East Asian Mathematical Journal
Vol. 28 (2012), No. 5, pp. 533–541
http://dx.doi.org/10.7858/eamj.2012.038

COMMON FIXED POINT THEOREM AND INVARIANT APPROXIMATION IN COMPLETE LINEAR METRIC SPACES

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Abstract. A common fixed point result of Gregus type for subcompatible mappings defined on a complete linear metric space is obtained. The considered underlying space is generalized from Banach space to complete linear metric spaces, which include Banach space and complete metrizable locally convex spaces. Invariant approximation results have also been determined as its application.

1. Introduction

In the realm of Best Approximation Theory, it is viable, meaningful and potentially productive to know whether some useful properties of the function being approximated are inherited by the approximation function. In this perspective, Meinardus [10] was the first to employ a fixed-point theorem of Schauder to establish the existence of an invariant approximation. Later, Brosowski [3] obtained a celebrated result and generalized the result of Meinardus [10]. Afterwards, several results have been established for commuting and noncommuting mappings in normed linear space, Banach space and locally convex space(for detail see [1, 2], [9, 11, 12, 13, 14, 15] and references therein).

Recently, Nashine and Khan [12] used the new type of noncommuting maps, known as subcompatible maps, which are compatible but converse is not true in general, to prove the common fixed point results in locally convex spaces and generalized the following theorem of Jungck [6], and the new result is applied to extend existence results in the area of best approximation.

Theorem 1.1. ([6]) *Let $\mathcal{T}$ and $\mathcal{S}$ be compatible self-maps of a closed convex subset $\mathcal{M}$ of a Banach space $\mathcal{X}$. Suppose $\mathcal{S}$ is linear, continuous, and that $\mathcal{T}(\mathcal{M}) \subseteq \mathcal{S}(\mathcal{M})$. If there exists $a \in (0,1)$ such that $x, y \in \mathcal{M}$*

$$\|\mathcal{T}x - \mathcal{T}y\| \leq a\|\mathcal{S}x - \mathcal{S}y\| + (1-a)\max\{\|\mathcal{T}x - \mathcal{S}x\|, \|\mathcal{T}y - \mathcal{S}y\|\}, \qquad (1)$$

then $\mathcal{T}$ and $\mathcal{S}$ have a unique common fixed point in $\mathcal{M}$.

Received May 18, 2010; Accepted August 22, 2000.
2000 *Mathematics Subject Classification.* 41A50, 47H10, 54H25.
Key words and phrases. Best approximation, invariant approximation, weakly compatible maps, subcompatible maps, complete linear metric spaces.

The present paper generalizes the results of Nashine and Khan [12] to complete linear metric space, which includes Banach spaces and complete metrizable locally convex spaces.

2. Preliminaries

In the material to be produced here, the following definitions have been used:

Theorem 2.1. *A linear topological space $\mathcal{X}$ is metrizable if and only if it has a countable base of neighbourhoods of zero. The topology of a linear metric space can always be defined by a real-valued function $F : \mathcal{X} \to \mathbb{R}$, called F-norm such that for all $x, y \in \mathcal{X}$ and scaler $\mathcal{K}$, we have*

(i) $F(x) \geq 0$;
(ii) $F(x) = 0 \Rightarrow x = 0$;
(iii) $F(x + y) \leq F(x) + F(y)$;
(iv) $F(\lambda x) \leq F(x)$ *for all* $\lambda \in \mathcal{K}$ *and* $|\lambda| \leq 1$;
(v) *if* $\lambda_n \to 0$, *and* $\lambda \in \mathcal{K}$, *then* $F(\lambda_n x) \to 0$.

Definition 1. Let $\mathcal{X}$ be a metric linear space. Then a nonempty subset $\mathcal{M}$ in $\mathcal{X}$ is said to be convex, if $\lambda x+(1-\lambda)y \in \mathcal{M}$, whenever $x, y \in \mathcal{M}$ and $0 \leq \lambda \leq 1$.

A subset $\mathcal{M}$ in $\mathcal{X}$ is said to be starshaped, if there exists at least one point $q \in \mathcal{M}$ such that the line segment $[x, q]$ joining x to q is contained in $\mathcal{M}$ for all $x \in \mathcal{M}$ (that is $\lambda x + (1 - \lambda)q \in \mathcal{M}$, for all $x \in \mathcal{M}$ and $0 \leq \lambda \leq 1$). In this case q is called the starcenter of $\mathcal{M}$.

Each convex set is starshaped with respect to each of its points, but not conversely.

Definition 2. ([5]) A pair $\{\mathcal{T}, \mathcal{S}\}$ of self-maps of a complete linear metric space $\mathcal{X}$ is said to be compatible, if $F(\mathcal{T}\mathcal{S}x_n - \mathcal{S}\mathcal{T}x_n) \to 0$, whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ such that $\mathcal{T}x_n, \mathcal{S}x_n \to t \in \mathcal{X}$.

Every commuting pair of mappings is compatible but the converse is not true in general.

Definition 3. Suppose that $\mathcal{M}$ is q-starshaped with $p \in Fix(\mathcal{S})$(set of fixed point) and is both $\mathcal{T}$- and $\mathcal{S}$-invariant. Then $\mathcal{T}$ and $\mathcal{S}$ are called $\mathcal{R}$-subcommuting on $\mathcal{M}$, if for all $x \in \mathcal{M}$ there exists a real number $\mathcal{R} > 0$ such that $F(\mathcal{S}\mathcal{T}x - \mathcal{T}\mathcal{S}x) \leq (\frac{\mathcal{R}}{k})F(((1 - k)q + k\mathcal{T}x) - \mathcal{S}x)$ for each $k \in (0, 1]$. If $\mathcal{R} = 1$, then the maps are called 1-subcommuting. The $\mathcal{S}$ and $\mathcal{T}$ are called $\mathcal{R}$-subweakly commuting on $\mathcal{M}$, if for all $x \in \mathcal{M}$ there exists a real number $\mathcal{R} > 0$ such that $F(\mathcal{S}\mathcal{T}x - \mathcal{T}\mathcal{S}x) \leq \mathcal{R}dist(\mathcal{S}x, [q, \mathcal{T}x])$, where $[q, x] = (1 - k)q + kx : 0 \leq k \leq 1$.

2.1. Subcompatible Mappings in Complete Linear Metric Space

We extend the concepts of subcompatible mappings to complete linear metric space in the following way:

Definition 4. ([1, 9]) Suppose that $\mathcal{M}$ is q-starshaped subset of a linear metric space $\mathcal{X}$. For the self maps $\mathcal{S}$ and $\mathcal{T}$ of $\mathcal{M}$ with $q \in Fix(\mathcal{S})$, define $\bigwedge_q(\mathcal{S}, \mathcal{T}) =$

$\bigcup\{\bigwedge(\mathcal{S},\mathcal{T}_k) : 0 \leq k \leq 1\}$ where $\mathcal{T}_k x = seg[\mathcal{T}x, q]$ and $\bigwedge(\mathcal{S},\mathcal{T}_k) = \{\{x_n\} \subset \mathcal{M} : \lim_n \mathcal{S}x_n = \lim_n \mathcal{T}_k x_n = t \in \mathcal{M}\}$. Then $\mathcal{S}$ and $\mathcal{T}$ are called subcompatible, if $\lim_n F(\mathcal{S}\mathcal{T}x_n - \mathcal{T}\mathcal{S}x_n) = 0$ for all sequences $x_n \in \bigwedge_q(\mathcal{S},\mathcal{T})$.

Obviously, subcompatible maps are compatible but the converse does not hold, in general, as the following example shows.

Example 2.2. ([1, 9, 11]) Let $\mathcal{X} = \mathbb{R}$ with usual metric and $\mathcal{M} = [1,\infty)$. Let $\mathcal{S}(x) = 2x - 1$ and $\mathcal{T}(x) = x^2$, for all $x \in \mathcal{M}$. Let $q = 1$. Then $\mathcal{M}$ is q-starshaped with $\mathcal{S}q = q$. As for sequences in $\mathcal{M}$ converging to 1, $\lim_n \|\mathcal{S}\mathcal{T}x_n - \mathcal{T}\mathcal{S}x_n\| = 0$, therefore $\mathcal{S}$ and $\mathcal{T}$ are compatible. For any sequence $\{x_n\}$ in $\mathcal{M}$ with $\lim_n x_n = 2$, we have, $\lim_n \mathcal{S}x_n = \lim_n \mathcal{T}_{\frac{2}{3}} x_n = 3 \in \mathcal{M}$. However, $\lim_n \|\mathcal{S}\mathcal{T}x_n - \mathcal{T}\mathcal{S}x_n\| \neq 0$. Thus $\mathcal{S}$ and $\mathcal{T}$ are not subcompatible.

Note that $\mathcal{R}$-subweakly commuting and $\mathcal{R}$-subcommuting maps are subcompatible. The following simple example reveals that the converse is not true, in general.

Example 2.3. Let $\mathcal{X} = \mathbb{R}$ with usual norm and $\mathcal{M} = [0,\infty)$. Let $\mathcal{S}(x) = \frac{x}{2}$ if $0 \leq x < 1$ and $\mathcal{S}x = x$ if $x \geq 1$, and $\mathcal{T}(x) = \frac{1}{2}$ if $0 \leq x < 1$ and $\mathcal{T}x = x^2$ if $x \geq 1$. Then $\mathcal{M}$ is 1-starshaped with $\mathcal{S}1 = 1$ and $\bigwedge_q(\mathcal{S},\mathcal{T}) = \{\{x_n\} : 1 \leq x_n < \infty\}$. Note that $\mathcal{S}$ and $\mathcal{T}$ are subcompatible but not $\mathcal{R}$-weakly commuting for all $\mathcal{R} > 0$. Thus $\mathcal{S}$ and $\mathcal{T}$ are neither $\mathcal{R}$-subweakly commuting nor $\mathcal{R}$-subcommuting maps.

Definition 5. Let $\mathcal{M}$ be a subset of a complete linear metric space $\mathcal{X}$. Let $x_0 \in \mathcal{X}$. An element $y \in \mathcal{M}$ is called a best approximant to $x_0 \in \mathcal{X}$, if

$$F(x_0 - y) = dist(x_0, \mathcal{M}) = \inf\{F(x_0 - z) : z \in \mathcal{M}\}.$$

Let $\mathcal{P}_{\mathcal{M}}(x_0)$ be the set of best $\mathcal{M}$-approximants to x_0 and so

$$\mathcal{P}_{\mathcal{M}}(x_0) = \{z \in \mathcal{M} : F(x_0 - z) = dist(x_0, \mathcal{M})\}.$$

The following result would also be used in the sequel:

Theorem 2.4. ([7]) *Let $\mathcal{T}$ and $\mathcal{S}$ be compatible self-maps of $\mathcal{M}$, a closed convex subset of a Banach space $\mathcal{X}$, satisfying:*

$$\begin{aligned}\|\mathcal{T}x - \mathcal{T}y\| \leq a\|\mathcal{S}x - \mathcal{S}y\| + b\max\{\|\mathcal{T}x - \mathcal{S}x\|, \|\mathcal{T}y - \mathcal{S}y\|\} \\ +c\max\{\|\mathcal{S}x - \mathcal{S}y\|, \|\mathcal{T}x - \mathcal{S}x\|, \|\mathcal{T}y - \mathcal{S}y\|\},\end{aligned} \tag{2}$$

for $x, y \in \mathcal{M}$, where $a, b, c > 0$ and $a + b + c = 1$. If $\mathcal{S}$ is linear and continuous in $\mathcal{M}$ and $\mathcal{T}(\mathcal{M}) \subseteq \mathcal{S}(\mathcal{M})$, then $\mathcal{T}$ and $\mathcal{S}$ have a unique common fixed point in $\mathcal{M}$.

3. Common fixed point theorem in complete linear metric space

In this section, we prove more general result in common fixed point theory for subcompatible mappings in complete linear metric space:

Theorem 3.1. *Let $\mathcal{M}$ be a nonempty closed convex subset of a complete linear metric space $\mathcal{X}$. Suppose the pair $\{\mathcal{T}, \mathcal{S}\}$ of self-mappings is subcompatible of $\mathcal{M}$ such that $\mathcal{T}(\mathcal{M}) \subseteq \mathcal{S}(\mathcal{M})$, and $\mathcal{S}$ is linear with $q \in Fix(\mathcal{S})$. If $\mathcal{S}$ is continuous and $\mathcal{T}$ and $\mathcal{S}$ satisfy for all $x, y \in \mathcal{M}$*

$$\begin{aligned} F(\mathcal{T}x-\mathcal{T}y) \leq\, & aF(\mathcal{S}x-\mathcal{S}y)+b\max\{dist([\mathcal{T}x,q],\mathcal{S}x), dist([\mathcal{T}y,q],\mathcal{S}y)\} \\ & +c\max\{F(\mathcal{S}x-\mathcal{S}y), dist([\mathcal{T}x,q],\mathcal{S}x), dist([\mathcal{T}y,q],\mathcal{S}y)\}, \end{aligned} \tag{3}$$

where $a, b, c > 0$ and $a + b + c = 1$, then $Fix(\mathcal{T}) \cap Fix(\mathcal{S}) \neq \emptyset$, provided one of the following conditions holds:

(C1) *$cl\mathcal{T}(\mathcal{M})$ is compact and $\mathcal{T}$ is continuous;*
(C2) *$Fix(\mathcal{S})$ is bounded and $\mathcal{T}$ is a compact map;*
(C3) *$\mathcal{M}$ is bounded and complete, $\mathcal{T}$ is hemicompact and $\mathcal{T}$ is continuous;*
(C4) *$\mathcal{M}$ is weakly compact, $\mathcal{S}$ is weakly continuous and $\mathcal{I} - \mathcal{T}$ is demiclosed at 0, where $\mathcal{I}$ is identity map;*
(C5) *$\mathcal{M}$ is weakly compact and $\mathcal{T}$ is completely continuous.*

Proof. Choose a sequence $\{k_n\} \subset (0, 1)$ with $k_n \to 1$ as $n \to \infty$. Define $\mathcal{G}_n : \mathcal{M} \to \mathcal{M}$ by

$$\mathcal{G}_n x = k_n \mathcal{T}x + (1 - k_n)q$$

for some $q \in \mathcal{M}$ and for all $x \in \mathcal{M}$. Then for each n, $\mathcal{G}_n(\mathcal{M}) \subseteq \mathcal{S}(\mathcal{M})$ as $\mathcal{M}$ is convex, $\mathcal{S}$ is linear, $q \in Fix(\mathcal{S})$ and $\mathcal{T}(\mathcal{M}) \subseteq \mathcal{S}(\mathcal{M})$. The subcompatibility of the pair $\{\mathcal{S}, \mathcal{T}\}$ implies that

$$\begin{aligned} 0 &\leq \lim_m F(\mathcal{G}_n\mathcal{S}x_m - \mathcal{S}\mathcal{G}_n x_m) \\ &\leq \lim_m F(k_n(\mathcal{T}\mathcal{S}x_m - \mathcal{S}\mathcal{T}x_m)) + \lim_m F((1 - k_n)(q - \mathcal{S}q)) \\ &\leq \lim_m F(\mathcal{T}\mathcal{S}x_m - \mathcal{S}\mathcal{T}x_m) \\ &= 0, \end{aligned}$$

for any $\{x_m\} \subset \mathcal{M}$ with $\lim_m \mathcal{G}_n x_m = \lim_m \mathcal{S}x_m = t \in \mathcal{M}$.

Thus the pair $\{\mathcal{G}_n, \mathcal{S}\}$ is compatible for each n. Also

$$\begin{aligned} F(\mathcal{G}_n x - \mathcal{G}_n y) &= F(k_n(\mathcal{T}x - \mathcal{T}y)) \\ &\leq F(\mathcal{T}x - \mathcal{T}y) \\ &\leq aF(\mathcal{S}x - \mathcal{S}y) + b\max\{dist([\mathcal{T}x,q],\mathcal{S}x), dist([\mathcal{T}y,q],\mathcal{S}y)\} \\ &\quad +c\max\{F(\mathcal{S}x - \mathcal{S}y), dist([\mathcal{T}x,q],\mathcal{S}x), dist([\mathcal{T}y,q],\mathcal{S}y)\} \\ &\leq aF(\mathcal{S}x - \mathcal{S}y) + b\max\{F(\mathcal{G}_n x - \mathcal{S}x), F(\mathcal{G}_n y - \mathcal{S}y),\} \\ &\quad +c\max\{F(\mathcal{S}x - \mathcal{S}y), F(\mathcal{G}_n x - \mathcal{S}x), F(\mathcal{G}_n y - \mathcal{S}y)\}, \end{aligned}$$

for all $x, y \in \mathcal{M}$.

(C1) Since $\mathcal{T}(\mathcal{M})$ is compact, $\mathcal{G}_n(\mathcal{M})$ is also compact. By Theorem 2.4, for each $n \geq 1$, there exists $y_n \in \mathcal{M}$ such that $y_n = \mathcal{S}y_n = \mathcal{G}_n y_n$. The compactness of $\mathcal{T}(\mathcal{M})$ implies that there exists a subsequence $\{\mathcal{T}y_m\}$ of $\{\mathcal{T}y_n\}$ such that $\mathcal{T}y_m \to y$ as $m \to \infty$. Then the definition of $\mathcal{G}_m y_m$ implies $y_m \to y$, so by the continuity of $\mathcal{T}$ and $\mathcal{S}$ we have $y \in Fix(\mathcal{T}) \cap Fix(\mathcal{S})$. Thus $Fix(\mathcal{T}) \cap Fix(\mathcal{S}) \neq \emptyset$.

(C2) As in (C1), there is a unique $y_n \in \mathcal{M}$ such that $y_n = \mathcal{G}_n y_n = \mathcal{S}y_n$. As $\mathcal{T}$ is compact and $\{y_n\}$ being in $Fix(\mathcal{S})$ is bounded so $\{\mathcal{T}y_n\}$ has a subsequence $\{\mathcal{T}y_m\}$ such that $\{\mathcal{T}y_m\} \to y$ as $m \to \infty$. Then the definition of $\mathcal{G}_m y_m$ implies $y_m \to y$, so by the continuity of $\mathcal{T}$ and $\mathcal{S}$ we have $y \in Fix(\mathcal{T}) \cap Fix(\mathcal{S})$. Thus $Fix(\mathcal{T}) \cap Fix(\mathcal{S}) \neq \emptyset$.

(C3) As in (C1), there exists $y_n \in \mathcal{M}$ such that $y_n = \mathcal{S}y_n = \mathcal{G}_n y_n$. And $\mathcal{M}$ is bounded, so $y_n \to \mathcal{T}y_n = (1 - (k_n)^{-1})(y_n - q) \to 0$ as $n \to \infty$ and hence $F(y_n - \mathcal{T}y_n) \to 0$ as $n \to \infty$. The hemicompactness of $\mathcal{T}$ implies that $\{y_n\}$ has a subsequence $\{y_j\}$ which converges to some $z \in \mathcal{M}$. By the continuity of $\mathcal{T}$ and $\mathcal{S}$ we have $z \in Fix(\mathcal{T}) \cap Fix(\mathcal{S})$. Thus $Fix(\mathcal{T}) \cap Fix(\mathcal{S}) \neq \emptyset$.

(C4) As in (C1), there exists $y_n \in \mathcal{M}$ such that $y_n = \mathcal{S}y_n = \mathcal{G}_n y_n$. Since $\mathcal{M}$ is weakly compact, we can find a subsequence $\{y_m\}$ of $\{y_n\}$ in $\mathcal{M}$ converging weakly to $y \in \mathcal{M}$ as $m \to \infty$ and as $\mathcal{S}$ is weakly continuous so $\mathcal{S}y = y$. By (C3), $\mathcal{I}y_m - \mathcal{T}y_m \to 0$ as $m \to \infty$. The demiclosedness of $\mathcal{I} - \mathcal{T}$ at 0 implies that $\mathcal{S}y = \mathcal{T}y$. Thus $Fix(\mathcal{T}) \cap Fix(\mathcal{S}) \neq \emptyset$.

(C5) As in (C4), we can find a subsequence $\{y_m\}$ of $\{y_n\}$ in $\mathcal{M}$ converging weakly to $y \to \mathcal{M}$ as $m \to \infty$. Since $\mathcal{T}$ is completely continuous, $\mathcal{T}y_m \to \mathcal{T}y$ as $m \to \infty$. Since $k_n \to 1$, $y_m = \mathcal{G}_m y_m = k_m \mathcal{T}y_m + (1 - k_m)q \to \mathcal{T}y$ as $m \to \infty$. Thus $\mathcal{T}y_m \to \mathcal{T}^2 y$ as $m \to \infty$ and consequently $\mathcal{T}^2 y = \mathcal{T}y$ implies that $\mathcal{T}w = w$, where $w = \mathcal{T}y$. Also, since $\mathcal{S}y_m = y_m \to \mathcal{T}y = w$, using the continuity of $\mathcal{I}$ and the uniqueness of the limit, we have $\mathcal{S}w = w$. Hence $Fix(\mathcal{T}) \cap Fix(\mathcal{S}) \neq \emptyset$.

□

An immediately consequence from Theorem 3.1 as,

Corollary 3.2. *Let $\mathcal{M}$ be a nonempty closed convex subset of a complete metrizable topological convex metric space $\mathcal{X}$. Suppose the pair $\{\mathcal{T}, \mathcal{S}\}$ of self-mappings is subcompatible of $\mathcal{M}$ such that $\mathcal{T}(\mathcal{M}) \subseteq \mathcal{S}(\mathcal{M})$, and $\mathcal{S}$ is linear with $q \in Fix(\mathcal{S})$. If $\mathcal{S}$ is continuous and $\mathcal{T}$ and $\mathcal{S}$ satisfy (3) for all $x, y \in \mathcal{M}$, where $a, b, c > 0$ and $a + b + c = 1$, then $Fix(\mathcal{T}) \cap Fix(\mathcal{S}) \neq \emptyset$, under each condition of Theorem 3.1.*

Corollary 3.3. *Let $\mathcal{M}$ be a nonempty closed convex subset of a Banach space $\mathcal{X}$. Suppose the pair $\{\mathcal{T}, \mathcal{S}\}$ of self-mappings is subcompatible of $\mathcal{M}$ such that $\mathcal{T}(\mathcal{M}) \subset \mathcal{S}(\mathcal{M})$, and $\mathcal{S}$ is linear with $q \in Fix(\mathcal{S})$. If $\mathcal{S}$ is continuous and $\mathcal{T}$ and $\mathcal{S}$ satisfy (3) for all $x, y \in \mathcal{M}$, where $a, b, c > 0$ and $a + b + c = 1$, then $Fix(\mathcal{T}) \cap Fix(\mathcal{S}) \neq \emptyset$, under each condition of Theorem 3.1.*

4. Invariant approximation in complete linear metric space

In this section, we establish more general results in invariant approximations theory with the aid of more general class of noncommuting mappings, known as, subcompatible mappings, as application of Theorem 3.1.

Theorem 4.1. *Let $\mathcal{X}$ be a complete linear metric space and $\mathcal{T}, \mathcal{S} : \mathcal{X} \to \mathcal{X}$. Let $\mathcal{M}$ be subset of $\mathcal{X}$ such that $\mathcal{T}(\partial\mathcal{M}) \subseteq \mathcal{M}$ and $x_0 \in Fix(\mathcal{T}) \cap Fix(\mathcal{S})$. Suppose $\mathcal{S}$ is linear on $\mathcal{P}_{\mathcal{M}}(x_0)$, $q \in Fix(\mathcal{S})$, $\mathcal{P}_{\mathcal{M}}(x_0)$ is nonempty closed and convex, and $\mathcal{S}(\mathcal{P}_{\mathcal{M}}(x_0)) = \mathcal{P}_{\mathcal{M}}(x_0)$. If $\mathcal{S}$ is continuous, the pair $\{\mathcal{T}, \mathcal{S}\}$ is subcompatible and satisfy (3) for all $x, y \in \mathcal{P}_{\mathcal{M}}(x_0)$ where $a, b, c > 0$ and $a+b+c = 1$. Suppose $F(\mathcal{T}x - x_0) \leq F(\mathcal{S}x - x_0)$ for all $x \in \mathcal{M}$, then $\mathcal{P}_{\mathcal{M}}(x_0) \cap Fix(\mathcal{T}) \cap Fix(\mathcal{S}) \neq \emptyset$, provided one of the following conditions holds:*

(BA1) *$cl\mathcal{T}(\mathcal{M})$ is compact and $\mathcal{T}$ is continuous;*
(BA2) *$Fix(\mathcal{S})$ is bounded and $\mathcal{T}$ is a compact map;*
(BA3) *$\mathcal{P}_{\mathcal{M}}(x_0)$ is bounded and complete, $\mathcal{T}$ is hemicompact and $\mathcal{T}$ is continuous;*
(BA4) *$\mathcal{P}_{\mathcal{M}}(x_0)$ is weakly compact, $\mathcal{S}$ is weakly continuous and $\mathcal{I} - \mathcal{T}$ is demiclosed at 0, where $\mathcal{I}$ is identity map;*
(BA5) *$\mathcal{P}_{\mathcal{M}}(x_0)$ is weakly compact and $\mathcal{T}$ is completely continuous.*

Proof. Let $y \in \mathcal{P}_{\mathcal{M}}(x_0)$. Then $y \in \partial\mathcal{M}$ and so $\mathcal{T}y \in \mathcal{M}$, because $\mathcal{T}(\partial\mathcal{M}) \subseteq \mathcal{M}$. Now since $\mathcal{T}x_0 = x_0 = \mathcal{S}x_0$, we have

$$F(\mathcal{T}y - x_0) \leq F(\mathcal{S}y - x_0) = dist(x_0, \mathcal{M}).$$

This shows that $\mathcal{T}y \in \mathcal{P}_{\mathcal{M}}(x_0)$. Consequently, $\mathcal{T}(\mathcal{P}_{\mathcal{M}}(x_0)) \subseteq \mathcal{P}_{\mathcal{M}}(x_0) = \mathcal{S}(\mathcal{P}_{\mathcal{M}}(x_0))$. Now Theorem 3.1 guarantees that $\mathcal{P}_{\mathcal{M}}(x_0) \cap Fix(\mathcal{T}) \cap Fix(\mathcal{S}) \neq \emptyset$. □

Define $\mathcal{C}^{\mathcal{S}}_{\mathcal{M}}(x_0) = \{x \in \mathcal{M} : \mathcal{S}x \in \mathcal{P}_{\mathcal{M}}(x_0)\}$ and $\mathcal{D}^* = \mathcal{P}_{\mathcal{M}}(x_0) \cap \mathcal{C}^{\mathcal{S}}_{\mathcal{M}}(x_0)$.

Theorem 4.2. *Let $\mathcal{X}$ be a complete linear metric space and $\mathcal{T}, \mathcal{S} : \mathcal{X} \to \mathcal{X}$. Let $\mathcal{M}$ be subset of $\mathcal{X}$ such that $\mathcal{T}(\partial\mathcal{M}) \subseteq \mathcal{M}$ and $x_0 \in Fix(\mathcal{T}) \cap Fix(\mathcal{S})$. Suppose $\mathcal{S}$ is linear on $\mathcal{D}^*$, $q \in Fix(\mathcal{S})$, $\mathcal{D}^*$ is nonempty closed and convex, $\mathcal{S}(\mathcal{D}^*) = \mathcal{D}^*$. If $\mathcal{S}$ is nonexpansive, the pair $\{\mathcal{T}, \mathcal{S}\}$ is subcompatible and satisfy (3) for all $x, y \in \mathcal{D}^*$ where $a, b, c > 0$ and $a+b+c = 1$. Suppose $F(\mathcal{T}x - x_0) \leq F(\mathcal{S}x - x_0)$ for all $x \in \mathcal{M}$, then $\mathcal{P}_{\mathcal{M}}(x_0) \cap Fix(\mathcal{T}) \cap Fix(\mathcal{S}) \neq \emptyset$, under each condition of Theorem 4.1.*

Proof. Let $y \in \mathcal{D}^*$, then $\mathcal{S}y \in \mathcal{D}^*$, since $\mathcal{S}(\mathcal{D}^*) = \mathcal{D}^*$. By the definition of $\mathcal{D}^*$, $y \in \partial\mathcal{M}$. Also $\mathcal{T}y \in \mathcal{M}$, since $\mathcal{T}(\partial\mathcal{M}) \subseteq \mathcal{M}$. Now since $\mathcal{T}x_0 = x_0 = \mathcal{S}x_0$,

$$F(\mathcal{T}y - x_0) \leq F(\mathcal{S}y - x_0) = dist(x_0, \mathcal{M}),$$

since $\mathcal{S}y \in \mathcal{P}_{\mathcal{M}}(x_0)$. This implies that $\mathcal{T}y$ is also closest to x_0, so $\mathcal{T}y \in \mathcal{P}_{\mathcal{M}}(x_0)$. As $\mathcal{S}$ is nonexpansive on $\mathcal{P}_{\mathcal{M}}(x_0) \cup \{x_0\}$,

$$F(\mathcal{S}\mathcal{T}y - x_0) = F(\mathcal{S}\mathcal{T}y - \mathcal{S}x_0) \leq F(\mathcal{T}y - x_0) \leq F(\mathcal{S}y - \mathcal{S}x_0) = F(\mathcal{S}y - x_0).$$

Thus, $\mathcal{S}\mathcal{T}y \in \mathcal{P}_{\mathcal{M}}(x_0)$. This implies that $\mathcal{T}y \in \mathcal{C}^{\mathcal{S}}_{\mathcal{M}}(x_0)$ and hence $\mathcal{T}y \in \mathcal{D}^*$. So $\mathcal{T}$ and $\mathcal{S}$ are selfmaps on $\mathcal{D}^*$. Hence, all the condition of the Theorem 3.1 are satisfied. Thus, there exists $z \in \mathcal{P}_{\mathcal{M}}(x_0)$ such that $z = \mathcal{S}z = \mathcal{T}z$. □

Theorem 4.3. *Let $\mathcal{X}$ be a complete linear metric space and $\mathcal{T}, \mathcal{S} : \mathcal{X} \to \mathcal{X}$. Let $\mathcal{M}$ be subset of $\mathcal{X}$ such that $\mathcal{T}(\partial\mathcal{M} \cap \mathcal{M}) \subseteq \mathcal{M}$ and $x_0 \in Fix(\mathcal{T}) \cap Fix(\mathcal{S})$. Suppose $\mathcal{S}$ is linear on $\mathcal{D}^*$, $q \in Fix(\mathcal{S})$, $\mathcal{D}^*$ is nonempty closed and convex, $\mathcal{S}(\mathcal{D}^*) = \mathcal{D}^*$. If $\mathcal{S}$ is nonexpansive, the pair $\{\mathcal{T}, \mathcal{S}\}$ is subcompatible and satisfy (3) for all $x, y \in \mathcal{D}^*$ where $a, b, c > 0$ and $a + b + c = 1$. Suppose $F(\mathcal{T}x - x_0) \leq F(\mathcal{S}x - x_0)$ for all $x \in \mathcal{M}$, then $\mathcal{P}_{\mathcal{M}}(x_0) \cap Fix(\mathcal{T}) \cap Fix(\mathcal{S}) \neq \emptyset$, under each condition of Theorem 4.1.*

Proof. Let $x \in \mathcal{D}^*$. Then, $x \in \mathcal{P}_{\mathcal{M}}(x_0)$ and hence $F(x - x_0) = dist(x_0, \mathcal{M})$. Note that for any $k \in (0, 1)$,

$$F(kx_0 + (1-k)x - x_0) = F((1-k)(x - x_0)) < dist(x_0, \mathcal{M}).$$

It follows that the line segment $\{kx_0 + (1-k)x : 0 < k < 1\}$ and the set $\mathcal{M}$ are disjoint. Thus x is not in the interior of $\mathcal{M}$ and so $x \in \partial\mathcal{M} \cap \mathcal{M}$. Since $\mathcal{T}(\partial\mathcal{M} \cap \mathcal{M}) \subset \mathcal{M}$, $\mathcal{T}x$ must be in $\mathcal{M}$. Along with the lines of the proof of Theorem 4.2, we have the result. □

Remark 1. It is observed that $\mathcal{S}(\mathcal{P}_{\mathcal{M}}(x_0)) \subset \mathcal{P}_{\mathcal{M}}(x_0)$ implies $\mathcal{P}_{\mathcal{M}}(x_0) \subset \mathcal{D}^*$ and hence $\mathcal{D}^* = \mathcal{P}_{\mathcal{M}}(x_0)$. Consequently, Theorem 4.2, 4.3 remain valid when $\mathcal{D}^* = \mathcal{P}_{\mathcal{M}}(x_0)$.

Remark 2. Theorem 3.1 - Theorem 4.3 generalize the results of Jungck and Hussain [8, Theorem 2.3 - Threom 2.5] in the sense that the more generalized noncommuting mappings, that is, subcompatible mappings have been used in place of compatible mappings.

Remark 3. Theorem 3.1 contains [2, Theorem 2.2] and [4, Theorem 1].

Remark 4. Theorem 4.1 - Theorem 4.3 contain Theorem 3.2 of Al - Thagafi [2], Theorem 3 of Sahab et al. [13] and Singh [14, 15, 16] in the sense that the more generalized noncommuting mappings(subcommuting mappings) and generalized relatively nonexpansive maps have been used in place of relatively nonexpansive commuting maps.

Recall that $\Im_0$ denotes the class of closed convex subset of $\mathcal{X}$ containing 0. For $\mathcal{M} \in \Im_0$, we define $\mathcal{M}_{x_0} = \{x \in \mathcal{M} : F(x) \leq 2F(x_0)\}$. It is clear $\mathcal{P}_{\mathcal{M}}(x_0) \subset \Im_0$.

Theorem 4.4. *Let $\mathcal{X}$ be a complete linear metric space and $\mathcal{T}, \mathcal{S} : \mathcal{X} \to \mathcal{X}$ with $x_0 \in Fix(\mathcal{T}) \cap Fix(\mathcal{S})$ and $\mathcal{M} \in \Im_0$ such that $\mathcal{T}(\mathcal{M}_{x_0}) \subset \mathcal{S}(\mathcal{M}) \subset \mathcal{M}$. Suppose $F(\mathcal{T}x - x_0) \leq F(\mathcal{S}x - x_0)$, $F(\mathcal{S}x - x_0) \leq F(x - x_0)$ for all $x \in \mathcal{M}$, the pair $\{\mathcal{T}, \mathcal{S}\}$ is continuous on $\mathcal{M}$ and one of the following two conditions is satisfied:*

(a) *$cl\mathcal{S}(\mathcal{M})$ is compact,*
(b) *$cl\mathcal{T}(\mathcal{M})$ is compact.*

Then

(i) *$\mathcal{P}_{\mathcal{M}}(x_0)$ is nonempty, closed and convex;*

(ii) $\mathcal{T}(\mathcal{P}_{\mathcal{M}}(x_0)) \subset \mathcal{S}(\mathcal{P}_{\mathcal{M}}(x_0)) \subset \mathcal{P}_{\mathcal{M}}(x_0)$ *provided* $F(\mathcal{S}x - x_0) \leq F(x - x_0)$ *for all* $x \in \mathcal{C}^{\mathcal{S}}_{\mathcal{M}}(x_0)$;

(iii) $\mathcal{P}_{\mathcal{M}}(x_0) \cap Fix(\mathcal{T}) \cap Fix(\mathcal{S}) \neq \emptyset$ *provided* $F(\mathcal{S}x - x_0) \leq F(x - x_0)$ *for all* $x \in \mathcal{C}^{\mathcal{S}}_{\mathcal{M}}(x_0)$, $\mathcal{S}$ *is linear on* $\mathcal{P}_{\mathcal{M}}(x_0)$, $\mathcal{P}_{\mathcal{M}}(x_0)$ *is closed and convex,* $\mathcal{S}(\mathcal{P}_{\mathcal{M}}(x_0)) = \mathcal{P}_{\mathcal{M}}(x_0)$, *the pair* $\{\mathcal{T}, \mathcal{S}\}$ *is subcompatible on* $\mathcal{P}_{\mathcal{M}}(x_0)$ *and satisfies (3) for all* $x, y \in \mathcal{P}_{\mathcal{M}}(x_0)$ *where* $a, b, c > 0$ *and* $a + b + c = 1$ *and* $q \in Fix(\mathcal{S})$.

Proof. (i) Let $r = dist(x_0, \mathcal{M})$. Then there is a minimizing sequence $\{y_n\}$ in $\mathcal{M}$ such that $\lim_m F(x_0 - y_n) = r$. As $cl\mathcal{S}(\mathcal{M})$ is compact, so $\{\mathcal{S}y_n\}$ has a convergent subsequence $\{\mathcal{S}y_m\}$ with $\mathcal{S}y_m = u$ (say) in $\mathcal{M}$. Now, by using $F(\mathcal{S}x - x_0) \leq F(x - x_0)$, we get

$$r \leq F(u - x_0) = \lim_m F(\mathcal{S}y_m - x_0) \leq \lim_m F(y_m - x_0) = \lim_m F(y_n - x_0) = r.$$

Hence $u \in \mathcal{P}_{\mathcal{M}}(x_0)$. Thus $\mathcal{P}_{\mathcal{M}}(x_0)$ is nonempty, closed and convex. Similarly, when $cl\mathcal{T}(\mathcal{M})$ is compact, we get the same conclusion by using inequalities $F(\mathcal{T}x - x_0) \leq F(\mathcal{S}x - x_0)$, $F(\mathcal{S}x - x_0) \leq F(x - x_0)$ for all $x \in \mathcal{M}$.

(ii) Let $y \in \mathcal{P}_{\mathcal{M}}(x_0)$. Then

$$F(\mathcal{T}x - x_0) \leq F(\mathcal{S}x - x_0) = dist(x_0, \mathcal{M}).$$

This implies that $\mathcal{T}y \in \mathcal{P}_{\mathcal{M}}(x_0)$ and so $\mathcal{T}(\mathcal{P}_{\mathcal{M}}(x_0)) \subset \mathcal{P}_{\mathcal{M}}(x_0)$. Also we have $\mathcal{S}(\mathcal{P}_{\mathcal{M}}(x_0)) \subset \mathcal{P}_{\mathcal{M}}(x_0)$. Let $y \in \mathcal{T}(\mathcal{P}_{\mathcal{M}}(x_0))$. Since $\mathcal{T}(\mathcal{M}_{x_0}) \subset \mathcal{S}(\mathcal{M})$ and $\mathcal{P}_{\mathcal{M}}(x_0) \subset \mathcal{M}_{x_0}$, there exist $y \in \mathcal{P}_{\mathcal{M}}(x_0)$ and $x \in \mathcal{M}$ such that $y = \mathcal{T}z = \mathcal{S}x$. Thus, we have

$$F(\mathcal{S}x - x_0) = F(\mathcal{T}z - x_0) \leq F(\mathcal{S}y - x_0) \leq F(y - x_0) = dist(x_0, \mathcal{M}).$$

Hence, $x \in \mathcal{C}^{\mathcal{S}}_{\mathcal{M}}(x_0) = \mathcal{P}_{\mathcal{M}}(x_0)$ and so (ii) holds.

(iii) (a) By (i), $\mathcal{P}_{\mathcal{M}}(x_0)$ is closed and (ii) and $\mathcal{P}_{\mathcal{M}}(x_0)$ is $\mathcal{S}$ and $\mathcal{T}$-invariant. Further, $\mathcal{P}_{\mathcal{M}}(x_0) \cap Fix(\mathcal{S}) \neq \emptyset$ implies that there exists $q \in \mathcal{P}_{\mathcal{M}}(x_0)$ such that $q \in Fix(\mathcal{S})$. By (ii), compactness of $cl\mathcal{S}(\mathcal{M})$ implies that $cl\mathcal{T}(\mathcal{P}_{\mathcal{M}}(x_0))$ is compact. The conclusion now follows from Theorem 3.1(i) applied to $\mathcal{P}_{\mathcal{M}}(x_0)$.

(b) By (ii), the compactness of $cl\mathcal{T}(\mathcal{M})$ implies that $cl\mathcal{T}(\mathcal{P}_{\mathcal{M}}(x_0))$ is compact. Theorem 3.1(i) further guarantees that $\mathcal{P}_{\mathcal{M}}(x_0) \cap Fix(\mathcal{T}) \cap Fix(\mathcal{S}) \neq \emptyset$. □

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