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GENERALIZED SYSTEM FOR RELAXED COCOERCIVE EXTENDED GENERAL VARIATIONAL INEQUALITIES

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Abstract. The approximate solvability of a generalized system for relaxed cocoercive extended general variational inequalities is studied by using the project operator technique. The results presented in this paper are more general and include many previously known results as special cases.

1. Introduction

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ respectively. Let K be nonempty closed and convex set in H, and $T : H \to H$ be given nonlinear operator.

In this paper, we consider, based on the projection method, the approximation solvability of a system of extended general variational inequalities with different (γ, r)-cocoercive mappings. The results obtained in this paper extend and improve the main ones in [2],[4],[5]. Let T_1, T_2, g, h be nonlinear mappings. We consider the problem of finding $(x^*, y^*) \in K \times K$ such that

$$\langle \rho T_1(y^*, x^*) + h(x^*) - g(y^*), g(x) - h(x^*)\rangle \geq 0, \forall x \in H : g(x) \in K, \rho > 0 \quad (1.1)$$

$$\langle \eta T_2(x^*, y^*) + g(y^*) - h(x^*), h(x) - g(y^*)\rangle \geq 0, \forall x \in H : h(x) \in K, \eta > 0 \quad (1.2)$$

which is called the system of extended general variational inequalities involving four different nonlinear operators (SEGVID).

We now discuss some special cases.

I. If $g = h$, then problem (SEGVID) is equivalent to the following system of variational inequalities: finding $(x^*, y^*) \in K \times K$ such that

$$\langle \rho T_1(y^*, x^*) + g(x^*) - g(y^*), g(x) - g(x^*)\rangle \geq 0, \forall x \in H : g(x) \in K, \rho > 0 \quad (1.3)$$

$$\langle \eta T_2(x^*, y^*) + g(y^*) - g(x^*), g(x) - g(y^*)\rangle \geq 0, \forall x \in H : g(x) \in K, \eta > 0 \quad (1.4)$$

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which is the system of general variational inequalities (SGVID).

II. For $g = h = I$, the identity operator, the problem (SEGVID) is equivalent to the following one: finding $(x^*, y^*) \in K \times K$ such that

$$\langle \rho T_1(y^*, x^*) + x^* - y^*, x - x^* \rangle \geq 0, \forall x \in K, \rho > 0 \tag{1.5}$$

$$\langle \eta T_2(x^*, y^*) + y^* - x^*, x - y^* \rangle \geq 0, \forall x \in K, \eta > 0 \tag{1.6}$$

which is called the system of variational inequalities (SNVID) and has been studied in [4].

III. If $T_1 = T_2 = T$, then the problem (SNVID) is equivalent to the following system of variational inequalities (SNVI): finding $(x^*, y^*) \in K \times K$ such that

$$\langle \rho T(y^*, x^*) + x^* - y^*, x - x^* \rangle \geq 0, \forall x \in K, \rho > 0 \tag{1.7}$$

$$\langle \eta T(x^*, y^*) + y^* - x^*, x - y^* \rangle \geq 0, \forall x \in K, \eta > 0 \tag{1.8}$$

which has been considered in [3],[4].

IV. If T_1, T_2 are univariate operators, then the problem (SEGVID) is equivalent to the following system of variational inequalities: finding $(x^*, y^*) \in K \times K$ such that

$$\langle \rho T_1(y^*) + h(x^*) - g(y^*), g(x) - h(x^*) \rangle \geq 0, \forall x \in K, \rho > 0 \tag{1.9}$$

$$\langle \eta T_2(x^*) + g(y^*) - h(x^*), h(x) - g(y^*) \rangle \geq 0, \forall x \in K, \eta > 0 \tag{1.10}$$

.

V. If $T_1 = T_2 = T$ is the univariate nonlinear operator, then the problem (1.9),(1.10) is equivalent to finding $u \in H, h(u) \in K$ such that

$$\langle Tu, g(v) - h(u) \rangle \geq 0, \qquad \forall v \in H, g(v) \in K. \tag{$*$}$$

An inequality of type $(*)$ is called extended general variational inequality involving three operators, which was introduced and studied by Noor [1]. The special cases of the extended general variational inequality have introduced in Noor [1]. Using a projection technique, Noor [1] established the equivalence between the extended general variational inequalities and the generalized nonlinear projection equation. Using this equivalent formulation, Noor discussed the existence of a solution of the extended general variational inequalities under suitable conditions. And Noor [1] emphasized that the problem $(*)$ is equivalent to that of finding $u \in H : h(u) \in K$ such that

$$\langle Tu + h(u) - g(u), g(v) - h(u) \rangle \geq 0, \qquad \forall v \in H, g(v) \in K. \tag{1.11}$$

We now recall the following well-known results and concepts.

Lemma 1.1. *For given $z \in H, u \in K$ satisfies the inequality*

$$\langle u - z, v - u \rangle \geq 0, \forall v \in K, \tag{1.12}$$

if and only if

$$u = P_K(z)$$

where P_K is the projection of H onto K. Also the projection operator P_K is nonexpansive.

Using Lemma 1.1, we can show that the extended general variational inequality (1.11) is equivalent to the fixed point problem. This result is mainly due to Noor [1].

Lemma 1.2. *The function $u \in H : h(u) \in K$ is a solution of the extended general variational inequality (1.8) if and only if $u \in H : h(u) \in K$ satisfies the relation*

$$h(u) = P_K[g(u) - \rho Tu], \tag{1.13}$$

where P_K is the projection operator and $\rho > 0$ is a constant.

It is clear from the Lemma 1.2 that the extended general variational inequality (1.11) and the fixed point problem (1.13) are equivalent. This alternative equivalent formulation has played a significant role in the studies of the variational inequalities and related optimization problems.

It is convenient to rewrite the relation (1.13) in the following form, which is very useful in obtaining our results:

$$u = (1 - \alpha_n)u + \alpha_n(u - h(u) + P_K[g(u) - \rho Tu]), \tag{1.14}$$

where $\alpha_n \in [0, 1]$, for all $n \geq 0$.

Using lemma 1.2 and (1.14), we can easily show that finding the solution $(x^*, y^*) \in K \times K$ of problem (SEGVID) is equivalent to finding $(x^*, y^*) \in K \times K$ such that :

$$x^* = (1 - \alpha_n)x^* + \alpha_n(x^* - h(x^*) + P_K[g(y^*) - \rho T_1(y^*, x^*)]). \tag{1.15}$$

$$y^* = (1 - \beta_n)y^* + \beta_n(y^* - g(y^*) + P_K[h(x^*) - \eta T_2(x^*, y^*)]). \tag{1.16}$$

We recall that the following definitions:

Definition 1. A mapping $T : K \to H$ is called μ-Lipschitzian if there exists a constant $\mu > 0$, such that

$$\|Tx - Ty\| \leq \mu\|x - y\|, \forall x, y \in K.$$

Definition 2. A mapping $T : K \to H$ is called r-strongly monotonic if there exists a constant $r > 0$, such that

$$\langle Tx - Ty, x - y\rangle \geq r\|x - y\|^2, \forall x, y \in K.$$

Definition 3. A mapping $T : K \to H$ is called α-cocoercive if there exists a constant $\alpha > 0$, such that

$$\langle Tx - Ty, x - y\rangle \geq \alpha\|Tx - Ty\|^2, \forall x, y \in K.$$

Clearly, every α-cocoercive mapping T is $(\frac{1}{\alpha})$-Lipschitz continuous.

Definition 4. A mapping $T : K \to H$ is called relaxed (γ, r)-cocoercive if there exist constants $\gamma > 0, r > 0$, such that

$$\langle Tx - Ty, x - y\rangle \geq -\gamma\|Tx - Ty\|^2 + r\|x - y\|^2, \forall x, y \in K.$$

For $\gamma = 0$, T is r-strongly monotone. This class of mappings is more general than the class of strongly monotone mapping.

In order to prove our results we need the following Lemma:

Lemma 1.3. ([4]) *Suppose* $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}, \{c_n\}_{n=0}^{\infty}$ *are nonnegative sequences satisfying the following inequality:*

$$a_{n+1} \leq (1 - \lambda_n)a_n + b_n + c_n, n \geq n_0$$

where n_0 *is some nonnegative integer,* $\lambda_n \in (0, 1), \sum_{n=0}^{\infty} \lambda_n = \infty$, *and* $b_n = o(\lambda_n)$ *and* $\sum_{n=0}^{\infty} c_n < \infty$, *then* $\lim_{n\to\infty} a_n = 0$.

2. Algorithms

In this section, we deal with an introduction of general two-step methods and its special form , which can be applied to the convergence analysis for projection operator technique in the context of the approximation solvability of the (SEGVID) problems(1.1),(1.2) and (1.9), (1.10) etc.

Algorithm 2.1. For arbitrary chosen initial points $x_0, y_0 \in H$, compute the sequence $\{x_n\}$ and $\{y_n\}$ by the iterative schemes

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n\{x_n - h(x_n) + P_K[g(y_n) - \rho T_1(y_n, x_n)]\}$$
$$y_n = (1 - \beta_n)x_n + \beta_n\{y_n - g(y_n) + P_K[h(x_n) - \eta T_2(x_n, y_n)]\},$$

where $\alpha_n, \beta_n \in [0, 1]$ for all $n \geq 0$.

If T_1 and T_2 are univariate mappings, then the Algorithm 2.1 reduces to the following method for solving the system of extended general variational inequalities (1.9), (1.10).

Algorithm 2.2. For arbitrary chosen initial points $x_0, y_0 \in H$, compute the sequence $\{x_n\}$ and $\{y_n\}$ by the iterative schemes

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n\{x_n - h(x_n) + P_K[g(y_n) - \rho T_1(y_n)]\}$$
$$y_n = (1 - \beta_n)x_n + \beta_n\{y_n - g(y_n) + P_K[h(x_n) - \eta T_2(x_n)]\},$$

where $\alpha_n, \beta_n \in [0, 1]$ for all $n \geq 0$.

For $\beta = 1$ in Algorithm 2.1, we arrive at

Algorithm 2.3. For arbitrary chosen initial points $x_0, y_0 \in H$, compute the sequence $\{x_n\}$ and $\{y_n\}$ by the iterative schemes

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n\{x_n - h(x_n) + P_K[g(y_n) - \rho T_1(y_n, x_n)]\}$$
$$g(y_n) = P_K[h(x_n) - \eta T_2(x_n, y_n)],$$

where $\alpha_n, \beta_n \in [0, 1]$ for all $n \geq 0$.

3. Main results

In this section, we investigate the strong convergence of Algorithm 2.1 under some suitable mild conditions and this is the main motivation as well as main result of this paper.

Theorem 3.1. *Let K be a closed convex subset of a real Hilbert space H. Let T_1 be relaxed (γ_1, r_1) cocoercive and μ_1-Lipschitzian mapping in the first variable, and T_2 be relaxed (γ_2, r_2) cocoercive and μ_2-Lipschitzian mapping in the first variable. Let g be a relaxed (γ_3, r_3) cocoercive and μ_3-Lipschitzian mapping of H into H and h be a relaxed (γ_4, r_4) cocoercive and μ_4-Lipschitzian mapping of K into H. Let $\{x_n\}, \{y_n\}$ be sequences defined by algorithm 2.1, for any initial point $x_0, y_0 \in K$, with conditions*

$$|\rho - \frac{r_1 - \gamma_1\mu_1^2}{\mu_1^2}| < \frac{\sqrt{((r_1 - \gamma_1\mu_1^2)^2 - \mu_1^2 k_1(2 - k_1)}}{\mu_1^2}, \qquad r_1 > \gamma_1\mu_1^2 + \mu_1\sqrt{k_1(2 - k_1)}, \quad k_1 < 1, \tag{3.1}$$

$$|\eta - \frac{r_2 - \gamma_2\mu_2^2}{\mu_2^2}| < \frac{\sqrt{((r_2 - \gamma_2\mu_2^2)^2 - \mu_2^2 k_2(2 - k_2)}}{\mu_2^2}, \qquad r_2 > \gamma_2\mu_2^2 + \mu_2\sqrt{k_2(2 - k_2)}, \quad k_2 < 1, \tag{3.2}$$

where

$$k_1 = \sqrt{1 + 2\gamma_3\mu_3^2 - 2r_3 + \mu_3^2},$$

$$k_2 = \sqrt{1 + 2\gamma_4\mu_4^2 - 2r_4 + \mu_4^2}.$$

$\alpha_n, \beta_n \in [0, 1]$ satisfy the following conditions
(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$, (ii)$\lim_{n\to\infty}(1 - \beta_n) = 0$.
Then (x_n, y_n) obtained from Algorithm 2.1 converges strongly to (x^*, y^*).

Proof. Since x^* and $y^* \in K$ are a solution to the problem (SEGVID), from (1.15), we have

$$\begin{aligned}
& \|x_{n+1} - x^*\| \\
= \;& \|(1 - \alpha_n)x_n + \alpha_n\{x_n - h(x_n) + P_K[g(y_n) - \rho T_1(y_n, x_n)]\} \\
& -(1 - \alpha_n)x^* - \alpha_n\{x^* - h(x^*) + P_K[g(x^*) - \rho T_1(y^*, x^*)]\}\| \\
\leq \;& (1 - \alpha_n)\|x_n - x^*\| + \alpha_n\|x_n - h(x_n) - x^* + h(x^*)\| \\
& +\alpha_n\|g(y_n) - \rho T_1(y_n, x_n) - g(x^*) + \rho T_1(y^*, x^*)\| \\
\leq \;& (1 - \alpha_n)\|x_n - x^*\| + \alpha_n\|x_n - h(x_n) - x^* + h(x^*)\| \\
& +\alpha_n\|g(y_n) - g(y^*) - y_n + y^*\| \\
& +\alpha_n\|y_n - y^* - \rho T_1(y_n, x_n) + \rho T_1(y^*, x^*)\|.
\end{aligned} \tag{3.3}$$

From the relaxed (γ_1, r_1)-cocoercive and μ_1-Lipschitzian definition on T_1,

$$\begin{aligned} & \|y_n - x^* - \rho(T_1(y_n, x_n) - T_1(y^*, x^*))\|^2 \\ = \ & \|y_n - x^*\|^2 - 2\rho\langle T_1(y_n, x_n) - T_1(y^*, x^*), y_n - x^*\rangle \\ & +\rho^2\|T_1(y_n, x_n) - T_1(y^*, x^*)\|^2 \\ \leq \ & \|y_n - x^*\|^2 - 2\rho[-\gamma_1\|T_1(y_n, x_n) - T_1(y^*, x^*)\|^2 + r_1\|y_n - y^*\|^2] \\ & +\rho^2\|T_1(y_n, x_n) - T_1(y^*, x^*)\|^2 \\ \leq \ & [1 + 2\rho\gamma_1\mu_1^2 - 2\rho r_1 + \rho^2\mu_1^2]\|y_n - y^*\|^2. \end{aligned} \tag{3.4}$$

In similar way, using the relaxed (γ_3, r_3)-cocoercivity and μ_3- lipschitzian of the operator g, and the relaxed (γ_4, r_4)-cocoercivity and μ_4- lipschitzian of the operator h, we have

$$\|y_n - y^* - [g(y_n) - g(y^*)]\| \leq \sqrt{1 + 2\gamma_3\mu_3^2 - 2r_3 + \mu_3^2}\|y_n - x^*\| = k_1\|y_n - y^*\|. \tag{3.5}$$

$$\|x_n ! - ! x^* - [h(x_n) - h(x^*)]\| \leq \sqrt{1 + 2\gamma_4\mu_4^2 - 2r_4 + \mu_4^2}\|x_n - x^*\| = k_2\|x_n - x^*\|. \tag{3.6}$$

From (3.3)-(3.5), we have

$$\|x_{n+1} - x^*\| \leq (1 - \alpha_n(1 - k_2))\|x_n - x^*\| + \alpha_n\theta_1\|y_n - y^*\|, \tag{3.7}$$

where $\theta_1 = k_1 + \sqrt{1 + 2\rho\gamma_1\mu_1^2 - 2\rho r_1 + \rho^2\mu_1^2}$. From (3.1) and (3.2), we have $\theta_1 < 1$.

From (1.16), we have

$$\begin{aligned} & \|y_n - y^*\| \\ \leq \ & (1 - \beta_n)\|x_n - y^*\| + \beta_n\|\{y_n - g(y_n) + P_K[h(x_n) - \eta T_2(x_n, y_n)]\} \\ & -\{y^* - g(y^*) + P_K[h(x^*) - \eta T_2(x^*, y^*)]\}\| \\ \leq \ & (1 - \beta_n)\|x_n - y^*\| + \beta_n\|y_n - y^* - g(y_n) + g(y^*)\| \\ & +\beta_n\|h(x_n) - \eta T_2(x_n, y_n) - h(x^*) + \eta T_2(x^*, y^*\| \\ \leq \ & (1 - \beta_n)\|x_n - y^*\| + \beta_n\|y_n - y^* - g(y_n) + g(y^*)\| \\ & +\beta_n\|x_n - x^* - h(x_n) + h(x^*)\| \\ & +\beta_n\|x_n - x^* - \eta(T_2(x_n, y_n) - T_2(x^*, y^*))\|, \end{aligned} \tag{3.8}$$

similarly, from the relaxed (γ_2, r_2)-cocoercive and μ_2-Lipschitzian definition on T_2,

$$\begin{aligned} & \|x_n - x^* - \eta(T_2(x_n, y_n) - T_2(x^*, y^*))\|^2 \\ = \ & \|x_n - x^*\|^2 - 2\eta\langle T_2(x_n, y_n) - T_2(x^*, y^*), x_n - x^*\rangle \\ & +\eta^2\|T_2(x_n, y_n) - T_2(x^*, y^*)\|^2 \\ \leq \ & \|x_n - x^*\|^2 - 2\eta[-\gamma_2\|T_2(x_n, y_n) - T_2(x^*, y^*)\|^2 + r_2\|x_n - x^*\|^2] \\ & +\eta^2\|T_2(x_n, y_n) - T_2(x^*, y^*)\|^2 \\ \leq \ & [1 + 2\eta\gamma_2\mu_2^2 - 2\eta r_2 + \eta^2\mu_2^2]\|x_n - x^*\|^2. \end{aligned} \tag{3.9}$$

From (3.6),(3.8),(3.9) we have

$$
\begin{aligned}
&\|y_n - y^*\| \\
\leq\ & (1-\beta_n)\|x_n - x^*\| + (1-\beta_n)\|x^* - y^*\| \\
& +\beta_n k_1\|y_n - y^*\| + \beta_n(k_2 + \sqrt{1 + 2\eta\gamma_2\mu_2^2 - 2\eta r_2 + \eta^2\mu_2^2})\|x_n - x^*\|, \\
=\ & \beta_n k_1\|y_n - y^*\| + (1-\beta_n(1-\theta_2))\|x_n - x^*\| + (1-\beta_n)\|x^* - y^*\|
\end{aligned}
$$

where $\theta_2 = k_2 + \sqrt{1 + 2\eta\gamma_2\mu_2^2 - 2\eta r_2 + \eta^2\mu_2^2}, \theta_2 < 1$, i.e.,

$$
\|y_n - y^*\| \leq \frac{1-\beta_n(1-\theta_2)}{1-\beta_n k_1}\|x_n - x^*\| + \frac{1-\beta_n}{1-\beta_n k_1}\|x^* - y^*\|. \tag{3.10}
$$

From(3.7)and (3.10), we obtain that

$$
\begin{aligned}
&\|x_{n+1} - x^*\| \\
\leq\ & (1-\alpha_n(1-k_2))\|x_n - x^*\| + \alpha_n\theta_1\|y_n - x^*\| \\
\leq\ & (1-\alpha_n(1-k_2))\|x_n - x^*\| + \alpha_n\theta_1\frac{1-\beta_n(1-\theta_2)}{1-\beta_n k_1}\|x_n - x^*\| \\
& +\alpha_n\theta_1\frac{1-\beta_n}{1-\beta_n k_1}\|x^* - y^*\| \\
=\ & \left\{1-\alpha_n\left[1-k_2-\theta_1\frac{1-\beta_n(1-\theta_2)}{1-\beta_n k_1}\right]\right\}\|x_n - x^*\| + \alpha_n\theta_1\frac{1-\beta_n}{1-\beta_n k_1}\|x^* - y^*\|,
\end{aligned}
$$

noticing $\lim_{n\to\infty}(1-\beta_n) = 0$, by Lemma 1.3, $\lim_{n\to\infty}\|x_n - x^*\| = 0$, i.e., $x_n \to x^*$. Furthermore it follows that $\lim_{n\to\infty}\|y_n - y^*\| = 0$, i.e., $y_n \to y^*$. □

References

[1] Muhammad. Aslam Noor, *Extended general variational inequalities*, Appl. Math. Lett. **22** (2009), 182–186.

[2] Muhammad. Aslam Noor and Khalida Inayat Noor, *Projection algorithms for solving a system of general variational inequalities*, Nonlinear Analysis **70** (2009), 2700–2706.

[3] R. U. Verma, *Generalized system for relaxed ccoercive variational inequalities and projection methods*, J. Optim. Theory Appl. **121** (2004), 203–210.

[4] S. S. Chang, H. W. Joseph Lee and C. K. Chan, *General system for relaxed cocoercive variational inequalities in Hilbert spaces*, Applied Mathematics Letters **20** (2007), 329–334.

[5] Zhenyu Huang and M. Asalm Noor, *An explicit projection method for s system of nonlinear variational inequalities with different*(γ, r)*-cocoercive mappings*, Applied Mathematics and Computation **190** (2007), 356–361.

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