

# Stochastic Scheduling Problems for Maximizing the Expected Number of Early Jobs with Common or Exchangeable Due Dates

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## ABSTRACT

In this paper, stochastic scheduling problems are considered when processing times and due dates follow arbitrary distributions and due dates are either common or exchangeable. For maximizing the expected number of early jobs, two policies, one, based on pairwise comparisons of the processing times, and the other, based on survivabilities, are introduced. In addition, it is shown that the former guarantees optimal solutions when the processing times and due dates are deterministic and that the latter guarantees optimal solutions when the due dates follow exponential distributions. Then a new approach, exploiting the two policies, is proposed and analyzed which turns out to give better job sequences in many situations. In fact, the new approach guarantees optimal solutions both when the processing times and due dates are deterministic and when the due dates follow exponential distributions.

Keywords: Stochastic Scheduling Problem, Early Job, Common Due Date, Exchangeable Due Dates

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## 1. INTRODUCTION

The problem of maximizing the expected number of early jobs with a common due date<sup>1)</sup> is usually applied to the situation where random breakdown of a machine is considered with the time of breakdown being the common due date. The number of job completions should be maximized before the machine breaks down.

In addition, the problem of maximizing the expected number of early jobs with exchangeable due dates<sup>2)</sup> frequently appears in agriculture, especially when perishable goods are involved (Sterna, 2006). For instance, crops

being collected from different stretches of land should be harvested. Crops perish if not collected prior to exchangeable due dates, determined by the mixture of land attributes and nature. Therefore maximizing the expected number of early jobs are the same as minimizing the amount of wasted crops.

The problems of maximizing the expected number of early jobs with a common due date or exchangeable due dates have been investigated in many literatures (Choi and Kim, 2011).

For instance, it is assumed that the processing times follow normal distributions with variances proportional to their respective processing time means and the common due date is deterministic. It is found that the optimal static policies are V or W-shaped where the shapes are based on their respective processing time means (Sarin *et al.*, 1991). Similar problems without assuming

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1) The same due date is assigned to all the jobs remaining to be processed.

2) The same distribution of due date is assigned to all the jobs remaining to be processed.

proportional variances are studied, where ‘normalized slack policy’ is introduced which is optimal for both static<sup>3)</sup> and dynamic<sup>4)</sup> scheduling in some cases (Jang, 2002).

Problems with arbitrary processing times and an exponential due date are considered. It is shown that if processing times are NBU (new-better-than-used),<sup>5)</sup> the optimal static policy is preserved in the optimal dynamic policy as well (De *et al.*, 1991). Similar problems with a more general objective function which includes earliness penalties are considered (Cai *et al.*, 1999). In addition, stochastic scheduling problems with general distributional assumptions on processing times are investigated (Li *et al.*, 1998).

In this paper, the problem of maximizing the expected number of early jobs through a single machine is considered. There are  $n$  jobs to be processed with processing times  $X_i, i=1, 2, \dots, n$ . It is assumed that jobs are not resumed once the due date occurs and that all the remaining unprocessed jobs cannot be processed thereafter. The due date for each job  $i, D_i, i=1, 2, \dots, n$ , is common ( $=D$ ) or exchangeable. Both processing times and due dates are assumed to follow arbitrary distributions. This paper aims to find a static policy of getting a sequence  $\pi(j_1, j_2, \dots, j_n)$  of  $n$  jobs in such a way that the expected number of early jobs  $E[\pi]$  is maximized.

The rest of this paper is organized as follows. Two policies are introduced in Section 2. One of them is based on pairwise comparisons of processing times, and the other is based on survivabilities. Then, in Section 3, a new policy, exploiting the two policies, is derived and analyzed, which gives better job sequences than the two policies in many situations. In conclusion, the proposed policy turns out to capture advantages of both policies, which is summarized in Table 1. Final remarks are addressed in Section 4.

Table 1. Comparison

Processing Times	Due Date	Pairwise Rank Based Policy	Probabilistic Slack Policy	Proposed Policy
Any	Exponential	Not optimal	Optimal	Optimal
Exponential	Any	Optimal	Optimal	Optimal
Deterministic	Deterministic	Optimal	Not optimal	Optimal
All distributions are non-overlapping		Optimal	Not optimal	Optimal

- 3) A set of jobs are scheduled before any one of them is processed, and once the first job is being processed, the sequence of the remaining jobs is never revisited until all the jobs are finished.
- 4) The order of jobs to be processed can be revisited at any time, utilizing all the information available up to that point in time.
- 5) The probability that a new unit will survive time  $t$  is greater than or equal to the probability that a used unit of age  $S$  will survive an additional time  $t$ .

## 2. ALGORITHMS

In this section, ‘pairwise rank based policy’ is introduced which is obtained from a sample path analysis. It is shown that the policy is optimal when the processing times are ordered in failure rate sense. Then ‘probabilistic slack policy’ is introduced as well. It is shown that the policy is optimal when due date is exponential. Finally, a new policy is proposed to exploit the two policies.

### 2.1 Pairwise Rank Based Policy

#### 2.1.1 Stochastic Ordering

The probability that  $t$  is between the processing time of job  $i, X_i$ , and the sum of processing times of jobs  $i$  and  $j, X_i + X_j$ , i.e.,  $g_i(i, j) = \Pr\{X_i \leq t < X_i + X_j\}$ , upon which optimal policy is based, is introduced. It is shown that if  $g_i(i, j) \geq g_i(j, i)$  holds for all  $t \geq 0$ , then sequence  $\pi(\pi_1, i, j, \pi_2)$  is better than sequence  $\pi'(\pi_1, j, i, \pi_2)$ , where the two adjacent jobs  $i$  and  $j$  in sequence  $\pi(\pi_1, i, j, \pi_2)$  are interchanged (Li *et al.*, 1998). Even though it seems that this sufficient condition is quite general, it turns out that the condition leads to a stochastic ordering. That is, if we denote the distribution function and probability density function of  $X_i$  by  $F_i$  and  $f_i$ , respectively, and  $\bar{F}_i = 1 - F_i$ , then

$$\begin{aligned}
 & \Pr\{X_i \leq t < X_i + X_j\} \\
 &= \int_0^\infty \Pr\{X_i \leq t < X_i + x \mid X_j = x\} dF_j(x) \\
 &= \int_0^\infty F_i(t) \bar{F}_i(t-x) dF_j(x) \\
 &= \int_0^t F_i(t) \bar{F}_i(t-x) dF_j(x) + \int_t^\infty F_i(t) dF_j(x) \\
 &= F_i(t) \left[ \int_0^t \bar{F}_i(t-x) dF_j(x) + \bar{F}_j(t) \right] \\
 &= \Pr\{X_i \leq t\} \Pr\{X_i + X_j > t\}
 \end{aligned}$$

Therefore, the condition  $g_i(i, j) \geq g_i(j, i)$ , for all  $t \geq 0$ , leads to the following:

$$\begin{aligned}
 & \Pr\{X_i \leq t < X_i + X_j\} \geq \Pr\{X_j \leq t < X_i + X_j\} \\
 \Leftrightarrow & F_i(t) \Pr\{X_i + X_j > t\} \geq F_j(t) \Pr\{X_i + X_j > t\} \\
 \Leftrightarrow & F_i(t) \geq F_j(t)
 \end{aligned}$$

The last inequality is just the stochastic ordering relationship  $X_i \leq_{st} X_j$  (Ross, 1996).

#### 2.1.2 More Likely Ordering

Even though the assumption that processing times are ordered in a stochastic ordering sense is useful, it is not likely to be satisfied. In other words, the data in practice do not come in a stochastic ordering sense. Consequently, a scheduling procedure based on such condi-

tions often leads to limited partial sequencing. Therefore, weaker ordering conditions are needed. The following lemma does not require the assumption of stochastic ordering. Thus, it can be used for any set of processing times that cannot be ordered in a stochastic ordering sense.

**Lemma 1**

Consider any two adjacent jobs  $i$  and  $j$  with any common due date. If  $\Pr\{X_i \leq X_j\} \geq \Pr\{X_i > X_j\}$ , then job  $i$  should precede job  $j$  in optimal sequence.

**Proof**

Consider a realization  $\varpi \in \Omega$  of processing times, and denote the realization of  $X_i$  by  $X_i(\varpi)$ . If  $X_i(\varpi) < X_j(\varpi)$ , then it is better to process job  $i$  first for any realization of due date. Otherwise, it is better to process job  $j$  first. Therefore, in this case, with probability  $\Pr\{X_i \leq X_j\}$ , it is better to process job  $i$  first for whatever the realization of due date is.

Denote a case where  $X_i$  is more likely smaller than  $X_j$ , i.e.,  $\Pr\{X_i \leq X_j\} \geq \Pr\{X_i > X_j\}$ , by  $X_i \prec X_j$ . In general, the relation  $X_i \prec X_j$  is not transitive. In other words, the conjunctive condition that  $X_i \prec X_j$  and  $X_j \prec X_k$  does not always imply  $X_i \prec X_k$ . For example, the following three discrete random variables  $X$ ,  $Y$  and  $Z$  are not transitive

$$\begin{cases} \Pr\{X = 2\} = 1/3 \\ \Pr\{X = 5\} = 2/3 \end{cases} \quad \begin{cases} \Pr\{Y = 3\} = 2/3 \\ \Pr\{Y = 6\} = 1/3 \end{cases} \quad Z = 4$$

since  $X \prec Y$  and  $Y \prec Z$ , but  $Z \prec X$ .

However, the transitivity does hold for exponential random variables. Let  $X_i, X_j$  and  $X_k$  be exponential random variables with respective rates,  $\lambda_i, \lambda_j$  and  $\lambda_k$ . Since  $\Pr\{X_i \leq X_j\} = \lambda_i / (\lambda_i + \lambda_j)$ , the conjunctive condition that  $X_i \prec X_j$  and  $X_j \prec X_k$ , implies that  $\lambda_i \geq \lambda_j$  and  $\lambda_j \geq \lambda_k$  which, in turn, leads to  $\lambda_i \geq \lambda_k$ . Thus we have  $X_i \prec X_k$ .

2.1.3 Failure Rate Ordering

A random variable  $X_i$  is said to be stochastically smaller than a random variable  $X_j$  in a failure rate sense if

$$\frac{f_i(t)}{\bar{F}_i(t)} \geq \frac{f_j(t)}{\bar{F}_j(t)} \quad \text{for all } t.$$

This ordering is denoted by  $X_i \leq_f X_j$ . The following lemma shows that this failure rate ordering implies more likely ordering.

**Lemma 2**

If  $X_i \leq_f X_j$ , then  $X_i \prec X_j$ .

**Proof**

$$\begin{aligned} & X_i \leq_f X_j \\ \Rightarrow & \frac{f_i(t)}{\bar{F}_i(t)} \geq \frac{f_j(t)}{\bar{F}_j(t)} \quad \text{for all } t \\ \Rightarrow & \int_0^\infty f_i(t) \bar{F}_j(t) dt \geq \int_0^\infty f_j(t) \bar{F}_i(t) dt \\ \Rightarrow & \Pr\{X_i \leq X_j\} \geq \Pr\{X_i > X_j\} \end{aligned}$$

The transitivity holds for the random variables that can be stochastically ordered in a failure rate sense by extending the above Lemma to the following corollary.

**Corollary 1**

Suppose  $X_i \leq_f X_j \leq_f X_k$ . Then  $X_i \prec X_k$ .

As a consequence of Lemma 1 and Lemma 2, for any non-negative due date, the optimal sequence is easily obtained if the processing times are stochastically ordered in a failure rate sense.

**Theorem 1**

If the processing times,  $X_i, i=1, \dots, n$ , are stochastically ordered in a failure rate sense in such a way that  $X_1 \leq_f X_2 \leq_f \dots \leq_f X_n$ , then the optimal policy processes jobs in an increasing order of index for any common due date.

Lemma 1 can be extended to problems with multiple common due date. Suppose there are multiple classes of jobs. All the jobs in the same class share their own common due date. Then partial sequence for each class of jobs can be obtained. These partial sequences are exactly the same as those obtained by applying pairwise rank based policy. Any two jobs in the same class are called class-wise adjacent if there are no other jobs in between.

**Lemma 3**

Consider any two class-wise adjacent jobs  $i$  and  $j$ . If  $X_i \prec X_j$ , then job  $i$  should precede job  $j$  in an optimal sequence.

**Proof**

Consider a realization  $\varpi \in \Omega$  of processing times. We denote the realization of  $X_i$  by  $X_i(\varpi)$ . Also, let  $S$  be the total processing time of any jobs between jobs  $i$  and  $j$ ,  $S(\varpi)$  be its realization, and  $D$  be the common due date for jobs  $i$  and  $j$ . If  $X_i(\varpi) < X_j(\varpi)$ , it is better to process job  $i$  first for any realization of due date  $D(\varpi)$ .

2.1.4 Pairwise Rank Based Policy

In many cases, stochastic ordering generates partial sequences only. This is from the fact that stochastic ordering requires a strong condition to be satisfied. The more likely ordering can generate partial sequences as well because the transitivity does not hold for some sets

of random variables. However, using the relation  $X_i \prec X_j$ , each and every job can be compared to every other, thus it can be ranked based on how many jobs should precede it via Lemma 1 or Lemma 3. The rank of job  $i$  is defined as the number of jobs that should precede job  $i$ . This policy of processing jobs in increasing order of rank is called ‘pairwise rank based policy.’

## 2.2 Probabilistic Slack Policy

In the previous subsection, the more likely smaller relation of any two random variables,  $X_i \prec X_j$ , was introduced. One drawback of the relation is that it is not transitive for some random variables. Thus pairwise rank based policy is merely a heuristic. In this subsection, the probability that the processing time of job  $i$  is less than or equal to the due date, i.e.,  $\Pr\{X_i \leq D\}$ , will be discussed. The probability  $\Pr\{X_i \leq D\}$  is called the probabilistic slack for job  $i$ . Intuition says that a sequence in which all the jobs are ordered in decreasing order of probabilistic slack might be optimal. This policy of sequencing jobs in decreasing order of probabilistic slack is called ‘probabilistic slack policy.’

When probabilistic slacks tie, the shortest processing time rule gives optimal sequence that may not be obtained by probabilistic slack policy. The ties in probabilistic slacks occur, for example, when the processing times and common due date are deterministic. This is because  $\Pr\{X_i \leq D\} = 1$  for the jobs whose processing times are smaller than the due date.

A policy, called normalized slack policy, and the concept of myopic optimality are introduced (Jang, 2002). Job  $i$  is said to be myopically better than job  $j$  if it is better to process job  $i$  first between the two. Job  $i$  is said to be myopically optimal if no other jobs are myopically better than job  $i$ .

Let  $E[\pi]$  be the expected number of early jobs for sequence  $\pi$ . By definition, job  $i$  is myopically better than job  $j$  if  $E[\pi(i, j, \dots)] > E[\pi(j, i, \dots)]$ . This condition leads to  $\Pr\{X_i \leq D\} > \Pr\{X_j \leq D\}$ . Since probabilistic slack policy sequences jobs in decreasing order of probabilistic slack, the following lemma is obvious.

### Lemma 4

Probabilistic slack policy is myopically optimal for any processing times and common due date if there are no ties in probabilistic slacks.

Problems in which the processing times follow normal distributions and the common due date is deterministic are considered. The normalized slack of job  $i$  is defined as

$$\frac{d - \mu_i}{\sigma_i},$$

where  $d$  is the deterministic common due date. Normalized slack policy sequences jobs in decreasing order of

normalized slack. This policy can be considered as a special case of probabilistic slack policy since normalized slack policy sequences jobs in non-decreasing order of normalized slack.

Problems in which the processing times are arbitrary and the common due date follows an exponential distribution are considered (Boxma and Forst, 1986; Cai and Zhou, 1999; and De *et al.*, 1991). It is shown to be optimal to process jobs in non-increasing order of  $E[e^{-\delta X_i}]$ , where  $\delta$  is the rate of the exponential random variable  $D$  which the common due date follows. Since the Laplace-Stieltjes transform of  $X_i$  at  $\delta$ , i.e.,  $E[e^{-\delta X_i}]$ , is  $\Pr\{X_i \leq D\}$ , this policy is the same as probabilistic slack policy.

When processing times can be stochastically ordered, probabilistic slack policy is optimal for any common due date. Indeed,  $X_i \leq_{st} X_j$  implies  $\Pr\{X_i \leq D\} \geq \Pr\{X_j \leq D\}$  since the former implies  $E[h(X_i)] \leq E[h(X_j)]$  for any function  $h(t)$  which is increasing in  $t$ .

$$\begin{aligned} X_i &\leq_{st} X_j \\ \Rightarrow \int_0^\infty F_D(t) f_i(t) dt &\leq \int_0^\infty F_D(t) f_j(t) dt \\ \Rightarrow \Pr\{X_i > D\} &\leq \Pr\{X_j > D\} \\ \Rightarrow \Pr\{X_i \leq D\} &\geq \Pr\{X_j \leq D\}, \end{aligned}$$

where  $F_D(t)$  is the distribution function of  $D$ .

### Theorem 2

Assume that processing times,  $X_i, i=1, \dots, n$ , are exponentially distributed and strictly stochastically ordered in such a way that  $X_1 \leq_{st} X_2 \leq_{st} \dots \leq_{st} X_n$  while the common due date  $D$  is any non-negative random variable. In this case, probabilistic slack policy is optimal. In addition, this policy sequences jobs in increasing order of index.

### Proof

Consider a sequence  $\pi = (\pi_1, j, i, \pi_2)$ , where  $X_i \leq_{st} X_j$ . We can obtain another sequence  $\pi'$  from sequence  $\pi$  by interchanging the two jobs  $i$  and  $j$ . The difference in the expected number of early jobs between these two sequences  $\pi$  and  $\pi'$  is  $E[\pi] - E[\pi'] = \Pr\{C_{\pi_1} + X_j \leq D\} - \Pr\{C_{\pi_1} + X_i \leq D\}$ , where  $C_{\pi_1}$  is the completion time of partial sequence  $\pi_1$ . By conditioning on  $C_{\pi_1}$ , the difference becomes

$$\begin{aligned} &\int_0^\infty \int_0^\infty (1 - F_D(t+s)) f_i(t) df_C(s) ds \\ &\quad - \int_0^\infty \int_0^\infty (1 - F_D(t+s)) f_j(t) df_C(s) ds \\ &= \int_0^\infty \int_0^\infty F_D(t+s) f_i(t) df_C(s) ds \\ &\quad - \int_0^\infty \int_0^\infty F_D(t+s) f_j(t) df_C(s) ds, \end{aligned}$$

where  $f_C(t)$  is the probability density function of  $C_{\pi_1}$ .

Since  $F_D(t+s)$  is increasing in  $t$  and  $X_i \leq_{st} X_j$ , the last quantity is less than 0. Therefore, we should interchange the two jobs  $i$  and  $j$  to maximize the expected number of early jobs. Applying the same procedure to all the remaining adjacent jobs in the sequence produces a new sequence which is stochastically ordered and maximizes the expected number of early jobs.

When processing times are not stochastically orderable, probabilistic slack policy can be optimal depending on the characteristic of the common due date. Let a random variable  $D_t$  denote the remaining time after  $t > 0$  time units have passed. If  $\Pr\{X_i \leq D_t\} > \Pr\{X_j \leq D_t\}$  for all  $t \geq 0$  and all  $i$  and  $j$ ,  $i \neq j$ , probabilistic slack policy sequences jobs in such a way that the expected number of early jobs is maximized.

**Lemma 5**

If  $\Pr\{X_i \leq D_t\} > \Pr\{X_j \leq D_t\}$  for all  $t \geq 0$  and all  $i$  and  $j$ ,  $i \neq j$ , probabilistic slack policy is optimal.

**Proof**

Consider a sequence  $\pi = (\pi_1, j, i, \pi_2)$ , where  $\Pr\{X_i \leq D_t\} > \Pr\{X_j \leq D_t\}$ . We obtain another sequence  $\pi'$  from sequence  $\pi$  by interchanging the two jobs  $i$  and  $j$ . The difference in the expected number of early jobs between these two sequences  $\pi$  and  $\pi'$  is  $E[\pi] - E[\pi'] = \Pr\{C_{\pi_1} + X_i \leq D - \Pr\{C_{\pi_1} + X_j \leq D\}$ , where  $C_{\pi_1}$  is the completion time of partial sequence  $\pi_1$ . Since  $\Pr\{X_i \leq D_t\} > \Pr\{X_j \leq D_t\}$  for all  $t > 0$ , the difference is less than 0 for any realization of  $C_{\pi_1}$ . Therefore, it is better to interchange the two jobs  $i$  and  $j$ .

A good example of the due date  $D$  that satisfies the condition of Lemma 5 is when the due date  $D$  follows an exponential distribution.

**2.3 Proposed Policy**

Previously, pairwise rank based policy and probabilistic slack policy are introduced. Each policy has both advantages and disadvantages over the other. For example, pairwise rank based policy generates optimal solutions when the processing times and due date are deterministic while probabilistic slack policy generates optimal solutions whenever the common due date follows an exponential distribution.

However, probabilistic slack policy has difficulty in generating proper sequences when there are ties in probabilistic slacks. If there is a means to break the ties, the performance of probabilistic slack policy will get better.

From the sample path analysis shown in Lemma 1, it is true that if  $X_i \prec X_j$ , then  $\Pr\{X_i + C \leq D\} \geq \Pr\{X_j + C \leq D\}$  for any non-negative random variable  $C$  and common due date  $D$ . Conversely, it is true as well that if  $\Pr\{X_i + C \leq D\} < \Pr\{X_j + C \leq D\}$ , then  $X_j \prec X_i$ .

From the discussions so far, it can be concluded

that the more likely smaller relation is a good way to break the ties in probabilistic slack policy. Therefore a new policy should sequence jobs in non-increasing order of probabilistic slacks and break ties with the more likely smaller relation.

Thus this policy is optimal even when the processing times and common due date are deterministic or non-overlapping. A set of distributions is defined to be non-overlapping if  $\Pr\{X_i \leq X_j\} = 0$  or 1 for all  $i$  and  $j$ ,  $i \neq j$  (Pinedo, 1983).

**3. PROBLEMS**

**3.1 Erlang Processing Times**

For any set of the processing times which follow Erlang distributions, the transitivity for pairwise rank based policy and stochastic ordering for probabilistic slack policy are satisfied for a special case as given in the following two propositions.

**Proposition 1**

Suppose  $X_i \sim \text{Erlang}(k_i, \lambda_i)$  and  $X_j \sim \text{Erlang}(k_j, \lambda_j)$ , where  $k_i$  is the number of phase and  $\lambda_i$  is the rate of each phase. If  $k_i \leq k_j$  and  $\lambda_i \geq \lambda_j$ , then  $X_i \prec X_j$ .

**Proof**

First, we obtain  $\Pr\{X_i \leq X_j\}$  as follows.

$$\begin{aligned} \Pr\{X_i \leq X_j\} &= \int_0^\infty \left(1 - \sum_{r=0}^{k_i-1} \frac{(\lambda_i t)^r e^{-\lambda_i t}}{r!}\right) \frac{\lambda_j^k t^{k_j-1} e^{-\lambda_j t}}{(k_j-1)!} dt \\ &= 1 - \int_0^\infty \sum_{r=0}^{k_i-1} \frac{(\lambda_i t)^r e^{-\lambda_i t}}{r!} \frac{\lambda_j^k t^{k_j-1} e^{-\lambda_j t}}{(k_j-1)!} dt \\ &= 1 - \sum_{r=0}^{k_i-1} \int_0^\infty \frac{\lambda_i^r \lambda_j^{k_j}}{r!(k_j-1)!} e^{-(\lambda_i+\lambda_j)t} t^{r+k_j-1} dt \\ &= 1 - \sum_{r=0}^{k_i-1} \frac{(r+k_j-1)!}{r!(k_j-1)!} \frac{\lambda_i^r \lambda_j^{k_j}}{(\lambda_i+\lambda_j)^{r+k_j}}. \end{aligned}$$

If we let  $\lambda_i/(\lambda_i+\lambda_j) = p \geq 0.5$ , then the last term in the above equation is decreasing in  $p \geq 0.5$  since

$$\begin{aligned} \frac{d}{dp} \sum_{r=0}^{k_i-1} \frac{(r+k_j-1)!}{r!(k_j-1)!} p^r (1-p)^{r+k_j} \\ = \sum_{r=0}^{k_i-1} \frac{(r+k_j-1)!}{r!(k_j-1)!} (1-p)^{k_j-1} p^{r-1} (r(1-p) - k_j p) \\ < 0. \end{aligned}$$

From the fact that  $\Pr\{X_i \leq X_j\} = 0.5$ , we have  $\sum_{r=0}^{k_i-1} \frac{(r+k_j-1)!}{r!(k_j-1)!} 0.5^r 0.5^{r+k_j} = 0.5$ . Therefore we have

$$\frac{(r+k_j-1)!}{r!(k_j-1)!} 0.5^r 0.5^{r+k_j} = 0.5. \text{ Therefore we have}$$

$$\sum_{r=0}^{k_j-1} \frac{(r+k_j-1)! \lambda_i^r \lambda_j^{k_j}}{r!(k_j-1)! (\lambda_i + \lambda_j)^{r+k_j}}$$

$$\leq \sum_{r=0}^{k_j-1} \frac{(r+k_j-1)! \lambda_i^r \lambda_j^{k_j}}{r!(k_j-1)! (\lambda_i + \lambda_j)^{r+k_j}}$$

The above equation implies  $\Pr\{X_i \leq X_j\} \geq 0.5$ .

**Proposition 2**

Suppose  $X_i \sim \text{Erlang}(k_i, \lambda_i)$  and  $X_j \sim \text{Erlang}(k_j, \lambda_j)$ , where  $k_i$  is the number of phase and  $\lambda_i$  is the rate of each phase. If  $k_i \leq k_j$  and  $\lambda_i \geq \lambda_j$ , then  $X_i \leq_{st} X_j$ .

**Proof**

If  $X \sim \text{Erlang}(k, \lambda)$ , then its distribution function can be represented by  $F(x) = 1 - \sum_{r=0}^{k-1} \frac{(\lambda x)^r}{r!} e^{-\lambda x}$ . From the following,

$$\frac{d}{d\lambda} \sum_{r=0}^{k-1} \frac{(\lambda x)^r}{r!} e^{-\lambda x}$$

$$= -xe^{-\lambda x} + \sum_{r=0}^{k-1} \frac{d}{d\lambda} \left( \frac{(\lambda x)^r}{r!} e^{-\lambda x} \right)$$

$$= -xe^{-\lambda x} + \sum_{r=0}^{k-1} xe^{-\lambda x} \left( \frac{(\lambda x)^{r-1}}{(r-1)!} - \frac{(\lambda x)^r}{r!} \right)$$

$$= xe^{-\lambda x} \left( \sum_{r=0}^{k-2} \frac{(\lambda x)^r}{r!} - \sum_{r=0}^{k-1} \frac{(\lambda x)^r}{r!} \right)$$

$$= -xe^{-\lambda x} \frac{(\lambda x)^{k-1}}{(k-1)!}$$

$$< 0,$$

it is noted that  $F(x)$  is increasing in  $\lambda$ . Therefore, it is obvious that  $F(x)$  is decreasing in  $k$ . Thus we have

$$1 - \sum_{r=0}^{k_i-1} \frac{(\lambda_i x)^r}{r!} e^{-\lambda_i x}$$

$$\geq 1 - \sum_{r=0}^{k_j-1} \frac{(\lambda_j x)^r}{r!} e^{-\lambda_j x}$$

$$\geq 1 - \sum_{r=0}^{k_j-1} \frac{(\lambda_j x)^r}{r!} e^{-\lambda_j x},$$

which leads to  $F_i(x) \geq F_j(x)$  for all  $x$ , where  $F_i(x)$  and  $F_j(x)$  are the distribution functions of  $X_i$  and  $X_j$ , respectively.

If only a subset of processing times can be stochastically ordered, both pairwise rank based policy and probabilistic slack policy cannot always generate optimal sequences. However, these two policies are believed to be quite good heuristics when the processing times

and common due date follow arbitrary distributions.

**3.2 Exponential Due Dates**

A good example of exchangeable due dates is the case where all the due dates are independently and identically distributed. Whereas there is only one realization of common due date, there might be many different realizations for exchangeable due dates.

The following theorem is proved for cases where processing times can be stochastically ordered in stochastic ordering sense (Boxma and Forst, 1986).

**Theorem 3**

Assume that processing times,  $X_i, i = 1, \dots, n$ , are stochastically ordered in such a way that  $X_1 \leq_{st} X_2 \leq_{st} \dots \leq_{st} X_n$  and all the due dates are any i.i.d. non-negative random variables. The optimal policy for maximizing the expected number of early jobs processes jobs in increasing order of index.

Theorem 3 implies that probabilistic slack policy is optimal when processing times can be stochastically ordered and due dates are exchangeable. Similarly, pairwise rank based policy is also applicable when due dates are exchangeable.

**Lemma 6**

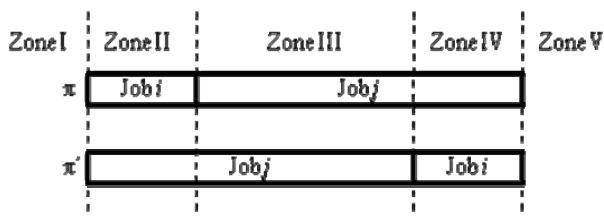
Consider any two adjacent jobs  $i$  and  $j$ , in a sequence. If  $X_i \prec X_j$ , optimal policy should sequence job  $i$  before job  $j$  for any exchangeable due dates.

**Proof**

Consider a realization,  $\varpi \in \Omega$ , of processing times. In addition, denote the realization of  $X_i$  by  $X_i(\varpi)$ . When  $X_i(\varpi) < X_j(\varpi)$ , there are three cases where processing job  $i$  first is worse than the other way around. Those cases are cancelled out since due dates are exchangeable. For example (see Figure 1), the probability that due dates for jobs  $i$  and  $j$  occurs in Zone I and Zone IV, respectively, is the same as the probability that due dates for jobs  $i$  and  $j$  occurs in Zone IV and Zone I, respectively. Consequently, given a realization of the processing times  $\varpi$ , the expected reward becomes larger when the shorter job is processed first. Therefore, with probability  $\Pr\{X_i \prec X_j\}$ , it is better to process job  $i$  first.

As long as the transitivity holds for processing times, pairwise rank based policy is guaranteed to be optimal for any exchangeable due dates. The new policy can be used in case of exchangeable due dates as well.

Just like the common due date case, multiple classes of exchangeable due dates can be considered in which a set of jobs within a class share any i.i.d. due dates. If any two jobs  $i$  and  $j$  are classwise-adjacent and  $X_i \prec X_j$ , then optimal policy should sequence job  $i$  before job  $j$  as given in the following lemma.



Due Date Occurrence		Difference
Job <i>i</i>	Job <i>j</i>	
Zone I	Zone I	0
Zone I	Zone II	0
Zone I	Zone III	0
Zone I	Zone IV	-1
Zone I	Zone V	0
Zone II	Zone I	0
Zone II	Zone II	0
Zone II	Zone III	0
Zone II	Zone IV	-1
Zone II	Zone V	0
Zone III	Zone I	1
Zone III	Zone II	1
Zone III	Zone III	1
Zone III	Zone IV	0
Zone III	Zone V	1
Zone IV	Zone I	1
Zone IV	Zone II	1
Zone IV	Zone III	1
Zone IV	Zone IV	0
Zone IV	Zone V	1
Zone V	Zone I	0
Zone V	Zone II	0
Zone V	Zone III	0
Zone V	Zone IV	-1
Zone V	Zone V	0

Figure 1. Illustration for the Proof of Lemma 6

**Lemma 7**

Consider any two class-wise adjacent jobs *i* and *j*. If  $X_i < X_j$ , then optimal policy should sequence job *i* before job *j*.

**Proof**

The proof is similar to that of Lemma 6. See Figure 2.

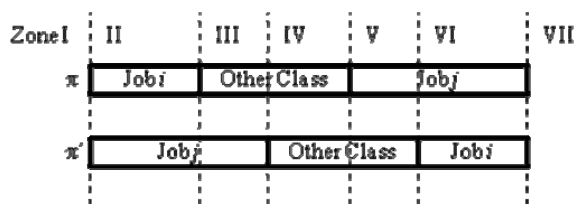


Figure 2. Illustration for the Proof of Lemma 7

**4. FINAL REMARKS**

In this paper, stochastic scheduling problems with a common due date and exchangeable due dates are considered. Based on a sample path analysis, pairwise rank

based policy which orders more likely smaller jobs first has been introduced. It is shown that pairwise rank based policy is optimal when processing times are stochastically ordered in failure rate sense.

In addition, probabilistic slack policy has also been introduced. It is shown that when either processing times are stochastically ordered or due date is exponential, probabilistic slack policy is optimal.

It is shown that both policies are complementary to each other, for example, if processing times and due date are deterministic, pairwise rank based policy is optimal, but probabilistic slack policy is not guaranteed to be optimal. On the other hand, as long as due date is exponential, probabilistic slack policy is optimal, but pairwise rank based policy is not always.

The proposed policy captures advantages of both policies. This policy sequences jobs in non-increasing order of probabilistic slacks and break ties by the more likely smaller relation. The new policy is optimal when either pairwise rank based policy or probabilistic slack policy is optimal

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