

## A compound Poisson risk model with variable premium rate

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### Abstract

We consider a general compound Poisson risk model in which the premium rate is surplus dependent. We analyze the joint distribution of the surplus immediately before ruin, the deficit at ruin and the time of ruin by solving the integro-differential equation for the Gerber-Shiu discounted penalty function.

*Keywords:* Compound Poisson model, Gerber-Shiu discounted penalty function, variable premium rate.

### 1. Introduction

We consider a general compound Poisson risk model in which the premium rate is dependent on the current surplus, i.e., the premium rate is a function of the surplus. It models the case where the company raises the premium once the surplus goes below some threshold, and/or it lowers the premium if the surplus goes above another threshold for attracting more customers, and/or it pays out dividend at a rate if the surplus is above some threshold (Asmussen, 2000). Let  $u \geq 0$  be the initial surplus. The aggregate claims constitute a compound Poisson process  $S(t)$ , given by the Poisson parameter  $\lambda$  and individual claim amount distribution function  $G(x)$  with  $G(0) = 0$ . That is,  $S(t)$  is described by

$$S(t) = \sum_{i=1}^{N(t)} X_i,$$

where  $\{N(t), t \geq 0\}$  is a Poisson process with mean  $\lambda$  per unit time and  $X_1, X_2, \dots$  are independent and identically distributed random variables with common distribution  $G(x)$ . Then the surplus process  $\{U(t), t \geq 0\}$  satisfies

$$U(t) = u + \int_0^t p(U(s))ds - S(t), \quad t \geq 0,$$

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where  $p(x)$  is the premium rate function. As usual, the time of (ultimate) ruin is defined as  $T := \inf\{t|U(t) < 0, t \geq 0\}$ , where  $T = \infty$  if ruin does not occur in finite time.

The Gerber-Shiu discounted penalty function (Gerber and Shiu, 1998) is defined as

$$m(u, \delta) := E[e^{-\delta T} w(U(T-), |U(T)|) I(T < \infty) | U(0) = u], \quad (1.1)$$

where  $w(x, y)$ ,  $x \geq 0$ ,  $y \geq 0$ , is the penalty at the time of ruin and  $I(E)$  denotes the indicator function assigning 1 if the event  $E$  occurs and 0 otherwise. It is well known that this discounted penalty function provides a unified approach to three important random variables: the surplus immediately before ruin  $U(T-)$ , the deficit at ruin  $|U(T)|$ , and the time of ruin  $T$  (Gerber and Shiu, 1998; Lin *et al.*, 2003).

Various stochastic processes are applied to analyze the characteristics of the risk processes in the insurance models (Klugman *et al.*, 2004; Kim and Kim, 2009; Hyeon and Cha, 2010). Asmussen (2000) discussed a few issues around the variable premium rate. Rong and Li (2004) obtained the ruin probability in the Cox risk model with variable premium rate and Lin and Pavlova (2006) studied the Poisson risk model in which the premium rate is a step function of the surplus. By Lin and Sendova (2008), it was extended into the compound Poisson risk model with variable premium rate. They derived the integro-differential equation for the Gerber-Shiu discounted penalty function such as

$$p(u) \frac{\partial}{\partial u} m(u, \delta) = (\lambda + \delta)m(u, \delta) - \lambda \int_0^u m(u - y, \delta) dG(y) - \lambda \zeta(u), \quad (1.2)$$

where

$$\zeta(x) := \int_x^\infty w(x, y - x) dG(y)$$

is the conditional expected penalty given that the surplus immediately prior to the time of ruin is  $x$ . Recently Park and Choi (2011) studied an asymptotic behavior of the finite-time ruin probability of the compound Poisson model when the initial surplus is large.

In this paper, we solve the above integro-differential equation (1.2) to obtain the Gerber-Shiu discounted penalty functions  $m(u, \delta)$  for the risk models with variable premium rate. And then by using the function  $m(u, \delta)$ , we derive the ultimate ruin probability, the Laplace transform of the time of ruin, the survival distribution function of the deficit at ruin and the Laplace transform of the first exit time from  $[0, \alpha)$ .

## 2. The Gerber-Shiu discounted penalty function

In this section, we obtain the solution to the integro-differential equation for the Gerber-Shiu discounted penalty function.

**Theorem 2.1** The solution  $m(u, \delta)$  to the equation (1.2) is given by

$$\begin{aligned} m(u, \delta) = & \frac{1 + \int_0^u K^*(x, 0, \delta) dx}{1 + \int_0^\infty K^*(x, 0, \delta) dx} \int_0^\infty \frac{\lambda \zeta(x)}{p(x)} \left( 1 + \int_x^\infty K^*(y, x, \delta) dy \right) dx \\ & - \int_0^u \frac{\lambda \zeta(x)}{p(x)} \left( 1 + \int_x^u K^*(y, x, \delta) dy \right) dx, \end{aligned} \quad (2.1)$$

where  $K^*(x, y, \delta)$  is defined in (2.4).

**Proof:** Since

$$m(u, \delta) = \int_0^u \frac{\partial}{\partial x} m(x, \delta) dx + m(0, \delta),$$

we have

$$\begin{aligned} \int_0^u m(u - y, \delta) dG(y) &= \int_0^u \int_0^{u-y} \frac{\partial}{\partial x} m(x, \delta) dx dG(y) + m(0, \delta)G(u) \\ &= \int_0^u \int_0^{u-x} \frac{\partial}{\partial x} m(x, \delta) dG(y) dx + m(0, \delta)G(u) \\ &= \int_0^u G(u - x) \frac{\partial}{\partial x} m(x, \delta) dx + m(0, \delta)G(u). \end{aligned}$$

Substituting the above in (1.2) and dividing both sides by  $p(u)$ , we have

$$\begin{aligned} &\frac{\partial}{\partial u} m(u, \delta) \\ &= \frac{\lambda + \delta}{p(u)} \left( \int_0^u \frac{\partial}{\partial x} m(x, \delta) dx + m(0, \delta) \right) - \lambda \int_0^u \frac{G(u - x)}{p(u)} \frac{\partial}{\partial x} m(x, \delta) dx \\ &\quad - \lambda \frac{m(0, \delta)G(u)}{p(u)} - \lambda \frac{\zeta(u)}{p(u)} \\ &= \int_0^u \frac{\delta + \lambda[1 - G(u - x)]}{p(u)} \frac{\partial}{\partial x} m(x, \delta) dx + \frac{\delta + \lambda[1 - G(u)]}{p(u)} m(0, \delta) - \lambda \frac{\zeta(u)}{p(u)}. \end{aligned}$$

Then it reduces to

$$\frac{\partial}{\partial u} m(u, \delta) = A(u, \delta) + \int_0^u K(u, x, \delta) d_x m(x, \delta), \tag{2.2}$$

where

$$K(x, y, \delta) := \frac{\delta + \lambda[1 - G(x - y)]}{p(x)}$$

and

$$\begin{aligned} A(x, \delta) &:= \frac{m(0, \delta)\{\delta + \lambda[1 - G(x)]\} - \lambda\zeta(x)}{p(x)} \\ &= m(0, \delta)K(x, 0, \delta) - \frac{\lambda\zeta(x)}{p(x)}. \end{aligned} \tag{2.3}$$

In a manner analogous to that of Harrison and Resnick (1976), we will now let

$$K^1(x, y, \delta) = K(x, y, \delta), \quad 0 \leq y < x, \delta \geq 0$$

and define its iterates recursively as

$$\begin{aligned} K^{n+1}(x, y, \delta) &= \int_y^x K^n(x, z, \delta) K^1(z, y, \delta) dz \\ &= \int_y^x K^1(x, z, \delta) K^n(z, y, \delta) dz, \quad 0 \leq y < x, \delta \geq 0, \end{aligned}$$

for  $n \geq 1$ . Using the bound  $K^1(x, y, \delta) \leq (\delta + \lambda)/p(x)$ , it follows easily by induction that

$$K^{n+1}(x, y, \delta) \leq \frac{(\lambda + \delta)^{n+1} \left[ \int_y^x \frac{1}{p(z)} dz \right]^n}{p(x)n!}, \quad 0 \leq y < x, \delta \geq 0,$$

for all  $n \geq 1$ . Thus the kernel

$$K^*(x, y, \delta) := \sum_{n=1}^{\infty} K^n(x, y, \delta), \quad 0 \leq y < x, \delta \geq 0 \quad (2.4)$$

is well-defined. Iterating the relation of (2.2)  $N - 1$  times gives

$$\frac{\partial}{\partial u} m(u, \delta) = A(u, \delta) + \int_0^u A(y, \delta) \sum_{n=1}^{N-1} K^n(u, y, \delta) dy + \int_0^u K^N(u, y, \delta) d_y m(y, \delta).$$

Letting  $N \rightarrow \infty$  and using the dominated convergence theorem, we then have

$$\frac{\partial}{\partial u} m(u, \delta) = A(u, \delta) + \int_0^u A(y, \delta) K^*(u, y, \delta) dy.$$

Substituting  $A(u, \delta)$  from (2.3) into the above equation and then integrating it with respect to  $u$ , we obtain

$$\begin{aligned} m(u, \delta) &= m(0, \delta) \left( 1 + \int_0^u K^*(x, 0, \delta) dx \right) \\ &\quad - \int_0^u \frac{\lambda \zeta(x)}{p(x)} \left( 1 + \int_x^u K^*(y, x, \delta) dy \right) dx. \end{aligned} \quad (2.5)$$

From the boundary condition that  $\lim_{u \rightarrow \infty} m(u, \delta) = 0$ , it follows that

$$m(0, \delta) = \frac{\int_0^{\infty} \frac{\lambda \zeta(x)}{p(x)} \left( 1 + \int_x^{\infty} K^*(y, x, \delta) dy \right) dx}{1 + \int_0^{\infty} K^*(x, 0, \delta) dx} \quad (2.6)$$

which completes (2.1).  $\square$

### 3. Some quantities of interest

The Gerber-Shiu discounted penalty function  $m(u, \delta)$  of (1.1) can lead to some important quantities of interest in the risk model. If we let  $\delta = 0$  and  $w(x, y) = 1$  for  $x, y \geq 0$  in (1.1), then we can obtain the ultimate ruin probability. If  $\delta > 0$  and  $w(x, y) = 1$  for  $x, y \geq 0$ , (1.1) becomes the Laplace transform of the time of ruin. In this section, by varying the transform parameter  $\delta$  and the penalty function  $w(x, y)$ , we derive the ultimate ruin probability, the Laplace transform of the time of ruin, the survival distribution function of the deficit at ruin and the Laplace transform of the first exit time from  $[0, \alpha)$ .

**3.1. The ultimate ruin probability**

Asmussen and Petersen (1988) connected this risk model with the dam model with general release rate by the time-reversion and then they showed that the ultimate ruin probability  $\psi(u) := P(T < \infty | U(0) = u)$  can be expressed by

$$\psi(u) = \int_u^\infty f(x)dx,$$

where  $f(x)$  is the stationary density of the dam content  $x$  and satisfies the storage equation

$$f(x) = \pi_0 Q(x, 0) + \int_0^x Q(x, y)f(y)dy,$$

where  $Q(x, y) := \lambda[1 - G(x - y)]/p(x)$  and  $\pi_0$  is the probability that the dam is empty.

Especially when  $\delta = 0$  and  $w(x, y) = 1$  for all  $x \geq 0, y \geq 0$ , we have  $\zeta(x) = 1 - G(x)$ . So

$$\frac{\lambda\zeta(x)}{p(x)} = K(x, 0, 0).$$

Since

$$\begin{aligned} \int_0^\infty \int_x^\infty K(x, 0, 0)K^*(y, x, 0)dydx &= \int_0^\infty \int_0^y K(x, 0, 0)K^*(y, x, 0)dx dy \\ &= \int_0^\infty \sum_{n=2}^\infty K^n(y, 0, 0)dy, \end{aligned}$$

the numerator of  $m(0, 0)$  with  $\delta = 0$  in (2.6) is given by

$$\int_0^\infty K(x, 0, 0)dx + \int_0^\infty \sum_{n=2}^\infty K^n(y, 0, 0)dy = \int_0^\infty K^*(x, 0, 0)dx.$$

Therefore it follows that

$$m(0, 0) = \frac{\int_0^\infty K^*(x, 0, 0)dx}{1 + \int_0^\infty K^*(x, 0, 0)dx}. \tag{3.1}$$

Notice that the ultimate ruin probability  $\psi(u)$  is coincident with  $m(u, 0)$  in this case. Since

$$\int_0^u K(x, 0, 0)dx + \int_0^u \int_x^u K^*(y, x, 0)dydx = \int_0^u K^*(x, 0, 0)dx,$$

inserting (3.1) into (2.5) gives

$$\begin{aligned} \psi(u) &= m(u, 0) \\ &= \frac{\int_u^\infty K^*(x, 0, 0)dx}{1 + \int_0^\infty K^*(x, 0, 0)dx}. \end{aligned}$$

### 3.2. The time of ruin

Assume that  $\delta > 0$  and  $w(x, y) = 1$  for all  $x \geq 0, y \geq 0$ . In this case  $m(u, \delta)$  becomes the Laplace transform of the time of ruin. Since  $\zeta(x) = 1 - G(x)$  and  $\frac{\lambda\zeta(x)}{p(x)} = K(x, 0, 0)$ , it follows from (2.5) and (2.6) that

$$\begin{aligned} m(u, \delta) &= E[e^{-\delta T} | U(0) = u] \\ &= m(0, \delta) \left( 1 + \int_0^u K^*(x, 0, \delta) dx \right) \\ &\quad - \int_0^u K(x, 0, 0) \left( 1 + \int_x^u K^*(y, x, \delta) dy \right) dx, \end{aligned}$$

where

$$m(0, \delta) = \frac{\int_0^\infty K(x, 0, 0) \left( 1 + \int_x^\infty K^*(y, x, \delta) dy \right) dx}{1 + \int_0^\infty K^*(x, 0, \delta) dx}.$$

### 3.3. The deficit at ruin

Let us define the survival distribution function of the deficit at ruin by

$$D(u, v) := P\{|U(T)| > v, T < \infty | U(0) = u\}.$$

If we assume that  $\delta = 0$  and  $w(x, y) = 1$  for all  $x \geq 0, y > v$ , then we have  $\zeta(x) = \int_x^\infty w(x, y - x) dG(y) = 1 - G(x + v)$ . Therefore from (2.5) and (2.6) it follows that

$$\begin{aligned} D(u, v) &= m(u, 0) \\ &= m(0, 0) \left( 1 + \int_0^u K^*(x, 0, 0) dx \right) \\ &\quad - \int_0^u \frac{\lambda[1 - G(x + v)]}{p(x)} \left( 1 + \int_x^u K^*(y, x, 0) dy \right) dx, \end{aligned}$$

where

$$m(0, 0) = \frac{\int_0^\infty \frac{\lambda[1 - G(x + v)]}{p(x)} \left( 1 + \int_x^\infty K^*(y, x, 0) dy \right) dx}{1 + \int_0^\infty K^*(x, 0, 0) dx}.$$

### 3.4. The first exit time

For  $0 \leq u \leq \alpha$ , we define

$$T_\alpha := \inf\{t \geq 0 | U(t) \notin [0, \alpha]\}$$

to represent the first exit time from  $[0, \alpha]$  (Lee, 2007).

Then, the Laplace transform of  $T_\alpha$  under the condition that  $U(0) = u$  is given by

$$\begin{aligned} \phi_\alpha(u, \delta) &:= E[e^{-\delta T_\alpha} | U(0) = u] \\ &= E[e^{-\delta T_\alpha} I(U(T_\alpha) < 0) | U(0) = u] + E[e^{-\delta T_\alpha} I(U(T_\alpha) = \alpha) | U(0) = u]. \end{aligned}$$

If we further define

$$T_\alpha^1 = \begin{cases} \infty & \text{if } U(T_\alpha) = \alpha \\ T_\alpha & \text{if } U(T_\alpha) < 0 \end{cases}$$

and

$$T_\alpha^2 = \begin{cases} T_\alpha & \text{if } U(T_\alpha) = \alpha \\ \infty & \text{if } U(T_\alpha) < 0, \end{cases}$$

then we obtain

$$\phi_\alpha(u, \delta) = E[e^{-\delta T_\alpha^1} | U(0) = u] + E[e^{-\delta T_\alpha^2} | U(0) = u].$$

For  $\delta > 0$ ,

$$\begin{aligned} E[e^{-\delta T_\alpha^1} | U(0) = u] &= E[e^{-\delta T_\alpha^1} I(T_\alpha^1 < \infty) | U(0) = u] \\ &:= m(u, \delta) \end{aligned}$$

with  $w(x, y) = 1$  for all  $x \geq 0, y \geq 0$ .

From (2.5) and the boundary condition that  $m(\alpha, \delta) = 0$  for all  $\delta \geq 0$ , it follows that

$$\begin{aligned} E[e^{-\delta T_\alpha^1} | U(0) = u] &= m(0, \delta) \left( 1 + \int_0^u K^*(x, 0, \delta) dx \right) \\ &\quad - \int_0^u K(x, 0, 0) \left( 1 + \int_x^u K^*(y, x, \delta) dy \right) dx, \end{aligned}$$

where

$$m(0, \delta) = \frac{\int_0^\alpha K(x, 0, 0) \left( 1 + \int_x^\alpha K^*(y, x, \delta) dy \right) dx}{1 + \int_0^\alpha K^*(x, 0, \delta) dx}.$$

Similarly, for  $\delta > 0$ ,

$$\begin{aligned} E[e^{-\delta T_\alpha^2} | U(0) = u] &= E[e^{-\delta T_\alpha^2} I(T_\alpha^2 < \infty) | U(0) = u] \\ &:= m(u, \delta) \end{aligned}$$

with

$$w(x, y) = \begin{cases} 1 & x = \alpha, y = 0 \\ 0 & \text{otherwise.} \end{cases}$$

We note that  $\zeta(x) = 0$  in this case. Using the boundary condition that  $m(\alpha, \delta) = 1$  for all  $\delta \geq 0$  in (2.5) we have

$$E[e^{-\delta T_\alpha^2} | U(0) = u] = \frac{1 + \int_0^u K^*(x, 0, \delta) dx}{1 + \int_0^\alpha K^*(x, 0, \delta) dx}.$$

Notice that here

$$m(u, 0) = \frac{1 + \int_0^u K^*(x, 0, 0) dx}{1 + \int_0^\alpha K^*(x, 0, 0) dx}$$

is the probability that the surplus reaches  $\alpha$  before it becomes empty, starting from  $u$ .

#### 4. Example

Suppose that the premium rate is constant, that is,  $p(x) = p$ . Then

$$\begin{aligned} K(x, y, \delta) &= K(x - y, 0, \delta) \\ &= \frac{\delta + \lambda[1 - G(x - y)]}{p}. \end{aligned}$$

Define

$$H_\delta(x) := \begin{cases} \sum_{n=0}^{\infty} W_\delta^{*n}(x) & x \geq 0 \\ 0 & x < 0, \end{cases}$$

with  $W_\delta(x) := \int_0^x \frac{\delta + \lambda[1 - G(y)]}{p} dy$ ,  $*n$  being the  $n$ -fold recursive Stieltjes convolution, and  $W_\delta^{*0}$  being the Heaviside function. Then we have

$$1 + \int_0^x K^*(y, 0, \delta) dy = H_\delta(x).$$

Hence the ultimate ruin probability in this case can be simplified to

$$\psi(u) = 1 - \left(1 - \frac{\lambda m_G}{p}\right) H_0(u), \quad (4.1)$$

where  $m_G := \int_0^\infty x dG(x)$ . The Laplace transform of the time of ruin and the Laplace transform of the first exit time from  $[0, \alpha)$  are given, respectively, by

$$E[e^{-\delta T} | U(0) = u] = \frac{\lambda m_G}{p} H_\delta(u) - (W_0 * H_\delta)(u)$$

and

$$E[e^{-\delta T_\alpha} | U(0) = u] = \frac{1}{H_\delta(\alpha)} [H_\delta(u)(W_0 * H_\delta)(\alpha) - H_\delta(\alpha)(W_0 * H_\delta)(u) + H_\delta(u)].$$

We further assume that the individual claim size is exponentially distributed with mean  $m_G$ . Then it reduces

$$H_0(x) = \frac{p - \lambda m_G e^{-\theta x}}{p - \lambda m_G},$$

where  $\theta := 1/m_G - \lambda/p$ . Substituting the above in (4.1) yields the well-known ultimate ruin probability

$$\psi(u) = \frac{\lambda m_G}{p} e^{-\theta u}.$$

We also obtain the survival distribution function of the deficit at ruin in this case which is given by

$$D(u, v) = \frac{\lambda m_G}{p} \exp\left(-\frac{v}{m_G} - \theta u\right).$$



## 5. Conclusions

We considered the general compound Poisson risk model which has a premium rule depending on the surplus. We solved the integro-differential equation to obtain the explicit form of the Gerber-Shiu discounted penalty function. We could derive the closed-form solutions for the ultimate ruin probability, the Laplace transform of the time of ruin, the survival distribution function of the deficit at ruin and the Laplace transform of the first exit time from  $[0, \alpha)$  by adopting the corresponding penalty function. The case of a constant premium rate and exponential claim sizes was treated as an example.

These closed-form solutions still remain complicated to use in many cases, albeit explicit. In the further study we investigate alternative approximation or numerical methods which are able to fit real data.

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