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PROPERTIES OF HYPERHOLOMORPHIC FUNCTIONS IN CLIFFORD ANALYSIS

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Abstract. The noncommutative extension of the complex numbers for the four dimensional real space is a quaternion. R. Fueter, C. A. Deavours and A. Subdery have developed a theory of quaternion analysis. M. Naser and K. Nôno have given several results for integral formulas of hyperholomorphic functions in Clifford analysis. We research the properties of hyperholomorphic functions on $\mathbb{C}^2 \times \mathbb{C}^2$.

1. Introduction

Let $m \in \mathbb{N}$, $m \geq 1$, and let $\mathcal{A}_n$ be the Clifford algebra constructed over a real anti-Euclidean quadratic n-dimensional vector space, $n \geq m$. Then $\mathcal{A}_n$ is a 2^n-dimensional real vector space with basis $\{e_A : A \subseteq \{1, ..., n\}\}$, where $e_\phi = e_0 = 1$, $e_A = e_{\alpha_1} ... e_{\alpha_h}$, $A = \{\alpha_1, ..., \alpha_h\}$, $\alpha_1 < \alpha_2 < ... < \alpha_h$; $e_{\alpha_j} e_{\alpha_k} = -e_{\alpha_k} e_{\alpha_j}$ when $j \neq k$ and $e^2_{\alpha_j} = -1$, $j = 1, ..., n$. For $a = \sum_A a_A e_A \in \mathcal{A}_n$, we put $\overline{a} = \sum_A a_A \overline{e}_A$ where for $A \neq \phi$, $\overline{e}_A = \overline{e}_{\alpha_h} ... \overline{e}_{\alpha_1}$, with $\overline{e}_i = -e_i$, $i = 1, ..., n$, while for $A = \phi$, $\overline{e}_0 = e_0$. If $x = (x_0, x_1, ..., x_m) \in \mathbb{R}^{m+1}$ is identified with $x = x_0 + x_1 e_1 + ... + x_m e_m$, then $\mathbb{R}^{m+1}$ may be considered as a subspace of $\mathcal{A}_n$.

2. Notations on Quaternion analysis

The field $\mathcal{T} = \mathcal{A}_2$ of quaternions

$$z = x_0 + ix_1 + jx_2 + kx_3, \; x_0, x_1, x_2, x_3 \in \mathbb{R} \tag{1}$$

is a four dimensional non-commutative $\mathbb{R}$-field of real numbers such that its four base elements 1, i, j and k satisfy the following :

$$i^2 = j^2 = k^2 = -1, \; ij = -ji = k, jk = -kj = i, ki = -ik = j. \tag{2}$$

The element 1 is the identity of $\mathcal{T}$. Identifying the element i with the imaginary unit $\sqrt{-1}$ in the $\mathbb{C}$-field of complex numbers, a quaternion z given by (1) is

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regarded as $z = z_1 + z_2 j \in \mathcal{T}$ where $z_1 := x_0 + ix_1$ and $z_2 := x_2 + ix_3$ are complex numbers in $\mathbb{C}$. Thus, we identify $\mathcal{T}$ with $\mathbb{C}^2$.

For the equation $z^3 + 8 = 0$ in the complex plane $\mathbb{C}$, the 3 solutions are $-2, 1 + \sqrt{3}i, 1 - \sqrt{3}i$ in $\mathbb{C}$. In the quaternion $\mathcal{T}$, the equation has solutions which forms $z = a + bi + cj + dk$ $(a, b, c, d \in \mathbb{R})$. Then it satisfies $z^3 = (a^3 - 3ab^2 - 3ac^2 - 3ad^2) + (3a^2b - b^3 - bc^2 - bd^2)i + (3a^2c - b^2c - c^3 - cd^2)j + (3a^2d - b^2d - c^2d - d^3)k$. That is, $a^3 - 3ab^2 - 3ac^2 - 3ad^2 = -8$, $3a^2b - b^3 - bc^2 - bd^2 = 0$, $3a^2c - b^2c - c^3 - cd^2 = 0$, $3a^2d - b^2d - c^2d - d^3 = 0$. Thus it has infinitely many solutions $z = 1 + bi + cj + dk$ with $b^2 + c^2 + d^2 = 3$ in $\mathcal{T}$.

We define the quaternionic multiplication of two quaternions $z = z_1 + z_2 j$ and $w = w_1 + w_2 j$ is defined by

$$zw = (z_1 w_1 - z_2 \overline{w_2}) + (z_1 w_2 + z_2 \overline{w_1})j,$$

where $\overline{w_1}$ and $\overline{w_2}$ are complex conjugations of w_1 and w_2, respectively. The quaternionic conjugate z^* and the absolute value $|z|$ of $z = z_1 + z_2 j$ are given by the following:

$$z^* = \overline{z_1} - z_2 j, \ |z| = \sqrt{|z_1|^2 + |z_2|^2}.$$

And every non-zero quaternion $z = z_1 + z_2 j$ has a unique inverse z^{-1} given by $z^{-1} = z^*/|z|^2$.

We use the following quaternionic differential operators :

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial z_1} - j\frac{\partial}{\partial \overline{z_2}} = \frac{1}{2}(\frac{\partial}{\partial x_0} - i\frac{\partial}{\partial x_1} - j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3}),$$
$$\frac{\partial}{\partial z^*} = \frac{\partial}{\partial \overline{z_1}} + j\frac{\partial}{\partial \overline{z_2}} = \frac{1}{2}(\frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} - k\frac{\partial}{\partial x_3})$$

where $\partial/\partial z_1$, $\partial/\partial \overline{z_1}$, $\partial/\partial z_2$, and $\partial/\partial \overline{z_2}$ are usual differential operators used in complex analysis. So, we have

$$\frac{\partial}{\partial z_1}j = \frac{1}{2}(\frac{\partial}{\partial x_0} - i\frac{\partial}{\partial x_1})j = \frac{1}{2}(j\frac{\partial}{\partial x_0} - ij\frac{\partial}{\partial\, x_1}) = \frac{1}{2}(j\frac{\partial}{\partial x_0} + ji\frac{\partial}{\partial x_1}) = j\frac{\partial}{\partial \overline{z_1}},$$
$$\frac{\partial}{\partial \overline{z_1}}j = \frac{1}{2}(\frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1})j = \frac{1}{2}(j\frac{\partial}{\partial x_0} + ij\frac{\partial}{\partial x_1}) = \frac{1}{2}(j\frac{\partial}{\partial x_0} - ji\frac{\partial}{\partial x_1}) = j\frac{\partial}{\partial z_1}.$$

In the space $\mathbb{C}^2 \times \mathbb{C}^2 \cong \mathcal{T} \times \mathcal{T}$ of two quaternion variables $z = z_1 + z_2 j$ and $w = w_1 + w_2 j$, where $z_1 = x_0 + ix_1$, $z_2 = x_2 + ix_3$, $w_1 = y_0 + iy_1$ and $w_2 = y_2 + iy_3$ in $\mathbb{C}$, we use the quaternion differential operators $\frac{\partial}{\partial z}$, $\frac{\partial}{\partial z^*}$ and

$$\frac{\partial}{\partial w} = \frac{\partial}{\partial w_1} - j\frac{\partial}{\partial \overline{w_2}}, \ \frac{\partial}{\partial w^*} = \frac{\partial}{\partial \overline{w_1}} + j\frac{\partial}{\partial \overline{w_2}}.$$

Let D be an open set in $\mathbb{C}^2$ and $f(z) = f_1(z) + f_2(z)j$ be a function defined in D with values in $\mathcal{T}$, where $z = (z_1, z_2)$ and $f_1(z)$ and $f_2(z)$ are complex valued functions.

Definition 1. Let D be an open set in $\mathbb{C}^2$. A function $f(z) = f_1(z) + f_2(z)j$ is said to be L(R)-hyperholomorphic in D, if
(a) f_1 and f_2 are continuously differentiable in D,
(b) $\frac{\partial}{\partial z^*} f = 0$ ($f \frac{\partial}{\partial z^*} = 0$) in D.

The above equations (b) of the definition 1 operate to f as follows

$$\frac{\partial}{\partial z^*} f = (\frac{\partial}{\partial \overline{z_1}} + j\frac{\partial}{\partial \overline{z_2}})(f_1 + f_2 j) = (\frac{\partial f_1}{\partial \overline{z_1}} - \frac{\partial \overline{f_2}}{\partial z_2}) + (\frac{\partial f_2}{\partial \overline{z_1}} + \frac{\partial \overline{f_1}}{\partial z_2})j,$$
$$(f\frac{\partial}{\partial z^*} = (f_1 + f_2 j)(\frac{\partial}{\partial \overline{z_1}} + j\frac{\partial}{\partial \overline{z_2}}) = (\frac{\partial f_1}{\partial \overline{z_1}} - \frac{\partial f_2}{\partial \overline{z_2}}) + (\frac{\partial f_2}{\partial z_1} + \frac{\partial f_1}{\partial z_2})j).$$

The function $f(z) = f_1(z) + f_2(z)j$ is L-hyperholomorphic function in $D \subset \mathbb{C}^2$, simply we say that $f(z)$ is a hyperholomorphic function on $D \subset \mathbb{C}^2$. The above equation for hyperholomorphic function $f(z)$ is equivalent to the following systems of equations :

$$\frac{\partial f_1}{\partial \overline{z_1}} = \frac{\partial \overline{f_2}}{\partial z_2}, \ \frac{\partial f_2}{\partial \overline{z_1}} = -\frac{\partial \overline{f_1}}{\partial z_2}. \tag{3}$$

We say that the equations (3) are the corresponding q-Cauchy-Riemann equations in $\mathcal{T}$.

Let Ω be an open set in $\mathbb{C}^2 \times \mathbb{C}^2$ and $f(z,w) = f_1(z,w) + f_2(z,w)j$ be a function defined in Ω with values in $\mathcal{T} \times \mathcal{T}$, where $(z,w) = (z_1, z_2, w_1, w_2) \in \Omega$.

Definition 2. Let Ω be an open set in $\mathbb{C}^2 \times \mathbb{C}^2$. A function $f(z,w) = f_1(z,w) + f_2(z,w)j$ is said to be hyperholomorphic in Ω, if
(a) f_1 and f_2 are continuously differentiable in Ω,
(b) $\frac{\partial}{\partial z^*} f = 0$ and $f\frac{\partial}{\partial w^*} = 0$ in Ω.

The above equations (b) of the definition 2 are equivalent to the following systems of equations :

$$\frac{\partial f_1}{\partial \overline{z_1}} = \frac{\partial \overline{f_2}}{\partial z_2}, \ \frac{\partial f_2}{\partial \overline{z_1}} = -\frac{\partial \overline{f_1}}{\partial z_2}, \ \frac{\partial f_1}{\partial \overline{w_1}} = \frac{\partial f_2}{\partial \overline{w_2}}, \ \frac{\partial f_2}{\partial w_1} = -\frac{\partial f_1}{\partial w_2}. \tag{4}$$

These are the corresponding q-Cauchy-Riemann equations in $\mathcal{T} \times \mathcal{T}$.

3. Properties of hyperholomorphic functions on $\mathcal{T}$

M. Naser [6] proved the following theorems.

Theorem 3.1. ([6]) *For any complex harmonic function $f_1(z)$ in a domain of holomorphy $D \subset \mathbb{C}^2$, we can find a function $f_2(z)$ so that $f(z) = f_1(z) + f_2(z)j$ is a hyperholomorphic function in D.*

Example 3.2. If we know a complex valued harmonic function

$$f_1(z) = \frac{\overline{z_1}}{|z|^4}$$

in a domain of holomorphy $D \subset \mathbb{C}^2$, then we can find a hyper-conjugate harmonic function $f_2(z)$ of $f_1(z)$ in D. That is

$$f_2(z) = -\frac{\overline{z_2}}{|z|^4},$$

and $f(z) = f_1(z) + f_2(z)j$ is a hyperholomorphic function in D.

Theorem 3.3. ([6]) *Let $\kappa = dz_1 \wedge dz_2 \wedge d\bar{z}_2 - dz_1 \wedge d\bar{z}_1 \wedge d\bar{z}_2 j$ and let a function $f(z)$ is hyperholomorphic in a domain G of $\mathcal{T}$. Then for any domain $D \subset G$ with smooth boundary ∂D,*

$$\int_{\partial D} \kappa f = 0$$

where κf is the quaternion product of the form κ on the function $f(z) = f_1(z) + f_2(z)j$.

K. Nôno [7] obtained the following theorem.

Theorem 3.4. ([7]) *Let G be a bounded domain in $\mathbb{C}^2$ with C^1-bounbary and $f(z) = f_1(z) + f_2(z)j$ be a continuous function on ∂G. Then the function*

$$g(z) = \frac{1}{4\pi^2} \int_{\partial G} \frac{-\{(\overline{\zeta_1} - \overline{z_1}) - (\overline{\zeta_2} - \overline{z_2})j\}}{(|\zeta_1 - z_1|^2 + |\zeta_2 - z_2|^2)^2} \kappa_\zeta f(\zeta), \ z \in \mathbb{C}^2 - \partial G$$

where $\kappa_\zeta = d\zeta_1 \wedge d\zeta_2 \wedge d\overline{\zeta_2} - d\zeta_1 \wedge d\overline{\zeta_1} \wedge d\overline{\zeta_2} j$ is hyperholomorphic in $\mathbb{C}^2 - \partial G$.

4. Properties of hyperholomorphic functions on $\mathcal{T} \times \mathcal{T}$

Example 4.1. If we know a complex valued harmonic function

$$f_1(z, w) = \frac{z_1\overline{z_1} + z_2\overline{z_2} + w_1\overline{w_1} + w_2\overline{w_2}}{|zw|^2}$$

in a pseudoconvex domain $\Omega \subset \mathbb{C}^2 \times \mathbb{C}^2$, then by the theorem of [5: Kajiwara-Li-Shon], we can find a hyper-conjugate harmonic function $f_2(z, w)$ of $f_1(z, w)$ in Ω. That is

$$f_2(z, w) = \frac{z_1\overline{z_1} - z_2\overline{z_2}}{|z|^2 z_1 z_2} + \frac{1}{\overline{w_1} w_2},$$

and $f(z, w) = f_1(z, w) + f_2(z, w)j$ is a hyperholomorphic function in Ω.

Theorem 4.2. *Let $\kappa_1 = dz_1 \wedge dz_2 \wedge d\overline{z_2} - dz_1 \wedge d\overline{z_1} \wedge d\overline{z_2} j$, $\kappa_2 = dw_1 \wedge dw_2 \wedge d\overline{w_2} - dw_1 \wedge d\overline{w_1} \wedge d\overline{w_2} j$ and let a function $f(z, w)$ is hyperholomorphic in a domain G of $\mathcal{T} \times \mathcal{T}$. Then for any domain $\Omega \subset G$ with smooth boundary $\partial\Omega$,*

$$\int_{\partial\Omega} (\kappa_1 f \kappa_2) = 0$$

where $\kappa_1 f \kappa_2$ is the quaternion product of forms κ_1 and κ_2 on the function $f(z, w) = f_1(z, w) + f_2(z, w)j$.

Proof. By the rule of the quaternion multiplication, we have

$$\begin{aligned}\kappa_1 f \kappa_2 = {} & f_1 dz_1 \wedge dz_2 \wedge d\overline{z_2} \wedge dw_1 \wedge dw_2 \wedge d\overline{w_2} \\ & - f_1 dz_1 \wedge dz_2 \wedge d\overline{z_2} \wedge dw_1 \wedge d\overline{w_1} \wedge d\overline{w_2} j \\ & + \overline{f_1} dz_1 \wedge d\overline{z_1} \wedge d\overline{z_2} \wedge dw_1 \wedge d\overline{w_1} \wedge dw_2 \\ & + \overline{f_1} dz_1 \wedge d\overline{z_1} \wedge d\overline{z_2} \wedge d\overline{w_1} \wedge dw_2 \wedge d\overline{w_2} j \\ & - f_2 dz_1 \wedge dz_2 \wedge d\overline{z_2} \wedge dw_1 \wedge d\overline{w_1} \wedge dw_2 \\ & - f_2 dz_1 \wedge dz_2 \wedge d\overline{z_2} \wedge d\overline{w_1} \wedge dw_2 \wedge d\overline{w_2} j \\ & + \overline{f_2} dz_1 \wedge d\overline{z_1} \wedge d\overline{z_2} \wedge dw_1 \wedge dw_2 \wedge d\overline{w_2} \\ & - \overline{f_2} dz_1 \wedge d\overline{z_1} \wedge d\overline{z_2} \wedge dw_1 \wedge d\overline{w_1} \wedge d\overline{w_2} j.\end{aligned}$$

Hence

$$\begin{aligned}d(\kappa_1 f \kappa_2) = {} & (-\frac{\partial f_1}{\partial \overline{z_1}} + \frac{\partial \overline{f_2}}{\partial z_2}) dz_1 \wedge d\overline{z_1} \wedge dz_2 \wedge d\overline{z_2} \wedge dw_1 \wedge dw_2 \wedge d\overline{w_2} \\ & + (\frac{\partial f_1}{\partial \overline{w_1}} - \frac{\partial f_2}{\partial \overline{w_2}}) dz_1 \wedge dz_2 \wedge d\overline{z_2} \wedge dw_1 \wedge d\overline{w_1} \wedge dw_2 \wedge d\overline{w_2} \\ & + (\frac{\partial \overline{f_1}}{\partial z_2} + \frac{\partial f_2}{\partial \overline{z_1}}) dz_1 \wedge d\overline{z_1} \wedge dz_2 \wedge d\overline{z_2} \wedge dw_1 \wedge d\overline{w_1} \wedge dw_2 \\ & + (\frac{\partial \overline{f_1}}{\partial \overline{w_2}} + \frac{\partial \overline{f_2}}{\partial \overline{w_1}}) dz_1 \wedge d\overline{z_1} \wedge d\overline{z_2} \wedge dw_1 \wedge d\overline{w_1} \wedge dw_2 \wedge d\overline{w_2} \\ & + (\frac{\partial f_1}{\partial \overline{z_1}} - \frac{\partial \overline{f_2}}{\partial z_2}) dz_1 \wedge d\overline{z_1} \wedge dz_2 \wedge d\overline{z_2} \wedge dw_1 \wedge d\overline{w_1} \wedge d\overline{w_2} j \\ & + (\frac{\partial f_1}{\partial w_2} + \frac{\partial f_2}{\partial w_1}) dz_1 \wedge dz_2 \wedge d\overline{z_2} \wedge dw_1 \wedge d\overline{w_1} \wedge dw_2 \wedge d\overline{w_2} j \\ & + (\frac{\partial \overline{f_1}}{\partial z_2} + \frac{\partial f_2}{\partial \overline{z_1}}) dz_1 \wedge d\overline{z_1} \wedge dz_2 \wedge d\overline{z_2} \wedge d\overline{w_1} \wedge dw_2 \wedge d\overline{w_2} j \\ & + (-\frac{\partial \overline{f_1}}{\partial w_1} + \frac{\partial \overline{f_2}}{\partial w_2}) dz_1 \wedge d\overline{z_1} \wedge d\overline{z_2} \wedge dw_1 \wedge d\overline{w_1} \wedge dw_2 \wedge d\overline{w_2} j.\end{aligned}$$

From the corresponding q-Cauchy-Riemann equations (4), we have $d(\kappa_1 f \kappa_2) = 0$. By Stoke's theorem we have

$$\int_{\partial\Omega} \kappa_1 f \kappa_2 = \int_{\Omega} d(\kappa_1 f \kappa_2) = 0.$$

□

Theorem 4.3. ([9]) *Let the function $f(z,w) = f_1(z,w) + f_2(z,w)j$ is hyperholomorphic in bounded domains U and V in $\mathbb{C}^2$ with C^1-boundaries and continuously differentiable in a neighborhood of $\overline{U} \times \overline{V}$. Then*

$$f(z,w) = \frac{1}{(4\pi^2)^2} \int_{\partial U} \int_{\partial V} H(z,\zeta)\kappa_\zeta f(\zeta,\eta)\kappa_\eta H(w,\eta) \tag{5}$$

where

$$H(z,\zeta)=\frac{(\overline{\zeta_1}-\overline{z_1})-(\overline{\zeta_2}-\overline{z_2})j}{|\zeta-z|^4},\ H(w,\eta)=\frac{(\overline{\eta_1}-\overline{w_1})-(\overline{\eta_2}-\overline{w_2})j}{|\eta-w|^4}$$

in $U\times V$.

Proposition 4.4. *The kernel of the integral formula* (5),

$$K(\zeta,z,\eta,w)=\frac{(\overline{\zeta_1}-\overline{z_1})-(\overline{\zeta_2}-\overline{z_2})j}{|\zeta-z|^4}\cdot\frac{(\overline{\eta_1}-\overline{w_1})-(\overline{\eta_2}-\overline{w_2})j}{|\eta-w|^4}$$

is hyperholomorphic with respect to ζ,z,η *and* w *for* $\zeta\neq z$, $\eta\neq w$.

Proof. It suffices to show that the functions $K(0,z,0,w)$ and $K(\zeta,0,\eta,0)$ are hyperholomorphic in $\mathbb{C}^2\times\mathbb{C}^2-\{0\}$. Since $K(0,z,0,w)=\frac{\overline{z_1}-\overline{z_2}j}{|z|^4}\cdot\frac{\overline{w_1}-\overline{w_2}j}{|w|^4}$, we have $\frac{\partial}{\partial z^*}K(0,z,0,w)=0$ and $K(0,z,0,w)\frac{\partial}{\partial w^*}=0$. □

Theorem 4.5. *Let* U *and* V *be bounded domains in* $\mathcal{T}\times\mathcal{T}$ *with* C^1*-boundaries and* $f(z,w)=f_1(z,w)+f_2(z,w)j$ *be a continuous function on* $\partial U\times\partial V$. *Then the function*

$$g(z,w)=\frac{1}{(4\pi^2)^2}\int_{\partial U}\int_{\partial V}H(z;\zeta)\kappa_\zeta f(\zeta,\eta)\kappa_\eta H(w;\eta)$$

is hyperholomorphic in $\mathcal{T}\times\mathcal{T}-\{\partial U\times\partial V\}$.

Proof. By the differentiation under the integral sign and Proposition 4.4, we have

$$\begin{aligned}\frac{\partial}{\partial z^*}g(z,w)&=\frac{\partial}{\partial z^*}\{\frac{1}{(4\pi^2)^2}\int_{\partial U}\int_{\partial V}\frac{(\overline{z_1}-\overline{z_2}j)}{(|z_1|^2+|z_2|^2)^2}\cdot\frac{(\overline{w_1}-\overline{w_2}j)}{(|w_1|^2+|w_2|^2)^2}\}\\&=\frac{1}{(4\pi^2)^2}\int_{\partial U}\int_{\partial V}\frac{\partial}{\partial z^*}\{\frac{(\overline{z_1}-\overline{z_2}j)}{(|z_1|^2+|z_2|^2)^2}\cdot\frac{(\overline{w_1}-\overline{w_2}j)}{(|w_1|^2+|w_2|^2)^2}\}\\&=0\end{aligned}$$

and

$$\begin{aligned}g(z,w)\frac{\partial}{\partial w^*}&=\{\frac{1}{(4\pi^2)^2}\int_{\partial U}\int_{\partial V}\frac{(\overline{z_1}-\overline{z_2}j)}{(|z_1|^2+|z_2|^2)^2}\cdot\frac{(\overline{w_1}-\overline{w_2}j)}{(|w_1|^2+|w_2|^2)^2}\}\frac{\partial}{\partial w^*}\\&=\frac{1}{(4\pi^2)^2}\int_{\partial U}\int_{\partial V}\{\frac{(\overline{z_1}-\overline{z_2}j)}{(|z_1|^2+|z_2|^2)^2}\cdot\frac{(\overline{w_1}-\overline{w_2}j)}{(|w_1|^2+|w_2|^2)^2}\}\frac{\partial}{\partial w^*}\\&=0\end{aligned}$$

in $\mathcal{T}\times\mathcal{T}-\{\partial U\times\partial V\}$.

Hence, $g(z,w)$ is hyperholomorphic in $\mathcal{T}\times\mathcal{T}-\{\partial U\times\partial V\}$. □

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