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ON ASYMPTOTICALLY DEMICONTRACTIVE MAPPINGS IN ARBITRARY BANACH SPACES

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Abstract. In this paper, the necessary and sufficient conditions for the strong convergence of a modified Mann iteration process to a fixed point of an asymptotically demicontractive mapping in real Banach spaces are considered. Presented results improve and extend the results of Igbokwe [3], Liu [4], Moore and Nnoli [6] and Osilike [7].

1. Introduction

Let K be a nonempty subset of a real normed space E and E^* be its dual space. We denote by J the normalized duality mapping from E into 2^{E^*} defined by

$$J(x) = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. If E is strictly convex, then J is single-valued. In the sequel, we shall denote the single-valued duality mapping by j.

Let T be a self-mapping of K.

Definition 1. T is called a k-strictly asymptotically pseudo-contractive mapping, with a sequence $\{k_n\} \subseteq [1, \infty)$, $\lim_{n\to\infty} k_n = 1$ if for all $x, y \in K$ there exist $j(x-y) \in J(x-y)$ and a constant $k \in [0, 1)$ such that

$$\begin{aligned} &\langle (I - T^n)x - (I - T^n)y,\ j(x-y)\rangle \\ &\geq \frac{1}{2}(1-k)\,\|(I-T^n)x - (I-T^n)y\|^2 - \frac{1}{2}(k_n^2 - 1)\,\|x-y\|^2, \end{aligned} \tag{1.1}$$

for all $n \in \mathbb{N}$.

Definition 2. T is called an asymptotically demicontractive mapping with a sequence $\{k_n\} \subseteq [0, \infty)$, $\lim_{n\to\infty} k_n = 1$, if $F(T) = \{x \in K : Tx = x\} \neq \emptyset$ and for

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all $x \in K$ and $x^* \in F(T)$, there exist $k \in [0,1)$ and $j(x-x^*) \in J(x-x^*)$ such that

$$\langle x - T^n x,\ j(x-x^*)\rangle \geq \frac{1}{2}(1-k)\left\|x - T^n x\right\|^2 - \frac{1}{2}(k_n^2 - 1)\left\|x - x^*\right\|^2 \quad (1.2)$$

for all $n \in \mathbb{N}$.

Definition 3. T is said to be uniformly L-Lipschitzian, if there exists a constant $L > 0$, such that

$$\left\|T^n x - T^n y\right\| \leq L\left\|x - y\right\|, \quad (1.3)$$

for all $x, y \in K$ and $n \in \mathbb{N}$.

A class of k-strictly asymptotically pseudo-contractive mappings and a class of asymptotically demicontractive mapping are introduced by Liu [4]. It is easy to see that a k-strictly asymptotically pseudo-contrative mapping with a non-empty fixed point set $F(T)$ is asymptotically demicontractive.

In Hilbert spaces, it is shown in [7] that (1.1) and (1.2) are equivalent to the following inequalities:

$$\left\|T^n x - T^n y\right\| \leq k_n^2\left\|x - y\right\|^2 + k\left\|(I - T^n)x - (I - T^n)y\right\|^2,$$

and

$$\left\|T^n x - T^n y\right\|^2 \leq k_n^2\left\|x - y\right\|^2 + \left\|x - T^n x\right\|^2,$$

respectively.

By using the modified Mann iteration method [5] introduced by Schu [8], Liu [4] proved a convergence theorem for the iterative approximation of fixed points of k-strictly asymptotically pseudo-contractive mappings and asymptotically demicontractive mappings in Hilbert spaces.

Osilike in [5], extended the results of Liu [4] about the iterative approximation of fixed points of k-strictly asymptotically demicontractive mappings from Hilbert spaces to much more general real q-uniformly smooth Banach spaces, $1 < q < \infty$ and specifically proved the following results.

Theorem 1.1. *Let $q > 1$ and let E be a real q-uniformly smooth Banach space. Let K be a closed convex and bounded subset of E and $T : K \to K$ a completely continuous uniformly L-Lipschitzian asymptotically demicontractive mapping with a sequence $k_n \subseteq [1,\infty)$ satisfying $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences satisfying the conditions*

(i) $0 \leq \alpha_n, \beta_n \leq 1$, $n \geq 1$;

(ii) $0 < \epsilon \leq c_q \alpha_n^{q-1}(1 + L\beta_n)^q \leq \frac{1}{2}q(1-k)(1+L)^{-(q-2)} - \epsilon$, *for all* $n \geq 1$ *and for some* $\epsilon > 0$; *and*

(iii) $\sum_{n=1}^{\infty}\beta_n < \infty$.

Then the sequence $\{x_n\}$ *generated from an arbitrary* $x_1 \in K$ *by*

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n T^n x_n, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T^n y_n, \ n \geq 1 \end{aligned}$$

converges strongly to a fixed point of T.

Remark 1. For Hilbert spaces, $q = 2$, $c_q = 1$ and, with $\beta_n = 0$ $\forall n$, Theorem 1 and Theorem 2 of Liu [4] follow from Theorem 1.1.

Recently Chidume and Măruşter [2] made a comprehensive and very useful survey on the main convergence properties of the modified Mann iteration method for the demicontractive mappings.

The purpose of this paper is to prove necessary and sufficient conditions for the strong convergence of the modified Mann iteration method to a fixed point of an asymptotically demicontractive mapping in real Banach spaces. Our results extend and improve the results of Igbokwe [3], Liu [4], Moore and Nnoli [6] and Osilike [7].

2. Preliminaries

In the sequel, we shall make use of the following lemmas.

Lemma 2.1. ([1]) *Let* E *be a real normed linear space. Then for all* $x, y \in E$ *and for* $j(x-y) \in J(x-y)$ *the following inequality holds:*

$$\|x+y\|^2 \leqslant \|x\|^2 + 2\langle y, j(x+y)\rangle .$$

Lemma 2.2. ([9]) *Let* $\{\sigma_n\}$ *and* $\{\beta_n\}$ *be sequences of nonnegative real numbers satisfying the following inequality*

$$\beta_{n+1} \leqslant (1+\sigma_n)\beta_n, \ n \geqslant 0.$$

If $\sum_{n\geqslant 0} \sigma_n < \infty$ *then* $\lim_{n\to\infty} \beta_n$ *exists and if there exists a subsequence of* $\{\beta_n\}$ *converging to* 0*, then* $\lim_{n\to\infty} \beta_n = 0$.

The following result is the special case of Lemma 2.1 proved by Igbokwe [3].

Lemma 2.3. ([3]) *Let* E *be a normed linear space and* K *a non-empty convex subset of* E*. Let* $T : K \to K$ *be an uniformly* L*-Lipschitzian mapping and let* $\{\alpha_n\}$ *be a sequence in* $[0,1]$*. For arbitrary* $x_1 \in K$*, generate the sequence* $\{x_n\}$ *by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, \ n \geq 1.$$

Then

$$\|x_n - Tx_n\| \leq \|x_n - T^n x_n\| + L(1+L)(1+L+L^2)\left\|x_{n-1} - T^{n-1}x_{n-1}\right\|.$$

3. Main results

We now prove our main results.

Lemma 3.1. *Let E be an arbitrary Banach space and K a non-empty convex subset of E. Let $T : K \to K$ be an uniformly L-Lipschitzian asymptotically demicontractive mapping with a sequence $\{k_n\} \subseteq [1, \infty)$, such that $\lim_{n\to\infty} k_n = 1$. For arbitrary $x_1 \in K$, generate the sequence $\{x_n\}$ by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, \ n \geq 1, \tag{3.1}$$

where $\{\alpha_n\}$ is a real sequence in $[0,1]$ satisfying
(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$, (ii) $\lim_{n\to\infty} \alpha_n = 0$. Then
(a) the sequence $\{x_n\}$ is bounded,
(b) $\liminf_{n\to\infty} \|x_{n+1} - T^n x_{n+1}\| = 0$,
(c) $\liminf_{n\to\infty} \|x_n - T^n x_n\| = 0$,
(d) $\liminf_{n\to\infty} \|x_n - T x_n\| = 0$.

Proof. Since T is asymptotically demicontractive, then

$$\langle x - T^n x, j(x - x^*)\rangle \geq \frac{1}{2}(1-k)\|x - T^n x\|^2 - \frac{1}{2}(k_n^2 - 1)\|x - x^*\|^2$$

and hence

$$\|x - T^n x\| \leqslant \sqrt{\frac{(2\|x - T^n x\| + (k_n^2 - 1)\|x - x^*\|)\|x - x^*\|}{1-k}}.$$

Therefore, by the triangle inequality,

$$\|x - x^*\| \leqslant \|T^n x - x^*\| + \sqrt{\frac{(2\|x - T^n x\| + (k_n^2 - 1)\|x - x^*\|)\|x - x^*\|}{1-k}}.$$

Now we shall prove that

$$\liminf_{n\to\infty} \|x_{n+1} - T^n x_{n+1}\| = 0. \tag{3.2}$$

If $x_n = Tx_n$ for all $n \geqslant m$ for some $m \in \mathbb{N}$, then (3.2) trivially holds, as we have for all $n \geqslant m$

$$\|x_{n+1} - T^n x_{n+1}\| = \|x_{n+1} - T^n T x_{n+1}\| = \left\|x_{n+1} - T^{n+1} x_{n+1}\right\| = 0.$$

Suppose now that there exists the smallest positive integer n_0 such that $x_{n_0} \neq Tx_{n_0}$. Put

$$a_0 := \|T^{n_0} x_{n_0} - x^*\| + \sqrt{\frac{\left(2\|x_{n_0} - T^{n_0} x_{n_0}\| + (k_{n_0}^2 - 1)\|x_{n_0} - x^*\|\right)\|x_{n_0} - x^*\|}{1-k}} + 1.$$

Then clearly

$$\|x_{n_0} - x^*\| \leqslant a_0. \tag{3.3}$$

To prove that $\liminf_{n\to\infty} \|x_{n+1} - T^n x_{n+1}\| = 0$, we shall assume, to the contrary, that

$\liminf_{n\to\infty} \|x_{n+1} - T^n x_{n+1}\| = 2\delta > 0$. Then there exists $n_0' \in \mathbb{N}$ such that $\|x_{n+1} - T^n x_{n+1}\| \geqslant \delta$ for all $n \geqslant n_0'$. Also, by $\lim\limits_{n\to\infty} k_n = 1$ and (ii), we may suppose that

$$\alpha_n \leq \min\left\{\frac{1}{2(1+L)}, \frac{(1-k)\delta^2}{16(1+L)^2 a_0^2}\right\}, \quad k_n^2 - 1 \leq \frac{(1-k)\delta^2}{16a_0^2} \quad \text{for all} \quad n \geqslant n_0'. \tag{3.4}$$

We now show that the sequence $\{x_n\}$ is bounded. By induction we shall show that

$$\|x_n - x^*\| \leqslant a_0 \quad \text{for all} \quad n \geqslant n_0'. \tag{3.5}$$

It is clear that (3.5) holds for $n = n_0$. Assume that it is true for some $n > N := \max\{n_0, n_0'\}$, that is, $\|x_n - x^*\| \leqslant a_0$ for some $n \geqslant N$. Then

$$\begin{aligned}
\|x_n - T^n x_n\| &\leq \|x_n - x^*\| + \|T^n x_n - x^*\| \\
&\leq (1+L)\|x_n - x^*\| \\
&\leq (1+L)a_0
\end{aligned}$$

and by the recursion formula (3.1),

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|(1-\alpha_n)x_n + \alpha_n T^n x_n - x^*\| \\
&= \|x_n - x^* - \alpha_n(x_n - T^n x_n)\| \\
&\leqslant \|x_n - x^*\| + \alpha_n \|x_n - T^n x_n\| \\
&\leqslant a_0 + (1+L)a_0\alpha_n \\
&\leqslant 2a_0.
\end{aligned} \tag{3.6}$$

On the other hand, by Lemma 2.1,

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|(1-\alpha_n)x_n + \alpha_n T^n x_n - x^*\|^2 \\
&= \|x_n - x^* - \alpha_n(x_n - T^n x_n)\|^2 \\
&\leqslant \|x_n - x^*\|^2 - 2\alpha_n \langle x_n - T^n x_n, j(x_{n+1} - x^*)\rangle \\
&= \|x_n - x^*\|^2 - 2\alpha_n \langle x_{n+1} - T^n x_{n+1}, j(x_{n+1} - x^*)\rangle \\
&\quad + 2\alpha_n \langle T^n x_n - T^n x_{n+1}, j(x_{n+1} - x^*)\rangle \\
&\quad + 2\alpha_n \langle x_{n+1} - x_n, j(x_{n+1} - x^*)\rangle .
\end{aligned} \tag{3.7}$$

Hence by (1.2) with $x = x_{n+1}$,

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 \leqslant\ & \|x_n - x^*\|^2 - (1-k)\alpha_n \|x_{n+1} - T^n x_{n+1}\|^2 \\
& + \left(k_n^2 - 1\right)\alpha_n \|x_{n+1} - x^*\|^2 \\
& + 2(1+L)\alpha_n \|x_{n+1} - x_n\| \|x_{n+1} - x^*\| .
\end{aligned} \tag{3.8}$$

Since

$$\begin{aligned}\|x_{n+1}-x_n\| &= \|(1-\alpha_n)x_n+\alpha_n T^n x_n - x_n\| \\ &= \alpha_n \|x_n - T^n x_n\| \\ &\leq (1+L)a_0\alpha_n,\end{aligned}$$

from (3.8), we get

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\leqslant \|x_n-x^*\|^2-(1-k)\alpha_n\|x_{n+1}-T^n x_{n+1}\|^2 \\ &\quad +\left(k_n^2-1\right)\alpha_n\|x_{n+1}-x^*\|^2 \\ &\quad +2(1+L)^2a_0\alpha_n^2\|x_{n+1}-x^*\|.\end{aligned}$$

Then, by (3.6) and (3.4),

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\leqslant \|x_n-x^*\|^2-(1-k)\delta^2\alpha_n \\ &\quad +4a_0^2\left[\left(k_n^2-1\right)+(1+L)^2\alpha_n\right]\alpha_n \\ &\leqslant \|x_n-x^*\|^2-(1-k)\delta^2\alpha_n+\frac{1}{2}(1-k)\delta^2\alpha_n\end{aligned}$$

and hence

$$\|x_{n+1}-x^*\|^2 \leqslant \|x_n-x^*\|^2-\frac{1}{2}(1-k)\delta^2\alpha_n. \tag{3.9}$$

Thus $\|x_{n+1}-x^*\| \leq \|x_n-x^*\| \leq a_0$ and so we proved (3.5). Therefore, we proved (a).

From (3.9), we have that for every $r > N$,

$$\begin{aligned}\frac{1}{2}(1-k)\delta^2\sum_{n=N}^{r}\alpha_n &\leqslant \sum_{n=N}^{r}\left(\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2\right) \\ &\leqslant \|x_N-x^*\|^2.\end{aligned}$$

Hence we have $\sum_{n=1}^{\infty}\alpha_n < \infty$, a contradiction with the condition (i). Therefore, our assumption $\delta > 0$ is wrong. Thus

$$\liminf_{n\to\infty}\|x_{n+1}-T^n x_{n+1}\| = 0.$$

Therefore, we proved (b).

Now according to Lemma 2.1, substituting $x = u+v$ and $y = -v$, we obtain

$$\|u+v\|^2 \geq \|u\|^2 + 2\langle v, j(u)\rangle, \tag{3.10}$$

which is mainly due to Igbokwe [3].

By (3.1) we have

$$\begin{aligned}\|x_{n+1}-T^n x_{n+1}\|^2 &= \|(1-\alpha_n)x_n+\alpha_n T^n x_n - T^n x_{n+1}\|^2 \\ &= \|x_n - T^n x_n - \alpha_n(x_n - T^n x_n) - (T^n x_{n+1} - T^n x_n)\|^2.\end{aligned}$$

Then by (3.10) we get

$$\begin{aligned}\|x_{n+1}-T^n x_{n+1}\|^2 &\geq \|x_n - T^n x_n\|^2 \\ &\quad -2\left\langle \alpha_n(x_n-T^n x_n)+(T^n x_{n+1}-T^n x_n), j(x_n-T^n x_n)\right\rangle.\end{aligned}$$

Thus

$$\begin{aligned}\|x_n - T^n x_n\|^2 \leq\ & \|x_{n+1} - T^n x_{n+1}\|^2 \\ & + 2\left\langle \alpha_n\left(x_n - T^n x_n\right) + \left(T^n x_{n+1} - T^n x_n\right), j(x_n - T^n x_n)\right\rangle \\ \leq\ & \|x_{n+1} - T^n x_{n+1}\|^2 \\ & + 2\left\|\alpha_n\left(x_n - T^n x_n\right) + \left(T^n x_{n+1} - T^n x_n\right)\right\| \|x_n - T^n x_n\|,\end{aligned} \tag{3.11}$$

Furthermore,

$$\begin{aligned}&\left\|\alpha_n\left(x_n - T^n x_n\right) + \left(T^n x_{n+1} - T^n x_n\right)\right\| \\ &\leq \alpha_n \|x_n - T^n x_n\| + \|T^n x_{n+1} - T^n x_n\| \\ &\leq (1+L)a_0\alpha_n + L\|x_{n+1} - x_n\| \\ &\leq (1+L)a_0\alpha_n + L(1+L)a_0\alpha_n \\ &= (1+L)^2 a_0\alpha_n.\end{aligned}$$

Therefore, from (3.11), we get

$$\|x_n - T^n x_n\|^2 \leq \|x_{n+1} - T^n x_{n+1}\|^2 + 2(1+L)^3 a_0^2 \alpha_n. \tag{3.12}$$

From (3.12), (ii) and (b),

$$\lim_{n\to\infty} \inf \|x_n - T^n x_n\| = 0. \tag{3.13}$$

Thus we proved (c).

At last, from (3.13) and Lemma 2.3, we obtain (d). □

Theorem 3.2. *Let E be an arbitrary Banach space and K a non-empty convex subset of E. Let $T : K \to K$ be uniformly L-Lipschitzian asymptotically demicontractive mapping with a sequence $\{k_n\} \subseteq [1, \infty)$, such that $\lim_{n\to\infty} k_n = 1$. For arbitrary $x_1 \in K$, let a sequence $\{x_n\}$ be generated as follows:*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n,\ n \geq 1,$$

where $\{\alpha_n\}$ is a real sequence in $[0,1]$ satisfying (i) $\sum_{n=1}^{\infty} \alpha_n = \infty$, (ii) $\lim_{n\to\infty} \alpha_n = 0$. If T is completely continuous, then $\{x_n\}$ converges strongly to some fixed point of T in K.

Proof. From Lemma 3.1, $\lim_{n\to\infty} \inf \|x_n - Tx_n\| = 0$. Therefore, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\lim_{j\to\infty} \left\|x_{n_j} - Tx_{n_j}\right\| = 0$. Since $\{x_{n_j}\}$ is bounded and T is completely continuous, $\{Tx_{n_j}\}$ has a subsequence $\{Tx_{n_{j_k}}\}$ which converges strongly. Hence $\{x_{n_{j_k}}\}$ converges strongly. Let $\lim_{k\to\infty} x_{n_{j_k}} = p$. Then $\lim_{k\to\infty} Tx_{n_{j_k}} = Tp$. Thus we have $\lim_{k\to\infty} \left\|x_{n_{j_k}} - Tx_{n_{j_k}}\right\| = \|p - Tp\| = 0$. Hence $p \in F(T)$. From (3.9) and Lemma 2.2 it follows that $\lim_{k\to\infty} \|x_n - p\| = 0$. This completes the proof. □

Corollary 3.3. *Let E be an arbitrary Banach space and K a nonempty convex subset of E. Let $T : K \to K$ be an uniformly L-Lipschitzian asymptotically demicontractive mapping with a sequence $\{k_n\} \subseteq [1, \infty)$, such that $\lim_{n\to\infty} k_n = 1$. For arbitrary $x_1 \in K$, let a sequence $\{x_n\}$ be generated by*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n,\ n \geq 1,$$

where $\{\alpha_n\}$ is a real sequence in $[0, 1]$ satisfying (i) $\sum_{n=1}^{\infty} \alpha_n = \infty$, (ii) $\lim_{n\to\infty} \alpha_n = 0$. Then $\{x_n\}$ converges strongly to some fixed point of T in K.

Corollary 3.4. *Let E be an arbitrary Banach space and K a nonempty convex subset of E. Let $T : K \to K$ be a k-strictly asymptotically pseudo-contractive mapping with $F(T) \neq \emptyset$ and a sequence $\{k_n\} \subseteq [1, \infty)$, such that $\lim_{n\to\infty} k_n = 1$. For arbitrary $x_1 \in K$, let a sequence $\{x_n\}$ be generated by*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n,\ n \geq 1,$$

where $\{\alpha_n\}$ is a real sequence in $[0, 1]$ satisfying (i) $\sum_{n=1}^{\infty} \alpha_n = \infty$, (ii) $\lim_{n\to\infty} \alpha_n = 0$. Then $\lim_{n\to\infty} \inf \|x_n - Tx_n\| = 0$.

Proof. Following Igbokwe [3], we obtain

$$\begin{aligned}
&\|(I - T^n)x - (I - T^n)y\| \, \|x - y\| \\
&\geq \frac{1}{2}[(1 - k) \|(I - T^n)x - (I - T^n)y\|^2 - (k_n^2 - 1) \|x - y\|^2] \\
&= \frac{1}{2}[\sqrt{1 - k}\, \|(I - T^n)x - (I - T^n)y\| \\
&\quad + \sqrt{k_n^2 - 1}\, \|x - y\|] \cdot [\sqrt{1 - k}\, \|(I - T^n)x - (I - T^n)y\| - \sqrt{k_n^2 - 1}\, \|x - y\|] \\
&\geq \frac{1}{2}[\sqrt{1 - k}\, \|(I - T^n)x - (I - T^n)y\|] \\
&\quad [\sqrt{1 - k}\, \|(I - T^n)x - (I - T^n)y\| - \sqrt{k^2 - 1}\, \|x - y\|].
\end{aligned}$$

Hence

$$\frac{1}{2}\sqrt{1 - k}[\sqrt{1 - k}\, \|(I - T^n)x - (I - T^n)y\|] - \sqrt{k^2 - 1}\, \|x - y\| \leq \|x - y\| .$$

Thus

$$\|(I - T^n)x - (I - T^n)y\| \leq [\frac{2 + \sqrt{(1 - k)(k_n^2 - 1)}}{1 - k}] \|x - y\| .$$

Furthermore,

$$\begin{aligned}
\|T^n x - T^n y\| - \|x - y\| &\leq \|(I - T^n)x - (I - T^n)y\| \\
&\leq [\frac{2 + \sqrt{(1 - k)(k_n^2 - 1)}}{1 - k}] \|x - y\| .
\end{aligned}$$

Hence we get

$$\|T^n x - T^n y\| \leq [1 + \frac{2 + \sqrt{(1 - k)(k_n^2 - 1)}}{1 - k}] \|x - y\| .$$

Since $\{k_n\}$ is bounded, $k_n \leq M$, $\forall n \geq 1$ for some M. Then

$$\begin{aligned} \|T^n x - T^n y\| &\leq [1 + \frac{2 + \sqrt{(1-k)(M^2-1)}}{1-k}] \|x - y\| \\ &\leq L \|x - y\|, \end{aligned}$$

where

$$L = 1 + \frac{2 + \sqrt{(1-k)(M^2-1)}}{1-k}.$$

Hence T is uniformly L-Lipschitzian. Since $F(T) \neq \emptyset$ and T is asymptotically demicontractive, the result follows from Lemma 3.1. □

Corollary 3.5. *Let E be an arbitrary Banach space and K a nonempty convex subset of E. Let $T : K \to K$ be a k-strictly asymptotically pseudo-contractive mapping with $F(T) \neq \emptyset$ and a sequence $\{k_n\} \subseteq [1, \infty)$, such that $\lim_{n\to\infty} k_n = 1$. For arbitrary $x_1 \in K$, let a sequence $\{x_n\}$ be generated by*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \; n \geq 1,$$

where $\{\alpha_n\}$ is a real sequence in $[0,1]$ satisfying (i) $\sum_{n=1}^{\infty} \alpha_n = \infty$, (ii) $\lim_{n\to\infty} \alpha_n = 0$. Then $\{x_n\}$ converges strongly to some fixed point of T in K.

Corollary 3.6. *Let E be an arbitrary Banach space and K a nonempty convex subset of E. Let $T : K \to K$ be a k-strictly asymptotically pseudo-contractive mapping with $F(T) \neq \emptyset$ and a sequence $\{k_n\} \subseteq [1, \infty)$, such that $\lim_{n\to\infty} k_n = 1$. For arbitrary $x_1 \in K$, let a sequence $\{x_n\}$ be generated by*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \; n \geq 1,$$

where $\{\alpha_n\}$ is a real sequence in $[0,1]$ satisfying (i) $\sum_{n=1}^{\infty} \alpha_n = \infty$, (ii) $\lim_{n\to\infty} \alpha_n = 0$. Then $\{x_n\}$ converges strongly to some fixed point of T in K.

Remark 2. Theorem 3.2 improves the results of Igbokwe [3], Liu [4], Moore and Nnoli [6] and Osilike [7].

References

[1] S. S. Chang, *On Chidume's open questions and approximate solutions of multivalued strongly accretive mapping equations in Banach spaces*, J. Math. Anal. Appl. **216** (1997), 94–111.

[2] C. E. Chidume and Ş. Măruşter, *Iterative methods for the computation of fixed points of demicontractive mappings*, J. Comput. Appl. Math. **234 (3)** (2010), 861-882.

[3] D. I. Igbokwe, *Approximation of fixed points of asymptotically demicontractive mappings in arbitrary Banach spaces*, J. Ineq. Pure Appl. Math. **3 (1)** (2002), Article 3.

[4] Q. H. Liu, *Convergence theorems of the sequence of iterates for asymptotically demicontractive and hemicontractive mappings*, Nonlinear Analysis **26 (11)** (1996), 1835–1842.

[5] W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953), 506–510.

[6] C. Moore and B. V. C. Nnoli, *Iterative sequence for asymptotically demicontractive maps in Banach spaces*, J. Math. Anal. Appl. **302 (2)** (2005), 557–562.

[7] M. O. Osilike, *Iterative approximations of fixed points of asymptotically demicontractive mappings*, Indian J. Pure Appl. Math. **29 (12)** (1998), 1291–1300.
[8] J. Schu, *Iterative construction of fixed points of asymptotically nonexpansive mappings*, J. Math. Anal. Appl. **158** (1991), 407–413.
[9] K. K. Tan and H. K. Xu, *Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process*, J. Math. Anal. Appl. **178** (1993), 301–308.

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