t) \vee L\}/S_2(t)]$.

4. Outside Floating-Strike Lookback Call Option

The proposed outside floating-strike lookback call option gives the holder the right to buy one underlying asset at its guaranteed floating-strike price that is some percentage times the smaller of a specific guaranteed amount and the lowest price of the other underlying asset. This section will derive an explicit pricing formula for the call option.

Let us take a close look at the payoff of the outside floating-strike lookback call option. Assume that $\lambda \, (> 0)$ is the percentage over the lesser of the lowest price and $L \leq S_2(0)$. The payoff of this call option is

$$(S_1(T) - \lambda \cdot \{\min(S_2(\tau), 0 \leq \tau \leq t) \wedge L\})_+. \tag{4.1}$$

By the fundamental theorem of asset pricing, the time-0 value of the payoff is

$$e^{-rT} E\left[\left(S_1(0)e^{X_1(T)} - \lambda \cdot S_2(0)e^{m_2(T) \wedge d} \right)_+ \right], \tag{4.2}$$

where $d = \ln(L/S_2(0)) \leq 0$. Calculating this discounted expectation (4.2) seems to require much complicated and tedious integration, but formula (2.14) and (2.16) can simplify and reduce many calculations.

Therefore, the time-0 value of the outside floating-strike lookback call option can be rewritten and decomposed into the sum of two expectations,

$$
e^{-rT} E\left[\left(S_1(0)e^{X_1(T)} - \lambda \cdot S_2(0)e^{m_2(T)\wedge d}\right)I\left((m_2(T) \wedge d) < X_1(T) + \ln\left(\frac{S_1(0)}{\lambda \cdot S_2(0)}\right)\right)\right]
$$

$$
= e^{-rT} S_1(0)E\left[e^{X_1(T)}I\left((m_2(T) \wedge d) < X_1(T) + \ln\left(\frac{S_1(0)}{\lambda \cdot S_2(0)}\right)\right)\right]
$$

$$
- \lambda e^{-rT} S_2(0)E\left[e^{m_2(T)\wedge d}I\left((m_2(T) \wedge d) < X_1(T) + \ln\left(\frac{S_1(0)}{\lambda \cdot S_2(0)}\right)\right)\right]. \tag{4.3}
$$

Applying (2.14), we have the second expectation in the RHS of (4.3)

$$
H\left(-d, -\left(\begin{matrix} 1 \\ \ln\left(\frac{S_1(0)}{\lambda \cdot S_2(0)}\right) \end{matrix}\right), -\left(\begin{matrix} r - \frac{1}{2}\sigma_1^2 \\ r - \frac{1}{2}\sigma_2^2 \end{matrix}\right), \left(\begin{matrix} \sigma_1 \\ \sigma_2 \end{matrix}\right), \rho, T\right). \tag{4.4}
$$

In addition, applying the factorization formula (2.9), (2.16) and the fact that $\{e^{-rt}S_i(t)\}$ is a martingale under the risk-neutral measure, the first term in the RHS of (4.3) will be

$$
e^{-rT} S_1(0)E\left[e^{X_1(T)}\right]\Pr\left((m_2(T) \wedge d) < X_1(T) + \ln\left(\frac{S_1(0)}{\lambda \cdot S_2(0)}\right); (1,0)'\right)
$$

$$
= S_1(0)H\left(-d, -\left(\begin{matrix} 0 \\ \ln\left(\frac{S_1(0)}{\lambda \cdot S_2(0)}\right) \end{matrix}\right), -\left(\begin{matrix} r + \frac{1}{2}\sigma_1^2 \\ r - \frac{1}{2}\sigma_2^2 + \rho\sigma_1\sigma_2 \end{matrix}\right), \left(\begin{matrix} \sigma_1 \\ \sigma_2 \end{matrix}\right), \rho, T\right). \tag{4.5}
$$

Hence, placing (4.4) and (4.5) into (4.3), we have the time-0 value of the outside floating-strike call option,

$$
S_1(0)H\left(-d, -\left(\begin{matrix} 0 \\ \ln\left(\frac{S_1(0)}{\lambda \cdot S_2(0)}\right) \end{matrix}\right), -\left(\begin{matrix} r + \frac{1}{2}\sigma_1^2 \\ r - \frac{1}{2}\sigma_2^2 + \rho\sigma_1\sigma_2 \end{matrix}\right), \left(\begin{matrix} \sigma_1 \\ \sigma_2 \end{matrix}\right), \rho, T\right)
$$

$$
- \lambda e^{-rT} S_2(0)H\left(-d, -\left(\begin{matrix} 1 \\ \ln\left(\frac{S_1(0)}{\lambda \cdot S_2(0)}\right) \end{matrix}\right), -\left(\begin{matrix} r - \frac{1}{2}\sigma_1^2 \\ r - \frac{1}{2}\sigma_2^2 \end{matrix}\right), \left(\begin{matrix} \sigma_1 \\ \sigma_2 \end{matrix}\right), \rho, T\right). \tag{4.6}
$$

For numerical results of pricing formula (4.6), see Table 2 showing that r, T and σ_1/σ_2 increase formula (4.6), but $L/S_2(0)$ and ρ decrease it. Thus, the interest rate, the maturity and the volatility of the first asset S_1 increase the call option price. Meanwhile, guaranteed strike price L and the correlation coefficient decrease the price.

Let us derive the time-t value of this call option. By the fundamental theorem of asset pricing, the time-t value of the payoff (4.1) will be expressed in terms of conditional expectation

$$
e^{-r(T-t)} E\left[(S_1(T) - \lambda \cdot \{\min(S_2(\tau), 0 \le \tau \le T) \wedge L\})_+ \,|\, (S_1(\tau), S_2(\tau)), 0 \le \tau \le t\right], \tag{4.7}
$$

whose floating-strike price can be calculated as follows:

$$
\lambda \cdot \{\min(S_2(\tau), 0 \le \tau \le t) \wedge \{\min(S_2(\tau), t \le \tau \le T) \wedge L\}
$$

$$
= \lambda \cdot S_2(t)e^{\min(X_2(\tau)-X_2(t),\, t \le \tau \le T) \wedge \ln\left[\frac{\min(S_2(\tau),\, 0 \le \tau \le t) \wedge L}{S_2(t)}\right]}. \tag{4.8}
$$

Table 2: Call option prices ($S_1(0) = S_2(0) = 100$, $\lambda = 1$, $\sigma_2 = 0.2$)

T	σ_1/σ_2	r	1	0.95	0.9	1	0.95	0.9
				$\rho = 0$			$\rho = 0.3$	
		0.04	8.55	9.37	12.10	8.42	9.29	12.09
	0.5	0.06	8.75	9.63	12.45	8.64	9.57	12.44
		0.08	8.96	9.90	12.80	8.86	9.85	12.79
		0.04	9.62	10.29	12.57	9.30	10.02	12.47
0.25	1.0	0.06	9.79	10.50	12.88	9.49	10.25	12.79
		0.08	9.97	10.72	13.19	9.68	10.50	13.11
		0.04	11.07	11.67	13.67	10.64	11.34	13.51
	1.5	0.06	11.23	11.86	13.94	10.81	11.55	13.80
		0.08	11.38	12.05	14.21	10.99	11.76	14.09
		0.04	12.14	12.71	14.68	11.98	12.57	14.64
	0.5	0.06	12.55	13.18	15.27	12.41	13.07	15.24
		0.08	12.97	13.67	15.88	12.86	13.59	15.86
		0.04	13.59	14.05	15.63	13.16	13.63	15.39
0.50	1.0	0.06	13.93	14.43	16.13	13.53	14.06	15.92
		0.08	14.28	14.83	16.64	13.92	14.50	16.47
		0.04	15.58	15.99	17.37	15.00	15.48	17.04
	1.5	0.06	15.88	16.32	17.79	15.33	15.85	17.50
		0.08	16.19	16.67	18.24	15.68	16.24	17.98

Note: The column group header above the six data columns is $L/S_2(0)$.

Here, note that

$$(\min(X_2(\tau) - X_2(t), t \le \tau \le T), X_1(T) - X_1(t)) \stackrel{d}{=} (m_2(T - t), X_1(T - t)). \qquad (4.9)$$

In addition, the vector in the LHS of (4.9) is independent of $(S_1(\tau), S_2(\tau))$ for $0 \le \tau \le t$. Thus, according to (3.9), (4.8), (4.9) and this independence, the time-t value (4.7) is the same as the time-0 value (4.2) except that $T = T - t$, $S_i(0) = S_i(t)$, and $L = \min(S_2(\tau), 0 \le \tau \le t) \wedge L$. Therefore, we have the time-t value of this call option,

$$S_1(t)H\left(-d_t^{\min}, -\begin{pmatrix} 0 \\ \ln\left(\frac{S_1(t)}{\lambda \cdot S_2(t)}\right) \end{pmatrix}, -\begin{pmatrix} r + \frac{1}{2}\sigma_1^2 \\ r - \frac{1}{2}\sigma_2^2 + \rho\sigma_1\sigma_2 \end{pmatrix}, \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}, \rho, T - t\right)$$

$$- \lambda e^{-r(T-t)}S_2(t)H\left(-d_t^{\min}, -\begin{pmatrix} 1 \\ \ln\left(\frac{S_1(t)}{\lambda \cdot S_2(t)}\right) \end{pmatrix}, -\begin{pmatrix} r - \frac{1}{2}\sigma_1^2 \\ r - \frac{1}{2}\sigma_2^2 \end{pmatrix}, \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}, \rho, T - t\right). \qquad (4.10)$$

where $d_t^{\min} = \ln[\{\min(S_2(\tau), 0 \le \tau \le t) \wedge L\}/S_2(t)]$.

5. Continuous Proportional Dividends

The previous sections derived the explicit pricing formulas for the outside floating-strike lookback options whose underlying assets pay no dividends. The pricing formulas in Section 3 and Section 4 can be extended to the case where each of the underlying assets pays dividends continuously at a rate proportional to its price. This section will derive explicit pricing formulas for this case.

Let $S_i(t)$ denote the time-t price of two underlying assets for $i = 1, 2$, respectively. Assume that δ_i is the constant nonnegative dividend yield rate such that the assets pay dividends $\delta_i S_i(t)dt$ between time t and time $t + dt$. If all dividends of asset i are reinvested in the asset, each share of the asset at time 0 grows to $e^{\delta_i t}$ shares at time t. We assume that the prices of these assets follow the model (2.1); therefore, if an investor buys one share of asset i at $S_i(0)$ and reinvests all dividends in the asset, his

fund value invested in asset i will be

$$e^{\delta_i t} S_i(t) = e^{\delta_i t} S_i(0) \exp(X_i(t)) \tag{5.1}$$

at time t. The risk-neutral measure is the Esscher measure of parameter vector $\boldsymbol{h} = \boldsymbol{h}^{**}$ with respect to which the process $\{e^{-(r-\delta_i)t} S_i(t)\}$ is a martingale. Therefore, $\boldsymbol{h}^{**}$ is the solution of

$$\boldsymbol{\mu} + V\boldsymbol{h}^{**} = \left(r - \delta_1 - \frac{\sigma_1^2}{2}, r - \delta_2 - \frac{\sigma_2^2}{2} \right)'. \tag{5.2}$$

Note that the process $\{X(t)\}$ is a Brownian motion with drift vector $\boldsymbol{\mu} + V\boldsymbol{h}^{**}$ and diffusion matrix V under the risk-neutral measure. For further discussion, see Section 9 of Gerber and Shiu (1996).

By the fundamental theorem of asset pricing, the time-0 values of the payoffs (3.1) and (4.1) are

$$
\begin{aligned}
& e^{-rT} E\left[\left(\lambda \cdot S_2(0) e^{M_2(T) \vee d} - S_1(0) e^{X_1(T)} \right)_+ ; h^{**} \right] \\
&= \lambda e^{-rT} S_2(0) E\left[e^{M_2(T) \vee d} I\left((M_2(T) \vee d) > X_1(T) + \ln\left(\frac{S_1(0)}{\lambda \cdot S_2(0)} \right) \right) ; h^{**} \right] \\
&\quad - e^{-\delta_1 T} e^{-(r-\delta_1)T} S_1(0) E\left[e^{X_1(T)} I\left((M_2(T) \vee d) > X_1(T) + \ln\left(\frac{S_1(0)}{\lambda \cdot S_2(0)} \right) \right) ; h^{**} \right]
\end{aligned} \tag{5.3}
$$

and

$$
\begin{aligned}
& e^{-rT} E\left[\left(S_1(0) e^{X_1(T)} - \lambda \cdot S_2(0) e^{m_2(T) \wedge d} \right)_+ ; h^{**} \right] \\
&= e^{-\delta_1 T} e^{-(r-\delta_1)T} S_1(0) E\left[e^{X_1(T)} I\left((m_2(T) \wedge d) < X_1(T) + \ln\left(\frac{S_1(0)}{\lambda \cdot S_2(0)} \right) \right) ; h^{**} \right] \\
&\quad - \lambda e^{-rT} S_2(0) E\left[e^{m_2(T) \wedge d} I\left((m_2(T) \wedge d) < X_1(T) + \ln\left(\frac{S_1(0)}{\lambda \cdot S_2(0)} \right) \right) ; h^{**} \right],
\end{aligned} \tag{5.4}
$$

respectively, of which two expectations are the same as ones of (3.3) and (4.3) except that the underlying stochastic process is a Brownian motion with drift vector (5.2) and $e^{-(r-\delta_1)T} S_1(0) E[e^{X_1(T)}; h^{**}] = S_1(0)$. Thus, the time-0 values of the outside floating-strike lookback put and call options are

$$
\begin{aligned}
& \lambda e^{-rT} S_2(0) H\left(d, \begin{pmatrix} 1 \\ \ln\left(\frac{S_1(0)}{\lambda \cdot S_2(0)} \right) \end{pmatrix}, \begin{pmatrix} r - \delta_1 - \frac{1}{2}\sigma_1^2 \\ r - \delta_2 - \frac{1}{2}\sigma_2^2 \end{pmatrix}, \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}, \rho, T \right) \\
&\quad - e^{-\delta_1 T} S_1(0) H\left(d, \begin{pmatrix} 0 \\ \ln\left(\frac{S_1(0)}{\lambda \cdot S_2(0)} \right) \end{pmatrix}, \begin{pmatrix} r - \delta_1 + \frac{1}{2}\sigma_1^2 \\ r - \delta_2 - \frac{1}{2}\sigma_2^2 + \rho\sigma_1\sigma_2 \end{pmatrix}, \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}, \rho, T \right)
\end{aligned} \tag{5.5}
$$

and

$$
\begin{aligned}
& e^{-\delta_1 T} S_1(0) H\left(-d, -\begin{pmatrix} 0 \\ \ln\left(\frac{S_1(0)}{\lambda \cdot S_2(0)} \right) \end{pmatrix}, -\begin{pmatrix} r - \delta_1 + \frac{1}{2}\sigma_1^2 \\ r - \delta_2 - \frac{1}{2}\sigma_2^2 + \rho\sigma_1\sigma_2 \end{pmatrix}, \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}, \rho, T \right) \\
&\quad - \lambda e^{-rT} S_2(0) H\left(-d, -\begin{pmatrix} 1 \\ \ln\left(\frac{S_1(0)}{\lambda \cdot S_2(0)} \right) \end{pmatrix}, -\begin{pmatrix} r - \delta_1 - \frac{1}{2}\sigma_1^2 \\ r - \delta_2 - \frac{1}{2}\sigma_2^2 \end{pmatrix}, \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}, \rho, T \right)
\end{aligned} \tag{5.6}
$$

Table 3: Put option prices (dividend) ($S_1(0) = S_2(0) = 100$, $\lambda = 1$, $\sigma_1 = \sigma_2 = 0.2$, $T = 0.5$, $r = 0.06$)

δ_1	δ_2	1	1.05	1.1	1	1.05	1.1
			$\rho = 0$			$\rho = 0.3$	
0.00		13.22	13.54	14.66	12.56	12.83	14.07
0.01	0.00	13.56	13.90	15.03	12.92	13.21	14.46
0.02		13.91	14.25	15.41	13.28	13.58	14.86
0.00		12.97	13.31	14.45	12.32	12.61	13.88
0.01	0.01	13.31	13.66	14.82	12.67	12.98	14.27
0.02		13.66	14.01	15.19	13.04	13.35	14.66
0.00		12.73	13.07	14.24	12.08	12.38	13.69
0.01	0.02	13.07	13.42	14.61	12.44	12.75	14.08
0.02		13.41	13.77	14.99	12.80	13.12	14.47

Column group header: $L/S_2(0)$

Table 4: Call option prices (dividend) ($S_1(0) = S_2(0) = 100$, $\lambda = 1$, $\sigma_1 = \sigma_2 = 0.2$, $T = 0.5$, $r = 0.06$)

δ_1	δ_2	1	0.95	0.9	1	0.95	0.9
			$\rho = 0$			$\rho = 0.3$	
0.00		13.93	14.43	16.13	13.53	14.06	15.92
0.01	0.00	13.53	14.02	15.69	13.11	13.63	15.47
0.02		13.13	13.61	15.26	12.71	13.21	15.03
0.00		14.09	14.58	16.24	13.69	14.20	16.02
0.01	0.01	13.69	14.16	15.80	13.27	13.78	15.57
0.02		13.29	13.76	15.37	12.86	13.35	15.13
0.00		14.26	14.73	16.35	13.86	14.35	16.13
0.01	0.02	13.85	14.32	15.91	13.44	13.92	15.68
0.02		13.46	13.91	15.48	13.03	13.50	15.23

Column group header: $L/S_2(0)$

respectively. Therefore, as done in Section 3 and Section 4, we have the time-t values of the outside floating-strike lookback put and call options,

$$
\lambda e^{-r(T-t)} S_2(t) H\left(d_t^{\max}, \begin{pmatrix} \frac{1}{\ln\left(\frac{S_1(t)}{\lambda \cdot S_2(t)}\right)} \end{pmatrix}, \begin{pmatrix} r - \delta_1 - \frac{1}{2}\sigma_1^2 \\ r - \delta_2 - \frac{1}{2}\sigma_2^2 \end{pmatrix}, \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}, \rho, T-t \right)
$$
$$
- e^{-\delta_1(T-t)} S_1(t) H\left(d_t^{\max}, \begin{pmatrix} 0 \\ \ln\left(\frac{S_1(t)}{\lambda \cdot S_2(t)}\right) \end{pmatrix}, \begin{pmatrix} r - \delta_1 + \frac{1}{2}\sigma_1^2 \\ r - \delta_2 - \frac{1}{2}\sigma_2^2 + \rho\sigma_1\sigma_2 \end{pmatrix}, \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}, \rho, T-t \right) \quad (5.7)
$$

and

$$
e^{-\delta_1(T-t)} S_1(t) H\left(-d_t^{\min}, -\begin{pmatrix} 0 \\ \ln\left(\frac{S_1(t)}{\lambda \cdot S_2(t)}\right) \end{pmatrix}, -\begin{pmatrix} r - \delta_1 + \frac{1}{2}\sigma_1^2 \\ r - \delta_2 - \frac{1}{2}\sigma_2^2 + \rho\sigma_1\sigma_2 \end{pmatrix}, \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}, \rho, T-t \right)
$$
$$
- \lambda e^{-r(T-t)} S_2(t) H\left(-d_t^{\min}, -\begin{pmatrix} \frac{1}{\ln\left(\frac{S_1(t)}{\lambda \cdot S_2(t)}\right)} \end{pmatrix}, -\begin{pmatrix} r - \delta_1 - \frac{1}{2}\sigma_1^2 \\ r - \delta_2 - \frac{1}{2}\sigma_2^2 \end{pmatrix}, \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}, \rho, T-t \right) \quad (5.8)
$$

respectively.

Finally, let us discuss numerical results of (5.5) and (5.6). For numerical results of put pricing formula (5.5), see Table 3 showing that δ_1 and $L/S_2(0)$ increase formula (5.5), but δ_2 and ρ decrease it. The dividend rate of the first asset increases the put price but the dividend rate of the second asset decreases it. In addition, for numerical results of call pricing formula (5.6), see Table 4. This table shows that δ_1, $L/S_2(0)$ and ρ decrease formula (5.6), but δ_2 increases it. The dividend rate of the first asset decreases the call price but the dividend rate of the second asset increases it.

6. Conclusion

We have derived the explicit pricing formulas for the proposed outside floating-strike lookback options and discussed some numerical results of the pricing formulas under either non-dividend assumption or continuous dividend assumption. More realistic assumptions in pricing and hedging outside floating-strike lookback options should be introduced in future research: stochastic interest rates, flexible monitoring periods and transaction costs.

Appendix: Proof of (2.13)

First of all, let us discuss the joint probability distribution function of random variables $M_2(T)$ and $X_1(T)$,

$$
\begin{aligned}
&\Pr(X_1(T) \le x, M_2(T) \le m) \\
&= \Phi_2\left(\frac{x - \mu_1 T}{\sigma_1 \sqrt{T}}, \frac{m - \mu_2 T}{\sigma_2 \sqrt{T}}; \rho\right) - e^{\frac{2\mu_2}{\sigma_2^2} m} \Phi_2\left(\frac{x - \mu_1 T}{\sigma_1 \sqrt{T}} - \frac{2\rho m}{\sigma_2 \sqrt{T}}, \frac{-m - \mu_2 T}{\sigma_2 \sqrt{T}}; \rho\right),
\end{aligned} \qquad (2.12)
$$

whose two bivariate normal distribution functions can be expressed as follows:

$$
\Phi_2\left(\frac{x - \mu_1 T}{\sigma_1 \sqrt{T}}, \frac{m - \mu_2 T}{\sigma_2 \sqrt{T}}; \rho\right) = \Pr(X_1(T) \le x, X_2(T) \le m) \qquad (A.1)
$$

and

$$
\Phi_2\left(\frac{x - \mu_1 T}{\sigma_1 \sqrt{T}} - \frac{2\rho m}{\sigma_2 \sqrt{T}}, \frac{-m - \mu_2 T}{\sigma_2 \sqrt{T}}; \rho\right) = \Pr\left(X_1(T) \le x - 2\rho\frac{\sigma_1}{\sigma_2} m, X_2(T) \le -m\right). \qquad (A.2)
$$

Hence, placing (A.1) and (A.2) into (2.12), we have

$$
\begin{aligned}
&\Pr(X_1(T) \le x, M_2(T) \le m) \\
&= \Pr(X_1(T) \le x, X_2(T) \le m) - e^{\frac{2\mu_2}{\sigma_2^2} m} \Pr\left(X_1(T) \le x - 2\rho\frac{\sigma_1}{\sigma_2} m, X_2(T) \le -m\right). \qquad (A.3)
\end{aligned}
$$

Next, let us derive two double integral formulas used many times for the proof of (2.13). Applying the factorization formula (2.9), one double integral can be expressed as follows:

$$
\begin{aligned}
&\iint\limits_{\substack{a \cdot x + b \cdot y < e \\ c \cdot x + d \cdot y < f}} e^{h_1 \cdot x} \frac{\partial^2}{\partial y \partial x} \Pr(X_1(T) \le x, X_2(T) \le y) dx dy \\
&= E\left[e^{h_1 \cdot X_1(T)} I\left(a \cdot X_1(T) + b \cdot X_2(T) < e, c \cdot X_1(T) + d \cdot X_2(T) < f\right)\right] \\
&= E\left[e^{h_1 \cdot X_1(T)}\right] \Pr\left(a \cdot X_1(T) + b \cdot X_2(T) < e, c \cdot X_1(T) + d \cdot X_2(T) < f; (h_1, 0)'\right) \\
&= e^{h_1 \mu_1 T + \frac{1}{2} h_1^2 \sigma_1^2 T} \\
&\quad \cdot \Phi_2\left(\frac{e - \left[a\left(\mu_1 + h_1 \sigma_1^2\right) + b(\mu_2 + h_1 \rho \sigma_1 \sigma_2)\right] T}{\sqrt{\left(a^2 \sigma_1^2 + b^2 \sigma_2^2 + 2ab\rho\sigma_1\sigma_2\right) T}}, \frac{f - \left[c\left(\mu_1 + h_1 \sigma_1^2\right) + d(\mu_2 + h_1 \rho \sigma_1 \sigma_2)\right] T}{\sqrt{\left(c^2 \sigma_1^2 + d^2 \sigma_2^2 + 2cd\rho\sigma_1\sigma_2\right) T}}; \rho^*\right), \quad (A.4)
\end{aligned}
$$

where $\rho^* = \{ac\sigma_1^2 + (ad+bc)\rho\sigma_1\sigma_2 + bd\sigma_2^2\} / \sqrt{(a^2\sigma_1^2 + b^2\sigma_2^2 + 2ab\rho\sigma_1\sigma_2)(c^2\sigma_1^2 + d^2\sigma_2^2 + 2cd\rho\sigma_1\sigma_2)}$.
Similarly, the other double integral c

$$
\iint_{\substack{a\cdot x+b\cdot y<e \\ c\cdot x+d\cdot y<f}} e^{h_2\cdot y} \frac{\partial^2}{\partial y \partial x} \Pr(X_1(T) \le x, X_2(T) \le y) dx dy
$$

$$
= e^{h_2\mu_2 T + \frac{1}{2}h_2^2\sigma_2^2 T}
$$

$$
\cdot \Phi_2\left(\frac{e - \left[a(\mu_1 + h_2\rho\sigma_1\sigma_2) + b\left(\mu_2 + h_2\sigma_2^2\right)\right]T}{\sqrt{\left(a^2\sigma_1^2 + b^2\sigma_2^2 + 2ab\rho\sigma_1\sigma_2\right)T}}, \frac{f - \left[c(\mu_1 + h_2\rho\sigma_1\sigma_2) + d\left(\mu_2 + h_2\sigma_2^2\right)\right]T}{\sqrt{\left(c^2\sigma_1^2 + d^2\sigma_2^2 + 2cd\rho\sigma_1\sigma_2\right)T}}; \rho^* \right). \quad (A.5)
$$

Now, let us derive (2.13). The expectation (2.13) can be expressed as

$$
E\left[e^{c\cdot(M_2(T)\vee d)} I((M_2(T) \vee d) > X_1(T) + k)\right]
$$

$$
= E\left[e^{c\cdot M_2(T)} I(M_2(T) > d, M_2(T) > X_1(T) + k)\right] + e^{c\cdot d} \Pr(M_2(T) \le d, X_1(T) \le d - k),
$$

which, applying (2.12), becomes

$$
\iint_{\substack{m>x+k \\ m>d}} e^{c\cdot m} \frac{\partial^2}{\partial m \partial x} \Pr(X_1(T) \le x, M_2(T) \le m) dx dy
$$

$$
+ e^{c\cdot d}\left[\Phi_2\left(\frac{d-k-\mu_1 T}{\sigma_1\sqrt{T}}, \frac{d-\mu_2 T}{\sigma_2\sqrt{T}}; \rho\right) - e^{\frac{2\mu_2}{\sigma_2^2}d} \Phi_2\left(\frac{d-k-\mu_1 T}{\sigma_1\sqrt{T}} - \frac{2\rho d}{\sigma_2\sqrt{T}}, \frac{-d-\mu_2 T}{\sigma_2\sqrt{T}}; \rho\right)\right]. \quad (A.6)
$$

If we place (A.3) into the double integral of (A.6), the first term of (A.6) can be decomposed into the sum of three double integrals as follows:

$$
\iint_{\substack{m>x+k \\ m>d}} e^{c\cdot m} \frac{\partial^2}{\partial m \partial x} \Pr(X_1(T) \le x, X_2(T) \le m) dx dm
$$

$$
- \frac{2\mu_2}{\sigma_2^2} \iint_{\substack{m>x+k \\ m>d}} e^{\left(c+\frac{2\mu_2}{\sigma_2^2}\right)m} \frac{\partial}{\partial x}\left[\Pr\left(X_1(T) \le x - 2\rho\frac{\sigma_1}{\sigma_2}m, X_2(T) \le -m\right)\right] dx dm.
$$

$$
- \iint_{\substack{m>x+k \\ m>d}} e^{\left(c+\frac{2\mu_2}{\sigma_2^2}\right)m} \frac{\partial^2}{\partial m \partial x}\left[\Pr\left(X_1(T) \le x - 2\rho\frac{\sigma_1}{\sigma_2}m, X_2(T) \le -m\right)\right] dx dm.
$$

$$
=: \text{(I)} - \frac{2\mu_2}{\sigma_2^2}\text{(II)} - \text{(III)}. \quad (A.7)
$$

Applying (A.5), the first double integral of (A.7) will be

$$
\text{(I)} = e^{c\mu_2 T + \frac{1}{2}c^2\sigma_2^2 T}
$$

$$
\cdot \Phi_2\left(\frac{-k - \left(\mu_1 + c\rho\sigma_1\sigma_2 - \mu_2 - c\sigma_2^2\right)T}{\sqrt{\left(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2\right)T}}, \frac{-d + \left(\mu_2 + c\sigma_2^2\right)T}{\sigma_2\sqrt{T}}; \frac{-\rho\sigma_1\sigma_2 + \sigma_2^2}{\sqrt{\left(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2\right)\sigma_2^2}} \right). \quad (A.8)
$$

Let us consider the second double integral of (A.7). Calculating the inside integral, the second double integral of of (A.7) will be written as follows:

$$
\begin{aligned}
(\text{II}) &= \iint_{\substack{m>x+k \\ m>d}} e^{(c+\xi)m} \frac{\partial}{\partial x} \left[\Pr\left(X_1(T) \le x - 2\rho \frac{\sigma_1}{\sigma_2} m, X_2(T) \le -m \right) \right] dx\, dm \\
&= \int_{m=d}^{m=\infty} e^{(c+\xi)m} \Pr\left(X_1(T) \le \left(1 - 2\rho \frac{\sigma_1}{\sigma_2} \right) m - k, X_2(T) \le -m \right) dm.
\end{aligned}
\tag{A.9}
$$

Here, assume that $\eta = 1 - 2\rho(\sigma_1/\sigma_2)$. If we apply integration by parts, (A.9) will be

$$
\begin{aligned}
&\frac{1}{c+\xi} \left[e^{(c+\xi)m} \Pr\left(X_1(T) \le \eta m - k, X_2(T) \le -m \right) \right]_{m=d}^{m=\infty} \\
&- \frac{1}{c+\xi} \int_{m=d}^{m=\infty} e^{(c+\xi)m} \frac{d}{dm} \Pr\left(X_1(T) \le \eta m - k, X_2(T) \le -m \right) dm \\
&=: (\text{II-1}) - \frac{1}{c+\xi}(\text{II-2}).
\end{aligned}
\tag{A.10}
$$

The first term of (A.10) is

$$
\begin{aligned}
(\text{II-1}) &= -\frac{1}{c+\xi} e^{(c+\xi)d} \Pr\left(X_1(T) \le \eta d - k, X_2(T) \le -d \right) \\
&= -\frac{1}{c+\xi} e^{(c+\xi)d} \Phi_2\left(\frac{\eta d - k - \mu_1 T}{\sigma_1 \sqrt{T}}, \frac{-d - \mu_2 T}{\sigma_2 \sqrt{T}}; \rho \right).
\end{aligned}
\tag{A.11}
$$

Here, we need to calculate (II-2), the last integral of (A.10). Assume that $\phi_2(x_1, x_2; \rho)$ denotes the joint density function of the bivariate standard normal distribution with correlation coefficient ρ. Let $g_1 = \eta m - k$ and $g_2 = -m$. The last integral of (A.10) can be expressed in terms of $\phi_2(x_1, x_2; \rho)$ as follows:

$$
\begin{aligned}
(\text{II-2}) &= \int_{m=d}^{m=\infty} e^{(c+\xi)m} \frac{d}{dm} \Pr\left(X_1(T) \le \eta m - k, X_2(T) \le -m \right) dm \\
&= \int_{m=d}^{m=\infty} e^{(c+\xi)m} \left[\frac{d}{dm} \int_{-\infty}^{g_1} \int_{-\infty}^{g_2} \frac{1}{\sigma_1 \sqrt{T} \sigma_2 \sqrt{T}} \phi_2\left(\frac{u - \mu_1 T}{\sigma_1 \sqrt{T}}, \frac{v - \mu_2 T}{\sigma_2 \sqrt{T}}; \rho \right) dv\, du \right] dm,
\end{aligned}
\tag{A.12}
$$

which, differentiating the inside double integral with respect to variable m, will be

$$
\begin{aligned}
&\eta \cdot \int_{m=d}^{m=\infty} \int_{-\infty}^{g_2} e^{(c+\xi)m} \frac{1}{\sigma_1 \sqrt{T} \sigma_2 \sqrt{T}} \phi_2\left(\frac{g_1 - \mu_1 T}{\sigma_1 \sqrt{T}}, \frac{v - \mu_2 T}{\sigma_2 \sqrt{T}}; \rho \right) dv\, dm \\
&- \int_{m=d}^{m=\infty} \int_{-\infty}^{g_1} e^{(c+\xi)m} \frac{1}{\sigma_1 \sqrt{T} \sigma_2 \sqrt{T}} \phi_2\left(\frac{u - \mu_1 T}{\sigma_1 \sqrt{T}}, \frac{g_2 - \mu_2 T}{\sigma_2 \sqrt{T}}; \rho \right) du\, dm.
\end{aligned}
\tag{A.13}
$$

Using a change of variables with $g_1 = \eta m - k$ and $g_2 = -m$ in the two double integrals of (A.13) and

applying (A.4) and (A.5), (A.13) can be calculated as follows:

$$
\frac{\eta}{|\eta|} \iint_{\substack{-\frac{1}{\eta}g_1 < \frac{k}{\eta} - d \\ \frac{1}{\eta}g_1 + v < -\frac{k}{\eta}}} e^{(c+\xi)\left(\frac{g_1+k}{\eta}\right)} \frac{1}{\sigma_1 \sqrt{T} \sigma_2 \sqrt{T}} \, \phi_2\left(\frac{g_1 - \mu_1 T}{\sigma_1 \sqrt{T}}, \frac{v - \mu_2 T}{\sigma_2 \sqrt{T}}; \rho\right) dv dg_1
$$

$$
- \iint_{\substack{u + \eta g_2 < -k \\ g_2 < -d}} e^{(c+\xi)(-g_2)} \frac{1}{\sigma_1 \sqrt{T} \sigma_2 \sqrt{T}} \, \phi_2\left(\frac{u - \mu_1 T}{\sigma_1 \sqrt{T}}, \frac{g_2 - \mu_2 T}{\sigma_2 \sqrt{T}}; \rho\right) du dg_2
$$

$$
= \frac{\eta}{|\eta|} e^{\frac{c+\xi}{\eta}k} e^{\frac{c+\xi}{\eta}\mu_1 T + \frac{1}{2}\left(\frac{c+\xi}{\eta}\right)^2 \sigma_1^2 T}
$$

$$
\times \Phi_2\left(\frac{k/\eta - d + \left(\left(\mu_1 + \frac{c+\xi}{\eta}\sigma_1^2\right)/\eta\right)T}{\sqrt{\left(\sigma_1^2/\eta^2\right)T}}, \frac{-k/\eta - \left(\left(\mu_1 + \frac{c+\xi}{\eta}\sigma_1^2\right)/\eta + \mu_2 + \frac{c+\xi}{\eta}\rho\sigma_1\sigma_2\right)T}{\sqrt{\left(\sigma_1^2/\eta^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2/\eta\right)T}}; \right.
$$

$$
\left. \frac{-\sigma_1^2/\eta^2 - \rho\sigma_1\sigma_2/\eta}{\sqrt{\left(\sigma_1^2/\eta\right)\left(\sigma_1^2/\eta^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2/\eta\right)}} \right) - e^{-(c+\xi)\mu_2 T + \frac{1}{2}(c+\xi)^2 \sigma_2^2 T}
$$

$$
\times \Phi_2\left(\frac{-k - \left(\mu_1 - (c+\xi)\rho\sigma_1\sigma_2 + \eta\mu_2 - \eta(c+\xi)\sigma_2^2\right)T}{\sqrt{\left(\sigma_1^2 + \eta^2\sigma_2^2 + 2\eta\rho\sigma_1\sigma_2\right)T}}, \frac{-d - \left(\mu_2 - (c+\xi)\sigma_2^2\right)T}{\sigma_2 \sqrt{T}}; \right.
$$

$$
\left. \frac{\rho\sigma_1\sigma_2 + \eta\sigma_2^2}{\sqrt{\left(\sigma_1^2 + \eta^2\sigma_2^2 + 2\eta\rho\sigma_1\sigma_2\right)\sigma_2^2}} \right)
$$

$$
= \text{(II-2)}. \tag{A.14}
$$

Let us calculate the last integral in the RHS of (A.7). Remind that $\xi = 2\mu_2/\sigma_2^2$, and $\eta = 1 - 2\rho(\sigma_1/\sigma_2)$. Using a change of variables with $u = x - 2\rho(\sigma_1/\sigma_2)$ and $v = -m$, the last double integral of (A.7) will be

$$
\text{(III)} = \iint_{\substack{v < -d \\ u + \eta v < -k}} e^{-(c+\xi)v} \frac{\partial^2}{\partial m \partial x} \left[\Pr\left(X_1(T) \le u, X_2(T) \le v\right)\right] du dv \tag{A.15}
$$

of which the second-order derivative with respect to m and x becomes

$$
-\frac{\partial^2}{\partial v \partial u}\left[\Pr\left(X_1(T) \le u, X_2(T) \le v\right)\right] - 2\rho\frac{\sigma_1}{\sigma_2}\frac{\partial^2}{\partial u^2}\left[\Pr\left(X_1(T) \le u, X_2(T) \le v\right)\right], \tag{A.16}
$$

if the chain rule is applied with $\partial u/\partial x = 1$, $\partial u/\partial m = -2\rho(\sigma_1/\sigma_2)$, $\partial v/\partial x = 0$ and $\partial v/\partial m = -1$. Hence, placing (A.16) into (A.15), we have

$$
\text{(III)} = -\iint_{\substack{v < -d \\ u + \eta v < -k}} e^{-(c+\xi)v} \frac{\partial^2}{\partial v \partial u}\left[\Pr\left(X_1(T) \le u, X_2(T) \le v\right)\right] du dv
$$

$$- 2\rho \frac{\sigma_1}{\sigma_2} \iint_{\substack{v<-d \\ u+\eta v<-k}} e^{-(c+\xi)v} \frac{\partial^2}{\partial u^2} \left[\Pr(X_1(T) \le u, X_2(T) \le v)\right] du dv$$

$$=: -(\text{III-1}) - 2\rho \frac{\sigma_1}{\sigma_2}(\text{III-2}). \tag{A.17}$$

Applying (A.5), the first double integral of (A.17) will be

$$(\text{III-1}) = e^{-(c+\xi)\mu_2 T + \frac{1}{2}(c+\xi)^2 \sigma_2^2 T}$$

$$\times \Phi_2 \left(\frac{-k - \left(\mu_1 - (c+\xi)\rho\sigma_1\sigma_2 + \eta\mu_2 - \eta(c+\xi)\sigma_2^2\right) T}{\sqrt{\left(\sigma_1^2 + \eta^2\sigma_2^2 + 2\eta\rho\sigma_1\sigma_2\right) T}}, \frac{-d - \left(\mu_2 - (c+\xi)\sigma_2^2\right) T}{\sigma_2 \sqrt{T}}; \right.$$

$$\left. \frac{\rho\sigma_1\sigma_2 + \eta\sigma_2^2}{\sqrt{\left(\sigma_1^2 + \eta^2\sigma_2^2 + 2\eta\rho\sigma_1\sigma_2\right)\sigma_2^2}} \right). \tag{A.18}$$

Now, let us consider (III-2), the second double integral of (A.17). The second-order derivative in the second double integral of (A.17) can be calculated as follows:

$$\frac{\partial^2}{\partial u^2} \left[\Pr(X_1(T) \le u, X_2(T) \le v)\right] = \frac{\partial^2}{\partial u^2} \left[\int_{-\infty}^{u} \int_{-\infty}^{v} \frac{1}{\sigma_1 \sqrt{T} \sigma_2 \sqrt{T}} \phi_2 \left(\frac{z - \mu_1 T}{\sigma_1 \sqrt{T}}, \frac{w - \mu_2 T}{\sigma_2 \sqrt{T}}; \rho\right) dw dz \right]$$

$$= \int_{-\infty}^{v} \frac{1}{\sigma_1 \sqrt{T} \sigma_2 \sqrt{T}} \frac{\partial}{\partial u} \phi_2 \left(\frac{u - \mu_1 T}{\sigma_1 \sqrt{T}}, \frac{w - \mu_2 T}{\sigma_2 \sqrt{T}}; \rho\right) dw. \tag{A.19}$$

Placing (A.19) into the second double integral of (A.17), we have a triple integral,

$$(\text{III-2}) = \iiint_{\substack{v<-d \\ u+\eta v<-k \\ w<v}} e^{-(c+\xi)v} \frac{1}{\sigma_1 \sqrt{T} \sigma_2 \sqrt{T}} \frac{\partial}{\partial u} \phi_2 \left(\frac{u - \mu_1 T}{\sigma_1 \sqrt{T}}, \frac{w - \mu_2 T}{\sigma_2 \sqrt{T}}; \rho\right) dw du dv$$

$$= \iint_{w<v<-d} e^{-(c+\xi)v} \frac{1}{\sigma_1 \sqrt{T} \sigma_2 \sqrt{T}} \left[\int_{u<-\eta v-k} \frac{\partial}{\partial u} \phi_2 \left(\frac{u - \mu_1 T}{\sigma_1 \sqrt{T}}, \frac{w - \mu_2 T}{\sigma_2 \sqrt{T}}; \rho\right) du \right] dw dv$$

$$= \iint_{w<v<-d} e^{-(c+\xi)v} \frac{1}{\sigma_1 \sqrt{T} \sigma_2 \sqrt{T}} \phi_2 \left(\frac{-\eta v - k - \mu_1 T}{\sigma_1 \sqrt{T}}, \frac{w - \mu_2 T}{\sigma_2 \sqrt{T}}; \rho\right) dw dv, \tag{A.20}$$

which, using a change of variables with $y = -\eta v - k$ and applying (A.4), can be calculated as follows:

$$\frac{1}{|\eta|} e^{\frac{c+\xi}{\eta}k} \iint_{\substack{-\frac{y}{\eta}<\frac{k}{\eta}-d \\ \frac{y}{\eta}+w<-\frac{k}{\eta}}} e^{\frac{c+\xi}{\eta}y} \frac{1}{\sigma_1 \sqrt{T} \sigma_2 \sqrt{T}} \phi_2 \left(\frac{y - \mu_1 T}{\sigma_1 \sqrt{T}}, \frac{w - \mu_2 T}{\sigma_2 \sqrt{T}}; \rho\right) dw dv$$

$$= \frac{1}{|\eta|} e^{\frac{c+\xi}{\eta}k} e^{\frac{c+\xi}{\eta}\mu_1 T + \left(\frac{c+\xi}{\eta}\right)^2 \sigma_1^2 T}$$

$$\times \Phi_2 \left(\frac{k/\eta - d - \left(-\left(\mu_1 + (c+\xi)\sigma_1^2/\eta\right)/\eta\right)T}{\sqrt{\left(\sigma_1^2/\eta^2\right)T}}, \frac{-k/\eta - \left(\left(\mu_1 + (c+\xi)\sigma_1^2/\eta\right)/\eta + \mu_2 + (c+\xi)\rho\sigma_1\sigma_2/\eta\right)T}{\sqrt{\left(\sigma_1^2/\eta^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2/\eta\right)T}}; \right.$$

$$\left. \frac{-\sigma_1^2/\eta^2 - \rho\sigma_1\sigma_2/\eta}{\sqrt{\left(\sigma_1^2/\eta^2\right)\left(\sigma_1^2/\eta^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2/\eta\right)}} \right)$$

$$= \text{(III-2)}. \tag{A.21}$$

Finally, according to (A.6), (A.8), (A.10) and (A.17), we have the expectation formula (2.13). Note that the normal distribution functions in (A.11), the first term and the second term of the RHS of (A.14) are the same as those in the last term of (A.6), (A.21) and (A.18), respectively.

References

Baxter, M. and Rennie, A. (1998). *Financial Calculus: An Introduction to Derivative Pricing*, Cambridge University Press, New York.

Conze, A. and Viswanathan (1991). Path dependent options: The case of lookback options, *Journal of Finance*, **46**, 1893–1907.

Gerber, H. U. and Shiu, E. S. W. (1994). Option pricing by Esscher transforms, *Transactions of the Society of Actuaries*, **46**, 99–140; Discussions 141–191.

Gerber, H. U. and Shiu, E. S. W. (1996). Actuarial bridges to dynamic hedging and option pricing, *Insurance: Mathematics and Economics*, **18**, 183–218.

Goldman, M. B., Sosin, H. B. and Gatto, M. A. (1979). Path dependent options: Buy at the low, sell at the high, *Journal of Finance*, **34**, 1111–1127.

Heynen, R. C. and Kat, H. M. (1994a). Crossing barriers, *Risk*, **7**, 46–50.

Heynen, R. C. and Kat, H. M. (1994b). Selective memory, *Risk*, **7**, 73–76.

Heynen, R. C. and Kat, H. M. (1997). *Lookback Options - Pricing and Applications in Exotic Options: The State of the Art*, edited by Clewlow, C. and Strickland, C. London: International Thomson.

Lee, H. (2008). Pricing floating-strike lookback options with flexible monitoring periods, *The Korean Journal of Applied Statistics*, **21**, 485–495.

Lee, H. (2009). Pricing outside floating-strike lookback options, *The Korean Journal of Applied Statistics*, **22**, 59–73.

Nelken, I. (1996). *The Handbook of Exotic Options: Instruments, Analysis, and Applications*, McGraw-Hill, New York.

Zhang, P. G. (1998). *Exotic Options: A Guide to Second Generation Options*, 2nd ed., World Scientific, Singapore.