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FINDING THE SKEW-SYMMETRIC SOLVENT TO A QUADRATIC MATRIX EQUATION

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Abstract. In this paper we consider the quadratic matrix equation which can be defined by

$$Q(X) = AX^2 + BX + C = 0,$$

where X is a $n \times n$ unknown real matrix; A, B and C are $n \times n$ given matrices with real elements. Newton's method is considered to find the skew-symmetric solvent of the nonlinear matrix equations $Q(X)$. We also show that the method converges the skew-symmetric solvent even if the Fréchet derivative is singular. Finally, we give some numerical examples.

1. Introduction

It is well-known that the main application of quadratic matrix equation

$$Q(X) = AX^2 + BX + C, \qquad A, B, C \in \mathbb{R}^{n\times n}, \tag{1}$$

arises in the quadratic eigenvalue problem

$$Q(\lambda)x = (\lambda^2 A + \lambda B + C)x = 0. \tag{2}$$

When $A = A^T$, $B = -B^T$, $C = C^T$ in the quadratic eigenvalue problem (2), it has a Hamiltonian eigenstructure, that is, the eigenvalues are symmetric with respect to both axes [9, 11]. Motivation for finding skew-symmetric solvent of the quadratic matrix equation (1) comes from the quadratic eigenvalue problem (2), because any skew-symmetric matrix has a pair of purely imaginary eigenvalues. For solving a skew-symmetric eigenvalue problem [10] we suggest an algorithm and convergent theory for finding the skew-symmetric solvent to the equation (1).

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2. Newton's methods for $Q(X)$

If we define E as the solution of the linear equation $Q(X) + Q_X(E) = 0$, where $Q_X(E)$ is the Fréchet derivative of Q at X in the direction E, then Newton's method for the quadratic matrix equations (1) with the given starting matrix X_0 can be written in the iteration form

$$\begin{cases} Q_{X_k}(E_k) = -Q(X_k), \\ \mathrm{X}_{k+1} = X_k + E_k, \end{cases} \qquad \text{where} \quad k = 0, 1, \cdots .$$

Thus, each step of Newton's method requires the finding of solution E of the linear equation

$$Q_X(E) = -Q(X). \tag{3}$$

The next theorem gives the conditions for the uniqueness of solution of the matrix equation (3).

Theorem 2.1. ([8]) *The Fréchet derivative Q_X is nonsingular iff*

i) *the pair $(AX + B, -A)$ is regular (that is,* $\det((AX + B) + \lambda A)$ *is not identically zero in λ),*

ii) $\lambda(AX + B, -A) \cap \lambda(X) = \emptyset$.

If A is nonsingular, the condition i) holds. Now we give some sufficient conditions for nonsingularity of Q_X at the $Q(X)$ solvent X.

Lemma 2.2. ([8, Lem. 3.1]) *If A is nonsingular then Q_X is nonsingular at*

i) *a dominant or minimal solvent S,*

ii) *all solvents S if the eigenvalues of $Q(\lambda) = \lambda^2 A + \lambda B + C (\lambda \in \mathbb{C})$ are distinct.*

To solve (3) we can apply several method for solving the generalized Sylvester equation described by Chu [1], Epton [5], Gardiner, Laub, Amato and Moler [6] and Golub, Nash and Van Loan [7]. Here, we describe a Schur algorithm for solving equation (3) which is proposed by Davis [2, 3]. First, we compute the Schur decomposition of $X \in \mathbb{C}^{n \times n}$,

$$W^* X W = T,$$

where W is unitary and T is upper triangular. Then, compute the generalized Schur decomposition of the matrices $AX + B$ and A,

$$M^*(AX + B)N = H, \qquad M^* A N = J,$$

where M and N are unitary, H and J are upper triangular.
Equating the kth columns and rearranging leads to

$$(H + t_{kk} J)\, y_k = g_k - \sum_{i=1}^{k-1} t_{ik} J y_i, \quad Y = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix}.$$

By solving these upper triangular systems in the order of $k = 1, \cdots, n$, Y can be obtained by column at a time.

3. Skew-symmetric solvents of the quadratic matrix equation $Q(X)$

Now, we present an algorithm to find skew-symmetric solution of the q-th Newton iteration (3).

Algorithm 1. *The matrices $A, B, C \in \mathbb{R}^{n\times n}$ and skew-symmetric matrix $X_q \in \mathbb{R}^{n\times n}$ are given. Iteration is started at skew-symmetric matrix $E_{q_0} \in \mathbb{R}^{n\times n}$.*

$k = 0$;

$$R_0 = -Q(X_q) - (AX_q + B) E_{q_0} - AE_{q_0} X_q$$
$$Y_0 = (AX_q + B)^T R_0 + A^T R_0 (X_q)^T$$
$$Q_0 = \frac{1}{2}\left(Y_0 - Y_0^T\right)$$
$$\gamma_0 = \frac{\|R_0\|^2}{\|Q_0\|^2}$$

while $R_k \neq 0$

$$\gamma_k = \frac{\|R_k\|^2}{\|Q_k\|^2}$$
$$E_{q_{k+1}} = E_{q_k} + \gamma_k Q_k$$
$$R_{k+1} = R_k - \gamma_k \left[(AX_q + B) Q_k + AQ_k X_q\right]$$
$$Y_{k+1} = (AX_q + B)^T R_{k+1} + A^T R_{k+1} (X_q)^T$$
$$\delta_k = \frac{\operatorname{tr}(Y_{k+1} Q_k)}{\|Q_k\|^2}$$
$$Q_{k+1} = \frac{1}{2}\left(Y_{k+1} - Y_{k+1}^T\right) + \delta_k Q_k$$

end.

Note that, the matrices E_{q_k} and Q_k in Algorithm 1 are all skew-symmetric. From Algorithm 1, we directly obtain the following basic lemmas.

Lemma 3.1. *Assume that the q-th Newton iteration* (3) *is consistent. The sequences $\{R_k\}$ and $\{Q_k\}$ are generated by Algorithm* 1*, and the integer number $l \geq 0$ such that $\|R_k\| \neq 0$ for all $k = 0, 1, \cdots, l$. Then, we have*

$$\operatorname{tr}\left(R_k^T R_j\right) = 0 \quad \textit{and} \quad \operatorname{tr}\left(Q_k^T Q_j\right) = 0 \quad \textit{for } k > j = 0, 1, \cdots, l. \tag{4}$$

Proof. We prove this theorem by principle induction.
Step 1. When $l = 1$, from Algorithm 1 we obtain

$$\begin{aligned}
\operatorname{tr}\left(R_1^T R_0\right) &= \operatorname{tr}\left\{\left[R_0 - \gamma_0 (AX_q + B) Q_0 - \gamma_0 A Q_0 X_q\right]^T R_0\right\} \\
&= \operatorname{tr}\left(R_0^T R_0\right) - \gamma_0 \operatorname{tr}\left\{\left[(AX_q + B) Q_0 + AQ_0 X_q\right]^T R_0\right\} \\
&= \|R_0\|^2 - \gamma_0 \operatorname{tr}\left\{Q_0^T \left[(AX_q + B)^T R_0 + A^T R_0 (X_q)^T\right]\right\} \\
&= \|R_0\|^2 - \gamma_0 \operatorname{tr}\left(Q_0^T Y_0\right) \\
&= \|R_0\|^2 - \gamma_0 \operatorname{tr}\left(Q_0^T \frac{Y_0 - Y_0^T}{2}\right) \\
&= \|R_0\|^2 - \gamma_0 \|Q_0\|^2 \\
&= 0,
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{tr}\left(Q_1^T Q_0\right) &= \operatorname{tr}\left[\left(\frac{Y_1 - Y_1^T}{2} + \delta_0 Q_0\right)^T Q_0\right] \\
&= \operatorname{tr}\left(Y_1^T Q_0\right) + \delta_0 \|Q_0\|^2 \\
&= \operatorname{tr}\left(Q_0^T Y_1\right) + \operatorname{tr}\left(Y_1 Q_0\right) \\
&= -\operatorname{tr}\left(Y_1 Q_0\right) + \operatorname{tr}\left(Y_1 Q_0\right) \\
&= 0.
\end{aligned}$$

Suppose that the result (4) holds for $l = s$. Then, when $l = s + 1$

$$\begin{aligned}
\operatorname{tr}\left(R_{s+1}^T R_s\right) &= \operatorname{tr}\left\{\left[R_s - \gamma_s \left(AX_q + B\right) Q_s - \gamma_s A Q_s X_q\right]^T R_s\right\} \\
&= \operatorname{tr}\left(R_s^T R_s\right) - \gamma_s \operatorname{tr}\left\{\left[\left(AX_q + B\right) Q_s + A Q_s X_q\right]^T R_s\right\} \\
&= \|R_s\|^2 - \gamma_s \operatorname{tr}\left\{Q_s^T \left[\left(AX_q + B\right)^T R_s + A^T R_s \left(X_q\right)^T\right]\right\} \\
&= \|R_s\|^2 - \gamma_s \operatorname{tr}\left(Q_s^T Y_s\right) \\
&= \|R_s\|^2 - \gamma_s \operatorname{tr}\left(Q_s^T \frac{Y_s - Y_s^T}{2}\right) \\
&= \|R_s\|^2 - \gamma_s \|Q_s\|^2 - \gamma_s \delta_{s-1} \operatorname{tr}\left(Q_s^T Q_{s-1}\right) \\
&= 0,
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{tr}\left(Q_{s+1}^T Q_s\right) &= \operatorname{tr}\left[\left(\frac{Y_{s+1} - Y_{s+1}^T}{2}\right)^T Q_s\right] + \delta_s \operatorname{tr}\left(Q_s^T Q_s\right) \\
&= \operatorname{tr}\left(Y_{s+1}^T Q_s\right) + \delta_s \|Q_s\|^2 \\
&= \operatorname{tr}\left\{\left[\left(AX_q + B\right)^T R_{s+1} + A^T R_{s+1} \left(X_q\right)^T\right]^T Q_s\right\} + \delta_s \|Q_s\|^2 \\
&= \operatorname{tr}\left\{R_{s+1}^T \left[\left(AX_q + B\right) Q_s + A Q_s X_q\right]\right\} + \delta_s \|Q_s\|^2 \\
&= \operatorname{tr}\left[R_{s+1}^T \frac{1}{\gamma_s} \left(R_s - R_{s+1}\right)\right] + \delta_s \|Q_s\|^2 \\
&= -\frac{1}{\gamma_s} \operatorname{tr}\left(R_{s+1}^T R_{s+1}\right) + \delta_s \|Q_s\|^2 \\
&= 0.
\end{aligned}$$

Step 2. Assume that $\operatorname{tr}\left(R_s^T R_j\right) = 0$, $\operatorname{tr}\left(Q_s^T Q_j\right) = 0$ for all $j = 0, 1, \cdots, s-1$. We show that $\operatorname{tr}\left(R_{s+1}^T R_j\right) = 0$ and $\operatorname{tr}\left(Q_{s+1}^T Q_j\right) = 0$ for $j = 0, 1, \cdots, s-1$. From Algorithm 1 and accompanying assumptions, we have

$$\operatorname{tr}\left(R_{s+1}^T R_j\right) = \operatorname{tr}\left(R_s^T R_j\right) - \gamma_s \operatorname{tr}\left\{\left[\left(AX_q + B\right) Q_s + A Q_s X_q\right]^T R_j\right\}$$

$$
\begin{aligned}
&= -\gamma_s \mathrm{tr}\left\{Q_s^T\left[(AX_q+B)^T R_j + A^T R_j (X_q)^T\right]\right\}\\
&= -\gamma_s \mathrm{tr}\left(Q_s^T Y_j\right)\\
&= -\gamma_s \mathrm{tr}\left(Q_s^T \frac{Y_j - Y_j^T}{2}\right)\\
&= -\gamma_s \mathrm{tr}\left[Q_s^T\left(Q_j - \delta_{j-1}Q_{j-1}\right)\right]\\
&= -\gamma_s \mathrm{tr}\left(Q_s^T Q_j\right) + \gamma_s\delta_{j-1}\mathrm{tr}\left(Q_s^T Q_{j-1}\right)\\
&= 0,
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{tr}\left(Q_{s+1}^T Q_j\right) &= \mathrm{tr}\left[\left(\frac{Y_{s+1}-Y_{s+1}^T}{2}\right)^T Q_j\right] + \delta_s \mathrm{tr}\left(Q_s^T Q_j\right)\\
&= \mathrm{tr}\left(Y_{s+1}^T Q_j\right)\\
&= \mathrm{tr}\left\{\left[(AX_q+B)^T R_{s+1} + A^T R_{s+1}(X_q)^T\right]^T Q_j\right\}\\
&= \mathrm{tr}\left\{R_{s+1}^T\left[(AX_q+B)Q_j + AQ_jX_q\right]\right\}\\
&= \frac{1}{\gamma_j}\mathrm{tr}\left[R_{s+1}^T\left(R_j - R_{j+1}\right)\right]\\
&= 0.
\end{aligned}
$$

Thus, the result (4) holds for $l = s+1$. Therefore, from Step 1 and Step 2 we complete the proof. □

Remark 1. If there exists a positive number l such that $R_k \neq 0$ for all $k = 0, 1, \cdots, l$, then the sequence $\{R_k\}$ which is generated by Algorithm 1 is orthogonal set.

Lemma 3.2. *Let E_q be a skew-symmetric solution of the q-th Newton iteration* (3)*, then*

$$
\mathrm{tr}\left[Q_k^T\left(E_q - E_{q_k}\right)\right] = \|R_k\|^2, \quad \textit{for} \quad k = 0, 1, \cdots. \tag{5}
$$

Proof. We prove the statement (5) by principle induction.
When $k = 0$, from Algorithm 1 we have

$$
\begin{aligned}
\mathrm{tr}\left[Q_0^T\left(E_q - E_{q_0}\right)\right] &= \mathrm{tr}\left[\left(\frac{Y_0 - Y_0^T}{2}\right)^T\left(E_q - E_{q_0}\right)\right]\\
&= \mathrm{tr}\left[Y_0^T\left(E_q - E_{q_0}\right)\right]\\
&= \mathrm{tr}\left\{\left[(AX_q+B)^T R_0 + A^T R_0 (X_q)^T\right]^T\left(E_q - E_{q_0}\right)\right\}\\
&= \mathrm{tr}\left\{R_0^T\left[(AX_q+B)\left(E_q - E_{q_0}\right) + A\left(E_q - E_{q_0}\right)X_q\right]\right\}
\end{aligned}
$$

$$= \mathrm{tr}\left\{R_0^T\left[-Q(X_q) - (AX_q + B)E_{q_0} - AE_{q_0}X_q\right]\right\}$$
$$= \|R_0\|^2.$$

Assume that the statement (5) holds for $k = l$, i.e., $\mathrm{tr}\left[Q_l^T\left(E_q - E_{q_l}\right)\right] = \|R_l\|^2$. Therefore, we can easily check that

$$\mathrm{tr}\left[Q_l^T\left(E_q - E_{q_{l+1}}\right)\right] = \mathrm{tr}\left[Q_l^T\left(E_q - E_{q_l}\right)\right] - \gamma_l \mathrm{tr}\left(Q_l^T Q_l\right) = 0.$$

From this fact, we obtain

$$\begin{aligned}
&\mathrm{tr}\left[Q_{l+1}^T\left(E_q - E_{q_{l+1}}\right)\right] \\
&= \mathrm{tr}\left\{\left[\frac{Y_{l+1} - Y_{l+1}^T}{2} + \delta_l Q_l\right]^T\left(E_q - E_{q_{l+1}}\right)\right\} \\
&= \mathrm{tr}\left[Y_{l+1}^T\left(E_q - E_{q_{l+1}}\right)\right] + \delta_l \mathrm{tr}\left[Q_l^T\left(E_q - E_{q_{l+1}}\right)\right] \\
&= \mathrm{tr}\left\{\left[(AX_q + B)^T R_{l+1} + A^T R_{l+1}(X_q)^T\right]^T\left(E_q - E_{q_{l+1}}\right)\right\} \\
&= \mathrm{tr}\left\{R_{l+1}^T\left[(AX_q + B)\left(E_q - E_{q_{l+1}}\right) + A\left(E_q - E_{q_{l+1}}\right)X_q\right]\right\} \\
&= \mathrm{tr}\left\{R_{l+1}^T\left[-Q(X_q) - (AX_q + B)E_{q_{l+1}} - AE_{q_{l+1}}X_q\right]\right\} \\
&= \|R_{l+1}\|^2,
\end{aligned}$$

which completes the proof. □

Remark 2. Lemma 3.2 implies that, the q-th Newton iteration (3) has a skew-symmetric solution if $R_k \neq 0$ leads to $P_k \neq 0$ for some integer number k. However, if $P_k \neq 0$ and $R_k = 0$, then the equation (3) is inconsistent.

Theorem 3.3. *If the q-th Newton iteration* (3) *has a skew-symmetric solution, then for a skew-symmetric starting matrix E_{q_0}, a skew-symmetric solution can be obtained, at most, in n^2 steps.*

Proof. Let $R_k \neq 0$ for all $k = 0, 1, \cdots, n^2 - 1$. Then from Lemma 3.1, the set $\{R_0, R_1, \cdots, R_{n^2-1}\}$ is an orthogonal basis of the matrix space $\mathbb{R}^{n\times n}$. Since, the q-th Newton iteration (3) has a skew-symmetric solution, $Q_k \neq 0$ for k by Lemma 3.2. Therefore, we can evaluate $E_{q_{n^2}}$ and R_{n^2} from Algorithm 1, and $\mathrm{tr}\left(R_{n^2}^T R_k\right) = 0$ for $k = 0, 1, \cdots, n^2 - 1$ by Lemma 3.1. However, $\mathrm{tr}\left(R_{n^2}^T R_k\right) = 0$ holds only when $R_{n^2} = 0$, which implies that $E_{q_{n^2}}$ is a solution of the q-th Newton iteration. By Algorithm 1, it is natural that $E_{q_{n^2}}$ is a skew-symmetric matrix. □

From Newton's method and Theorem 3.3, we obtained the following convergence theory.

Theorem 3.4. *Assume that the quadratic matrix equation* (1) *has a skew-symmetric solvent and each Newton iteration is consistent for a skew-symmetric* *starting matrix X_0. The sequence $\{X_k\}$ is generated by Newton's method with X_0 such that*

$$\lim_{k\to\infty} X_k = S,$$

and the matrix S is the solvent of $Q(X)$, then S is a skew-symmetric matrix.

Proof. Let $E_0, E_1, \cdots, E_k$ be skew-symmetric solution of first, second, $\cdots$, kth Newton iteration, respectively. Then, from Newton's method we can obtain $(k+1)$th approximation matrix

$$X_{k+1} = X_0 + E_0 + \cdots + E_k,$$

which is also skew-symmetric. Since, the Newton sequence $\{X_k\}$ converges to a solvent S, so, it is a skew-symmetric solvent. □

4. Numerical experiments

The relative residual $\rho_Q(X_k)$ and $\rho_P(X_k)$, stop condition $\|R_k\|$ are same as in Section 4.3. We first consider the quadratic matrix equation

$$Q_1(X) \equiv X^2 + \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} X + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = 0 \tag{6}$$

which is dealt by Dennis, Traub and Weber [4]. It has an infinite number of solvents which have a form:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} i & 0 \\ -1-i & 0 \end{bmatrix}, \begin{bmatrix} -i & 0 \\ -1+i & 1 \end{bmatrix}, \begin{bmatrix} -z-1-i & i(z-1) \\ iz-1 & z \end{bmatrix},$$

$$\begin{bmatrix} -z+1+i & -i(z-1) \\ -zi-1 & z \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1+i & i \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1-i & -i \\ -1 & 0 \end{bmatrix}, \tag{7}$$

where $i = \sqrt{-1}$ and $z \in \mathbb{C}$. There are three skew-symmetric solvents in (6), that is,

$$\begin{bmatrix} \frac{1}{2}+\frac{1}{2}i & -\frac{1}{2}-\frac{1}{2}i \\ \frac{1}{2}+\frac{1}{2}i & \frac{1}{2}+\frac{1}{2}i \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} -\frac{3}{2}-\frac{1}{2}i & \frac{1}{2}-\frac{1}{2}i \\ -\frac{1}{2}+\frac{1}{2}i & \frac{1}{2}-\frac{1}{2}i \end{bmatrix}.$$

Since our researches are progressed in real matrix spaces, we examine a skew-symmetric solvent $S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. First, we select the skew-symmetric starting matrix $X_0 = \begin{bmatrix} 0 & 1.001 \\ -1.001 & 0 \end{bmatrix}$. It is sufficiently close to S, since a scalar number $\|S - X_0\| \approx 4.4721e-005$ can be sufficiently small. Sure enough we expected, the skew-symmetric solvent S can be obtained using Newton's method with Algorithm 1 with the starting matrix X_0. The convergence result is displayed in Table 1.

No.iterations	$\rho_Q(X_k)$ of Newton's method
1	$1.41e-007$
2	$3.54e-014$
3	$1.26e-016$

TABLE 1. The relative residual of problem (6).

Next, we consider when the Fréchet derivative is singular. Let the quadratic matrix equation be

$$Q_2(X) \equiv \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} X^2 + \begin{bmatrix} 0 & -4 \\ 0 & -4 \end{bmatrix} X + \begin{bmatrix} 5 & -25 \\ 5 & -25 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \quad (8)$$

Starting Newton's method with Algorithm 1 at the matrix $\begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}$, then we can be obtained a skew-symmetric solvent $\begin{bmatrix} 0 & 5 \\ -5 & 0 \end{bmatrix}$. Figure 1 shows our Newton's method with the starting matrix converges to a solvent. Therefore, we can know without difficulty this starting matrix enough close to the solvent.

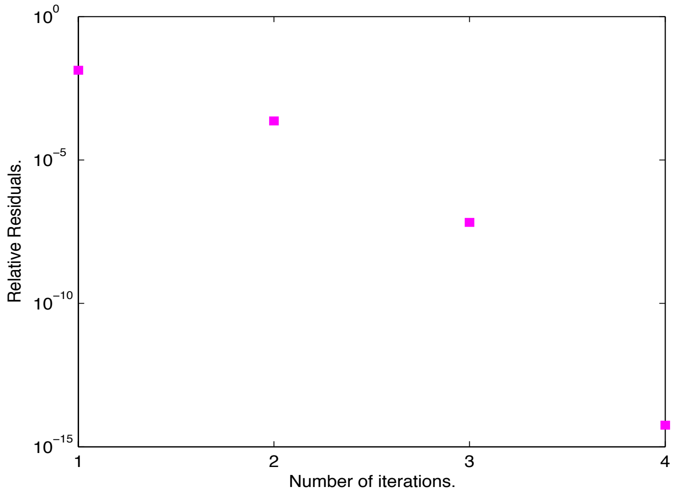

FIGURE 1. The convergence result for problem (8) with skew-symmetric matrices.

In this paper, we introduced a iterative method for solving Newton steps (3) and (3) over skew-symmetric. Then we incorporated the method into Newton's method to find the skew-symmetric solvent. Our algorithm can be worked even if the Fréchet derivative is singular.

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