한국수학사학회지 *The Korean Journal for History of Mathematics*
제24권 제3호 (2011년 8월), 59–76

A reconstruction of the Gödel's proof of the consistency of GCH and AC with the axioms of Zermelo-Fraenkel set theory

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Starting from a collection V as a model which satisfies the axioms of NBG, we call the elements of V as sets and the subcollections of V as classes.

We reconstruct the Gödel's proof of the consistency of GCH and AC with the axioms of Zermelo-Fraenkel set theory by using Mostowski-Shepherdson mapping theorem, reflection principles in Tarski-Vaught theorem and Montague-Levy theorem and the fact that NBG is a conservative extension of ZF.

NBG의 공리들을 충족시키는 모델로서의 집합 V를 도입하고 그것의 요소들을 sets라 부르고 그것의 부분집합들을 classes라 부른다.

일반연속체가설(GCH)와 선택공리(AC)가 ZF 집합론과 무모순이라는 것에 대한 괴델의 증명을 그 이후 나온 Mostowski-Shepherdson mapping 정리, Tarski-Vaught 정리 및 Montague-Levy 정리의 반사원리들, NBG가 ZF의 보존적 확장이라는 정리 등을 이용하여 재구성해 본다.

Keywords: NBG, ZF, Mostowski-Shepherdson mapping theorem, reflection principles, constructibility, absoluteness, relative consistency.

0 Introduction

(1) Cantor had conjectured the proposition, now called the *well-ordering theorem*, that every set can be well-ordered. In 1904 Zermelo gave a proof of this conjecture, using in an essential way the following mathematical principle: for every set X there is a *choice function*, f, which is defined on the collection of non-empty subsets of X, such that for every set A in its domain we have $f(A) \in A$. Subsequently, in 1908, Zermelo presented an axiomatic version of set theory in which his proof of the well-ordering theorem could be carried out. One of the axioms

This paper was supported by research funds of Chonbuk National University in 2008. I deeply appreciate for the useful comments of the anonymous referees.

MSC: 03E35

Receiveded on Apr. 7th, revised on Jul. 20th, accepted on Jul. 25th.

was the principle just stated, which Zermelo referred to as the "Axiom der Auswahl"(the *axiom of choice*, abbreviated AC).

Zermelo's proof was the subject of considerable controversy. The well-ordering theorem is quite remarkable, since, for example, there is no obvious way to define a well-ordering of the set of real numbers. Nor is such an explicit well-ordering provided by Zermelo's proof. Thus many people who thought Zermelo's result implausible cast doubt upon the validity of AC.

The work of Gödel (Gödel 1939, 1940) showed that AC is "safe" in the following sense: If the usual axioms of set theory (ZF or the closely related system NBG, including the axiom of foundation but excluding AC) do not lead to a contradiction, then they remain consistent when AC is adjoined as an additional axiom.

In his theory of infinite cardinals Cantor proved (making essential but implicit use of AC) that the totality of all infinite cardinal numbers is well-ordered (and in fact is order-isomorphic to the totality of all ordinal numbers). However, an important question was left open by Cantor 's work. Let $\mathfrak{c}$ be the cardinal number of the set of real numbers (or, as this set is sometimes referred to, the *continuum*). Cantor showed that $\mathfrak{c}$ is not the first infinite cardinal, but he was unable to determine its precise place in the hierarchy of infinite cardinals. He conjectured, however, that $\mathfrak{c}$ is precisely equal to $\aleph_1$, the second infinite cardinal. This conjecture became known as the *continuum hypothesis*(*CH*). It is easily shown that $\mathfrak{c} = 2^{\aleph_0}$, and so CH is equivalent to the statement $2^{\aleph_0} = \aleph_1$. A natural generalization, considered later by Hausdorff and called the generalized hypothesis (GCH), asserts that for every ordinal α, $2^{\aleph_\alpha} = \aleph_{\alpha+1}$.

Gödel did not succeed in settling whether or not CH (and GCH) is true. But he was able to show that the usual axioms of set theory do not disprove CH (and GCH), so that if they settle its truth value at all, it must be a theorem. As it turned out, Cohen was able to show in 1963 that the latter also does not hold—he showed that the negation of GCH (in fact even the negation of the special CH) is consistent in ZF; he also showed that AC is not provable in ZF and the most remarkable result of Cohen is that even if we add AC to ZF axioms it is still not possible to prove the CH.

So the usual axioms of set theory are *not strong enough* to settle the problems of the continuum problem. It is perfectly possible that new principles of set theory may one day be found which, though not derivable from the present axioms, are nevertheless self-evident and which might settle the continuum hypothesis one way or the other. Indeed Gödel, as a mathematical realist or Platonist,—despite his proof of the formal consistency of the continuum hypothesis—has conjectured that when such a principle is found the continuum hypothesis will then be seen to be false (Gödel, 1947).

(2) As we shall see later, Gödel's constructible hierarchy (which is used to define the class of constructible sets) can be viewed as a variant of the cumulative rank hierarchy. The rank hierarchy had first been clearly stated by Zermelo in 1930, in the context of second-order models of set theory. In a sense, Gödel combined the rank hierarchy of Zermelo with the first-order perspective of Skolem in order to obtain the hierarchy of constructible sets.

Another antecedent of Gödel's constructible hierarchy is the ramified theory of types of Russell and Whitehead. Indeed Gödel explicitly states that his constructible hierarchy can be viewed as the natural prolongation to transfinite levels of the ramified theory of types (Gödel 1944 p.147). In a letter to Hao Wang, he attests to the fruitfulness of his platonistic attitudes for his research in the foundations of mathematics. Referring to his work on the consistency of CH, he says, "However, as far as, in particular, the continuum hypothesis is concerned, there was a special obstacle which really made it practically impossible for constructivists to discover my consistency proof. It is the fact that the ramified hierarchy, which had been invented expressly for constructive purposes, has to be used in an entirely nonconstructive way" (Wang 1974 p. 10). The essentially nonconstructive element lies in the use of arbitrary ordinals as the levels in Gödel's extension of the ramified theory.

(3) As a price for having given up Frege's unlimited abstraction principle, which caused Russell's paradox, for the limited abstraction principle, or separation principle—also known as Aussonderungsprinzip—which is this: given any property P and given any set a there exists the set of all elements *of the set* a that have property P, Zermelo in 1908 had to take the existence of the sets $\varnothing$, $\{a, b\}$, $\bigcup a$, $\mathrm{P}(a)$ as separate axioms. He also took an axiom of infinity, which provided for the existence of the set ω of natural numbers. Let's call this (excluding AC) *Zermelo* set theory. Later Fraenkel, and independently Skolem, added a powerful axiom known as the *axiom of substitution*, or the *axiom of replacement*, which roughly says that given any set x, one can form a new set by simply replacing each element of x by some element. The resulting system is known as *Zermelo-Fraenkel* set theory—abbreviated ZF.

One might also wish to state axioms providing for the existence of various *properties* of sets. Zermelo did *not* do this, and to that extent his system was not a completely formal axiom system in the modern sense of the term. It was Thoralf Skolem who proposed to identify properties with *first-order properties*, by which is meant conditions defined by *first-order formulas*—i.e., well formed expressions built from the set-membership symbol "$\in$," the variables $x, y, \ldots$ ranging over all sets, the connectives of propositional logic, and the quantifiers for the set variables $x, y, \ldots$. In Skolem's formulation, it was necessary to express Zermelo's separation principle as an *infinite* number of axioms, one for every first-order formula.

Now, Zermelo protested vigorously against this interpretation by Skolem. For Zermelo, properties were to be thought of as *all* meaningful conditions, not just those conditions given by first-order formulas (or higher order formulas, for that matter). A still stronger system of set theory called *Morse-Kelly* set theory forces more properties into the picture—properties defined by quantifying over properties as well as sets, but this still does not provide for *all* the properties of sets that there are. In fact (by incompleteness results of Gödel) there is *no* formal axiomatization of properties that capture the entire picture, and this might be a key factor in the difficulty of deciding whether the continuum hypothesis is true, or false.

(4) Conceptually, Zermelo-Fraenkel set theory is a simple one, but technically it is in many ways quite awkward and inelegant. A far more attractive system was developed by *V*on Neumann, later revised by Robinson, Bernays, and Gödel and is now known as *NBG*. This is the main system of this paper (and it is the system Gödel 1940 worked with). The basic idea is that certain collections of things are called *classes* and certain collections are called *sets*. The term "class" is the more comprehensive one, since every set is also a class, but not every class is a set. Which classes are sets? Our philosophical position is that these notions are *relative* to any given *model* of the axioms of class-set theory. That is, a collection V is called a *model* of class-set theory if it satisfies the axioms of NBG, which will be given below. The elements of V are called the sets of the model and the subcollections of V are called the classes of the model. When the model V is fixed for the discussion, then the sets of the model are more briefly called "sets" and the classes of the model are simply called "classes."

In this paper, ZF is the object theory, NBG is the meta-theory, and the description of NBG is done in the meta-meta-theory (English).

The aim of this paper is to reconstruct Gödel's original proof of the consistency of GCH and AC with the axioms of Zermelo-Fraenkel set theory, by using later results such as Mostowski-Shepherdson mapping theorem, reflection principles in Tarski-Vaught theorem and Montague-Levy theorem and the fact that NBG is a conservative extension of ZF, thereby obtaining a more modern proof.

1 Some basics of class-set theory

We introduce the class-set theory of von Neumann, Bernays, and Gödel (NBG).

We define $A \subseteq B$ (A is a "subclass" of B) as $\forall x\,(x \in A \supset x \in B)$.

We now fix a class V (an *NBG universe*, or a *genuine Zermelo-Fraenkel universe*), which satisfies P_1, P_2, $A_1, \ldots, A_8$ below. For the rest, a "class" means a subclass of V. The elements of V will be called "sets".

We henceforth use capital letters, A, B, C, $D, \ldots$ as standing for classes and small letters x, y, z, a, b, $c, \ldots$ for sets. By a "first-order" property of sets we mean one defined by a formula in which we quantify only over sets. We include $=$ as a logical symbol in the first-order logic.

P_1: [Axiom of extensionality] $\forall x\,(x \in A \equiv x \in B) \supset A = B$.

P_2: [Separation] $\forall A_1, \ldots, \forall A_n\, \exists B\, \forall x\,(x \in B \equiv \Phi(A_1, \ldots, A_n, x))$.

A_1: Every set is a class (V is *transitive*).[1)]

A_2: Every subclass of a set is a set (V is *swelled*).

A_3: The class $\varnothing$ is a set.

A_4: For any sets a, b, the class $\{a, b\}$ is a set.

A_5: If x is a set, so is $\cup x$.

A_6: For any set x, $\mathrm{P}(x)$ is a set.

A_7: ω (the class of natural numbers) is a set.

A_8: For any function F and a set x, $F''(x)$ (i.e., the image of x under F) is a set (axiom of substitution).[2)]

We presuppose the concepts of ordinals, cardinals, the transfinite induction principle, and the definition by transfinite recursion as familiar.

2 Rank

For each ordinal α we define the set R_α by the following transfinite recursion.

$R_0 = \varnothing$.

$R_{\alpha+1} = \mathrm{P}(R_\alpha)$.

$R_\lambda = \bigcup_{\alpha<\lambda} R_\alpha$ (for each limit ordinal λ).

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1) A class A is called *transitive* if every element of A is itself a class of elements of A—in other words, every element of A is a subclass of A.

2) For a detailed formalization of NBG, see Mendelson (2010) chapter 4. One interesting feature of NBG is that P_2, which consists of an infinite set of axioms—one for each formula $\Phi(X, X_1, \ldots, X_N)$—can be replaced by only finitely many axioms. This can be done in many different ways—one of which consists in asserting, for any classes A and B, the existence of the following seven classes:

B_1: The class of all ordered pairs $\langle x, y\rangle$ such that $x \in y$.

B_2: The intersection $A \cap B$.

B_3: The complement $\overline{A}$.

B_4: The class of all x such that $\langle x, y\rangle \in A$ for some y.

B_5: The class $A \times V$.

B_6: The class of all triples $\langle x, y, z\rangle$ such that $\langle y, z, x\rangle \in A$.

B_7: The calss of all triples $\langle x, y, z\rangle$ such that $\langle x, z, y\rangle \in A$.

So NBG is finitely axiomatizable. But the system (first-order) ZF is not (for this see Montague(1961)).

We say a set has *rank* if it is a member of some R_α, and if a set x does have rank, we define its rank to be the first ordinal α such that $x \in R_{\alpha+1}$ (or, what is the same thing, the first α such that $x \subseteq R_\alpha$). We define R_Ω be the union of all the R_α's. Then we have:

Proposition 2.1. *Each R_α is transitive.*

Proposition 2.2. *For each ordinal α, $R_\alpha \in R_{\alpha+1}$.*

Proposition 2.3. *For each α, $R_\alpha \subseteq R_{\alpha+1}$, but $R_\alpha \neq R_{\alpha+1}$.*

Let O be an *On*-sequence[3] $O_0, O_1, \ldots, O_\alpha, \ldots$ of subsets of a class A. We shall call O an *ordinal hierarchy* on A if the following three conditions hold.

(1) $O_0 = \varnothing$.
(2) For each ordinal α, $O_\alpha \subseteq O_{\alpha+1}$.
(3) For each limit ordinal λ, $O_\lambda = \bigcup_{\alpha<\lambda} O_\alpha$.

(We see that R_α sequence is an ordinal hierarchy.)

Suppose now that O is an ordinal hierarchy on A. We shall say that an element x of A has *O-rank* if it is a member of O_α, in which case we define its O-rank to be the first ordinal α such that $x \in O_{\alpha+1}$ (as we did for the R_α sequence).

Theorem 2.4. *For any ordinal hierarchy O on A the following conditions hold:*

- O_1: *$\alpha \leq \beta$ implies $O_\alpha \subseteq O_\beta$.*
- O_2: *If for each α O_α is a proper subset of $O_{\alpha+1}$. then for all ordinals α and β, $\alpha < \beta$ iff $O_\alpha \subset O_\beta$.*
- O_3: *For any $x \in A$ and any α, $x \in O_\alpha$ iff $x \in O_{\beta+1}$ for some $\beta < \alpha$.*
- O_4: *$x \in O_\alpha$ iff x has O-rank $< \alpha$.*
- O_5: *x has O-rank α iff $x \in O_{\alpha+1} - O_\alpha$.*
- O_6: *For any set S of elements with O-rank, there is an ordinal α such that $S \subseteq O_\alpha$.*

3 Mostowski-Shepherdson mappings

Given a relational system (A, R), for any element $a \in A$, by $a*$ we shall mean the class of all x in A such that $x\mathrm{R}a$ holds. We refer to the elements of $a*$ as the *components* of a.

If A is any transitive class, then in $(A, \in)$, for any $a \in A$, $a* = a$—more generally, whether A is transitive or not, $a* = a \cap A$ (and hence $a* \subseteq a$).

(A, R) is called a *proper* relational system if, for every $a \in A$, the class $a*$ is a set.

3) An *On*-sequence means a function whose domain is the class *On* of ordinal numbers.

(A, R) is called *extensional* if the following condition holds: for any elements x, y in A, $x* = y*$ implies $x = y$. (One sometimes says that A is extensional, meaning that $(A, \in)$ is extensional.)

For a relational system (A, R) and any subclass B of A, an element x of B is called an *initial* element of B if $x * \cap B = \varnothing$. We now call (A, R) *well founded* if every non-empty subclass of A contains an initial element.

By an isomorphism from (A_1, R_1) to (A_2, R_2) is meant an 1-1 function F from A_1 onto A_2 such that for any elements x and y of A_1, $x\mathrm{R}_1 y$ iff $F(x)\mathrm{R}_2 F(y)$.

Theorem 3.1 (Mostowski-Shepherdson mapping theorem). *Suppose (A, R) is a relational system that is proper, extensional, and well founded. Then:*

(1) there is the Mostowski-Shepherdson map F for (A, R) (a function on A such that $F(x) = F''(x*)$ for every x in A);
(2) F maps A onto some transitive class M;
(3) (A, R) is isomorphic to $(M, \in)$ under F (for any x and y in A, $x\mathrm{R}y$ iff $F(x) \in F(y)$, and F is 1-1).

Our main application of the theorem is the case that R is the $\in$-relation.

Since $(A, \in)$ is automatically proper and $x* = x \cap A$, we have:

Theorem 3.2. *For any well founded, extensional class A, there is a unique map F with domain A such that for every $x \in A$, $F(x) = F''(x \cap A)$, and this F is an $\in$-isomorphism from A onto a transitive class.*

Theorem 3.3. *For any well founded, extensional class A, if T is a transitive subclass of A, then for each element x of T, $F(x) = x$.*

4 Reflection principles

We shall consider the set of all formulas based on the primitive connectives $\neg, \wedge$, and $\exists$ and involving only one binary predicate P (which in application to set theory will be $\in$) besides $=$.

Given a class A, free(bound) occurrences of variables, atomic formula, an A-formula (a formula with constants in A[4]), a pure formula (an A-formula involving no constants), an A-sentence (an A-formula with no free occurrences of variables), a pure sentence, truth in the relational system (A, R) etc. are as familiar.

A pure formula $\Phi(x_1, \ldots, x_n)$ is said to be *satisfiable* in (A, R) if there are elements $a1, \ldots, a_n$ of A such that the sentence $\Phi(a_1, \ldots, a_n)$ is true in (A, R).

(A_1, R_1) and (A_2, R_2) are called *elementarily equivalent* if for every pure sentence X, X is true in (A_1, R_1) iff X is true in (A_2, R_2).

4) We use the elements of A as names of themselves.

(A_1, R_1) is called an *elementary subsystem* of (A_2, R_2) if $A_1 \subseteq A_2$ and $\mathrm{R}_2 \restriction A_1 = R_1$ and for every sentence X with constants in A_1, X is true in (A_1, R_1) iff X is true in (A_2, R_2).

A sentence X with constants in A is called *true over* A if it is true in $(A, \in)$ (we express this as $\models_A X$).

Given a set A and a subset B, we will say that B *reflects* A if for every sentence X whose constants are all in B, X is true over B iff X is true over A.

Given a set A, a subset B and a formula $\Phi(x_1, \ldots, x_n)$ whose constants are all in B, we say that B *reflects* A *with respect to the formula* Φ if for all constants $b_1, \ldots, b_n$ of B, the sentence $\Phi(b_1, \ldots, b_n)$ is true over B iff it is true over A.

We shall also say that B *completely reflects* A *with respect to* Φ if B reflects A with respect to all subformulas of Φ (which includes Φ itself).

Theorem 4.1 (The class version of the Tarski-Vaught theorem). *Suppose K is extensional, well orderable and A_0 is an infinite subset of K of cardinality $\mathfrak{c}$. If X is a pure sentence that is true over K, then A_0 can be extended to an extensional subset B of K of cardinality $\mathfrak{c}$ such that X is true over B.*

For any class K, a formula $\Phi(x)$ whose constants are all in K is said to *define over* K the class of all elements $k \in K$ such that $\Phi(k)$ is true over K. And a subclass K_1 of K is called *definable over* K (more completely, *first-order definable over* K) if it is defined over K by some formula $\Phi(x)$ whose constants are all in K. For any *set*[5)] A we let $\mathrm{Def}(A)$ be the set of all definable (over A) subsets of A.

We now combine the Mostowski-Shepherdson mapping theorem with the Tarski-Vaught theorem to obtain:

Theorem 4.2 (M.S.T.V.). *Let K be an extensional, well founded class that can be well ordered, and let X be a pure sentence that is true over K. Then any infinite transitive subset A of K is a subset of some transitive set T of the same cardinality as A, such that X is true over T.*

Proof. Assume the hypothesis. Suppose that A is an infinite transitive subset of K of cardinality $\mathfrak{c}$. By theorem 4.1 (Tarski-Vaught theorem), A is a subset of an extensional subset B of K of cardinality $\mathfrak{c}$ such that X is true over B. Since K is well founded, so is its subset B, and so B is extensional and well founded, and so by theorem 3.2 (Mostowski-Shepherdson) B is $\in$-isomorphic to a transitive set T. Obviously T also has cardinality $\mathfrak{c}$, and since T is $\in$-isomorphic to B and X is a pure sentence that is true over B, then X is also true over T. Finally, since A is transitive, then by theorem 3.3, the Mostowski-Shepherdson mapping from B onto T carries every element of A to itself, and so A is a subset of T. □

5) If A is a class, we cannot say that the class of all definable subsets of A is a set. But if A is a set, since $\mathrm{Def}(A) \subseteq P(A)$ and $P(A)$ is a set (A_6), $\mathrm{Def}(A)$ is indeed a set by A_2.

We let W be an *On*-sequence $w_0, w_1, \ldots, w_\alpha, \ldots,$ of sets satisfying the following two conditions.

(1) for $\alpha \leq \beta$, $w_\alpha \subseteq w_\beta$;
(2) for any limit ordinal λ, $w_\lambda = \bigcup_{\alpha<\lambda} w_\alpha$.

As an example, any ordinal hierarchy satisfies these conditions. We let W be the union of all the w_α's.

Theorem 4.3 (Montague-Levy). *For any pure formula Φ and any ordinal α, there exists a limit ordinal $\beta > \alpha$ such that w_β completely reflects W with respect to Φ.*

5 Constructible sets

For each ordinal α, we define the set L_α by the following transifinite recursion.

(1) $L_0 = \varnothing$,
(2) $L_{\alpha+1} = \text{Def}(L_\alpha)$,
(3) $L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha$ (for λ a limit ordinal).

We let L be the union of all the L_α—the elements of L are called the *constructible sets*. For any constructible set x, by its *order* is meant the first α such that $x \in L_{\alpha+1}$.

Now we list some key properties of the constructible sets.

C_0 Each L_α is transitive. (It follows that L is transitive.)
C_1 $L_\alpha \in L_{\alpha+1}$ and $L_\alpha \subseteq L_{\alpha+1}$.
C_2 The L_α-sequence is an ordinal hierarchy.
C_3 If $\alpha \leq \beta$, then $L_\alpha \subseteq L_\beta$.
C_4 For each α, $L_\alpha \subseteq R_\alpha$.
C_5 L is well founded.
C_6 $L_\alpha \neq L_{\alpha+1}$.
C_7 The set of all elements of order α is $L_{\alpha+1} - L_\alpha$.
C_8 Each L_α is constructible and of order α.
C_9 L_α is the set of all constructible sets of order $< \alpha$.
C_{10} If x is constructible, then every element of x is constructible and has order less than the order of x.
C_{11} Every set of constructible sets is a subset of some constructible set.
C_{12} For any constructible sets a and b, the following sets are constructible:

(1) $a \cap b$,
(2) $a \cup b$,
(3) $a - b$,
(4) $\{a, b\}$,

(5) $\langle a, b\rangle$.

C_{13} For any constructible sets $x_1, \ldots, x_n$, the set $\langle x_1, \ldots, x_n\rangle$ is constructible ($n \geq 2$).

C_{14} Every ordinal α is constructible, and of order α.

Consider a class K and a sentence X whose constants (if there are any) are all in K. If the truth value of X over K is the same as its real truth value (its truth value over V), then we say that X is *absolute* over K. Consider a formula $\Phi(x_1, \ldots, x_n)$ with no constants. We say that the formula Φ is absolute over K if for any elements $a_1, \ldots, a_n$ of K, the sentence $\Phi(a_1, \ldots, a_n)$ is absolute over K. We shall say that a class (or a property, or a relation) is absolute over K if it is defined over V by at least one formula with no constants that is absolute over K. Finally, we shall say that a formula or a class is *absolute* if it is absolute over every transitive class K.

We will eventually show that the very property of constructibility is absolute over L(i.e., that L is absolute over L).

To do this we first need to establish, of many classes and relations, that they are absolute.[6)]

Here are some of them.

(1) $x \in y$
(2) $x \subseteq y$
(3) $x = y$
(4) $x \in (y \cup z)$
(5) $x = (y \cup z)$
(6) $x = (y \cap z)$
(7) x is empty
(8) $x \cap y$ is empty
(9) $z = \{x, y\}$
(10) $y = \{x\}$
(11) $y = x+$
(12) $\varnothing \in x$
(13) $y+ \in x$
(14) $y = \bigcup x$
(15) x is transitive
(16) x is $\in$-connected
(17) x is an ordinal, Ordinal(x)
(18) x is a successor ordinal
(19) x is a limit ordinal
(20) x is a natural number, Num(x)
(21) For each natural number n, the conditions $x \in n$, $x = n$, $n \in x$
(22) $\{x, y\} \in z$
(23) $y \in \{x\}$, $y = \{x\}$, $\{x\} \in y$
(24) $z \in \langle x, y\rangle$, $z = \langle x, y\rangle$, $\langle x, y\rangle \in z$

6) This can be done by showing that every Δ_0-class and relations are absolute over any transitive class and that each of the 35 items can be defined over V by Δ_0-formulas. By a Δ_0-formula is meant any formula formed by the rules:

(1) Any atomic formula $x \in y$ or $x = y$ is Δ_0.
(2) If Φ, Ψ are Δ_0, so are $\neg\Phi$, $\Phi \wedge \Psi$ (and hence also $\Phi \vee \Psi$, $\Phi \supset \Psi$, $\Phi \equiv \Psi$).
(3) If Φ is Δ_0, then for any distinct variables x, y, $(\exists x \in y)\Phi$ and $(\forall x \in y)\Phi$ are Δ_0.

As for (16), a class A is called $\in$-*connected* if for any two distinct elements x, y of A either $x \in y$ or $y \in x$. As for (17), we note that x is an ordinal iff x is well founded, transitive, and $\in$-connected (Raphael Robinson's characterization of ordinals). We assume for the rest that V is well founded. Now using (17) we can prove C_{14}.

(25) For each $n \leq 2$, the relations
$z \in \langle x_1, \ldots, x_n \rangle$,
$z = \langle x_1, \ldots, x_n \rangle$,
$\langle x_1, \ldots, x_n \rangle \in z$

(26) x is an unordered pair

(27) x is an ordered pair

(28) x is a binary relation

(29) x is a function, Fun(x)

(30) y is the domain of x
($y = \text{Dom}(x)$)

(31) y is the range of x
($y = \text{Ran}(x)$)

(32) f is a function from x onto y

(33) f is a 1-1 function from x onto y

(34) f is a 1-1 function from x into y

(35) $z \subseteq x \times y$, $x \times y \subseteq z$, $z = x \times y$.

6 L is a well founded first-order universe.

We have defined a class K to be *swelled* if K contains with each element x, all subsets of x as well. We shall now define K to be *first-order swelled* if K contains with every element x, all subsets of x which are first-order definable over K.

We now define K to be a *first-order Zermelo-Fraenkel universe*—more briefly a *first-order universe*—if the following conditions hold:

F_1 K is transitive;

*F_2 K is first-order swelled;

F_3 $\varnothing \in K$;

F_4 For every x, y in K, $\{x, y\} \in K$;

F_5 For every x in K, $\bigcup x \in K$;

*F_6 For every x in K, $P(x) \cap K \in K$;

F_7 $\omega \in K$;

*F_8 For every function F from elements of K to elements of K, if F is first-order definable over K, then for every x in K, $F''(x)$ is in K.

In addition, we say K is a *well founded first-order Zermelo-Fraenkel universe* or a *well founded first-order universe* if, in addition to the conditions above, we also have:

F_9 Every non-empty member of K contains an initial member.

We have an * before those conditions which differ from the corresponding conditions defining a "genuine" Zermelo-Fraenkel universe (an NBG universe).

Now consider the following formal axioms of Zermelo-Fraenkel set theory (and the well foundedness axiom).

Ax1 [Extensionality] $\forall z\,(z \in x \equiv z \in y) \supset x = y$.

Ax2 [Separation] This is an axiom schema—an infinite collection of axioms—one for each formula $\Phi(x, y_1, \ldots, y_n)$ (n may be 0). $\forall x\,\exists z\,\forall y\,(y \in z \equiv (y \in x \wedge \Phi(y, y_1, \ldots, y_n)))$.

Ax3 [Axiom of the empty set] $\exists x \neg \exists y\,(y \in x)$.

Ax4 [Axiom of unordered pair] $\exists y \, \forall z \, (z \in y \equiv (z = x_1 \vee z = x_2))$.

Ax5 [Union axiom] $\forall x \, \exists y \, \forall z \, (z \in y \equiv \exists w \, (z \in w \wedge w \in x))$.

Ax6 [Power axiom] $\forall x \, \exists y \, \forall z \, (z \in y \equiv z \subseteq x)$.

Ax7 [Axiom of infinity][7)]
$\exists w \, (\varnothing \in w \wedge$
$\forall x \, (x \in w \supset \exists z \, (z \in w \wedge \forall v \, (v \in z \equiv v \in x \vee v = x)))$.

Ax8 [Axioms of substitution] All formulas of the form:
$\forall x \, \forall y \, \forall z \, (\Phi(x,y) \wedge \Phi(x,z) \supset y = z) \supset$
$\exists v \, \forall y \, (y \in v \equiv \exists x \, (x \in w \wedge \Phi(x,y)))$.

In addition to the ZF axioms, we generally will also want to consider the following.

Ax9 [Axiom of well foundedness]
$\forall a \, \{\exists y \, (y \in a) \supset \exists y \, [y \in a \wedge \forall z \, (z \in a \supset \neg(z \in y))]\}$.

We can see that for any transitive class K, K is a (well founded) first-order universe if and only if all the formal axioms of Zermelo-Fraenkel set theory (and the well foundedness axiom) are true over K. (For an open formula $\Phi(x_1, \ldots, x_n)$ with free variables, when we say that Φ is true over K we mean that the sentence $\forall x_1, \ldots, \forall x_n \; \Phi(x_1, \ldots, x_n)$ is true over K.) And we can prove the following:

Theorem 6.1. *A sufficient condition for a transitive class K to be a first-order universe is that every subset of K that is definable over K is an element of K.*

In application to the class L, theorem 6.1 means that to show L is a first-order universe, it suffices to show that every set of constructible sets that is definable over L is a constructible set. We can show this using the Montague-Levy reflection principle as follows.

Suppose a is a definable subset of L; let $\Phi(x, a_1, \ldots, a_n)$ be a formula, whose constants $a_1, \ldots, a_n$ are in L, that defines a over L. Let α_0 be an ordinal greater than the orders of all elements of a so that $a \subseteq L_{\alpha_0}$. Let $\alpha_1, \ldots, \alpha_n$ be the respective orders of $a_1, \ldots, a_n$. Let α be the greatest of the ordinals $\alpha_0, \alpha_1, \ldots, \alpha_n$.

We can now use the Montague-Levy reflection principle for the sequence L_0, L_1, $\ldots$, L_α, $\ldots$. By this principle there is a limit ordinal $\beta > \alpha$ (hence greater than the ordinals $\alpha_0, \alpha_1, \ldots, \alpha_n$) such that L_β reflects L with respect to the formula $\Phi(x, y_1, \ldots, y_n)$. Now all of the constants $\alpha_1, \ldots, \alpha_n$ are in L_β and the set a is a subset of L_β. Then, since L_β reflects L with respect to $\Phi(x, y_1, \ldots, y_n)$, it follows that $\Phi(x, a_1, \ldots, a_n)$ defines a over L_β. [$a = \{b \in L : \vDash_L \Phi(b, a_1, \ldots, a_n)\} = \{b \in L_\beta : \vDash_L \Phi(b, a_1, \ldots, a_n)\} = \{b \in L_\beta : \vDash_{L_\beta} \Phi(b, a_1, \ldots, a_n)\}$.] So $a \in L_{\beta+1}$ and so a is constructible, i.e., $a \in L$.

Moreover we have already seen in C_5 that L is well founded. Hence we have:

Theorem 6.2. *L is a well founded first-order universe.*

7) "$\varnothing \in w$" is simply an abbreviation of "$\exists y \, (y \in w \wedge \exists z \, (z \in y))$".

7 Constructibility is absolute over L, i.e., L is absolute over L.

We will say that a formula $\Phi(x_1, \ldots, x_n)$ with no constants is *absolute upwards over a class* K if for every elements $a_1, \ldots, a_n$ of K, if $\Phi(a_1, \ldots, a_n)$ is true over K, then it is true over V. We will also say that Φ is *absolute downwards over* K, if for every $a_1, \ldots, a_n$ of K, if $\Phi(a_1, \ldots, a_n)$ is true over V, then it is true over K. Thus Φ is absolute over K iff it is both upwards and downwards absolute over K. We shall say that a *relation* $\mathrm{R}(x_1, \ldots, x_n)$ is absolute upwards(downwards) over K if it is defined by at least one formula that is absolute upwards(downwards) over K. And we shall say that a formula or relation is *absolute upwards*(*downwards*) if it is absolute upwards (downwards) over all *transitive* classes.

And we inductively define the class of Σ-*formulas* by the following rules.

(1) Every Δ_0-formula is a Σ-formula.
(2) If Φ and Ψ are Σ-formulas, so are $\Phi \wedge \Psi$ and $\Phi \vee \Psi$.
(3) If Φ is a Σ-formula, so is $\exists x \Phi$.
(4) If Φ is a Σ-formula, so are $(\forall x \in y)\Phi$ and $(\exists x \in y)\Phi$.

By a Σ-*relation* we mean a relation that is defined over V by a Σ-formula.

Theorem 7.1. *All Σ-formulas (and hence all Σ-relations) are absolute upwards.*

Let us say a formula $\Phi(x, y)$ is *function-like* over a transitive class K if $\forall x\, \exists y\, [\Phi(x, y) \wedge \forall z(\Phi(x, z) \supset z = y]$ is true over K.

Theorem 7.2. *Suppose $\Phi(x, y)$ is a Σ-formula that is function-like over both the transitive class K and over V. Then $\Phi(x, y)$ is absolute over K.*

Proof. Let $a, b \in K$. If $\Phi(a, b)$ is true over K, since Φ is Σ, $\Phi(a, b)$ is true over V because Σ-formulas are absolute upwards and K is transitive. Now suppose $\Phi(a, b)$ is true over V. Since $a \in K$ and Φ is function-like over K, there must be some $c \in K$ such that $\Phi(a, c)$ is true over K. Since Φ is absolute upwards, $\Phi(a, c)$ is true over V, but so is $\Phi(a, b)$, so since Φ is function-like over V, $b = c$ is true over V. Thus b and c are the same set, hence $\Phi(a, b)$ is true over K. $\square$

Theorem 7.3. *Let F be a function defined on V and let F^* be the function on On defined from F by the following transfinite recursion:*

(1) $F^*(0) = \varnothing$,
(2) $F^*(\alpha + 1) = F(F^*(\alpha))$,
(3) $F^*(\lambda) = \bigcup_{\alpha<\lambda} F^*(\alpha)$ *(λ a limit ordinal).*

Then if F is Σ, F^* is also Σ.

Proof. Suppose F is Σ. Then using the fact that (1)–(35) of chapter 5 are Δ_0, we see that the relation $F^*(\alpha) = x$ is also Σ, because it can be written as:

$$\begin{aligned}&\text{Ordinal}(\alpha) \wedge \exists f[\text{Fun}(f) \wedge \text{Dom}(f) = \alpha + 1 \wedge f(\alpha) = x \wedge f(0)\\&\quad = \varnothing \wedge (\forall \beta \in \alpha + 1)(f(\beta + 1) = F(f(\beta))) \wedge (\forall \beta \in \alpha + 1)\\&\quad (\beta \text{ is a limit ordinal} \supset f(\beta) = \bigcup f''(\beta))].\end{aligned}$$ □

Now we apply theorem 7.3 to the case where F is the function Def (that assigns to each set x the set of subsets of x that are first-order definable over x). Note that in this case F^* assigns to each ordinal α the set L_α.

It can be shown that the relation $y = \text{Def}(x)$ is Σ.[8] So it follows that the function F^*—that assigns to each ordinal α the set L_α—is also Σ.

Suppose we extend this F^* to a function A, on all sets, by mapping non-ordinals to 0. We let $M(x, y)$ be a Σ-formula, fixed for the discussion, that defines over V the relation $L_x = y$; that is, $M(x, y)$ defines over V the graph of the function F^*. Also let $N(x, y)$ be the formula $[\text{Ordinal}(x) \wedge M(x, y)] \vee [\neg\text{Ordinal}(x) \wedge y = 0]$. Then $N(x, y)$ defines over V the function A. We have given an informal NBG proof of the fact that $N(x, y)$ defines over V a function, but the informal NBG proof could be formalized in ZF.[9] By theorem 6.2, L is a first-order universe, so it follows that $N(x, y)$ is function-like over L.[10] Clearly $N(x, y)$ is also function-like over V.[11] Then by theorem 7.2 $N(x, y)$ is absolute over L. It follows that $M(x, y)$ is also absolute over L, because it is equivalent to $\text{Ordinal}(x) \wedge N(x, y)$. We thus have established the following.

Theorem 7.4. *The formula $M(x, y)$ is Σ and absolute over L.*

We now let $L(x, y)$ be the Σ-formula $\exists z\, (M(y, z) \wedge x \in z)$. This formula defines over V the relation $x \in Ly$.

Theorem 7.5. *The formula $L(x, y)$ and hence also the relation $x \in Ly$ defined by it is absolute upwards—in fact it is Σ.*

Theorem 7.6. *The formula $L(x, y)$ and hence also the relation $x \in Ly$ defined by it is absolute over L.*

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8) For all the details of the proof using the coding of all formulas into sets, see Devlin(1984) pp. 56–71.

9) Our informal NBG proof could, of course, be formalized into a formal NBG proof. And NBG is a conservative extension of ZF (that is, if Φ is a sentence in the language of ZF, then Φ is a theorem of ZF if and only if it is a theorem of NBG) (for this, see Flannagan(1976)).

10) Since $N(x, y)$ defines a function, $\vdash_{\text{NBG}} \forall x\, \exists y\, [N(x, y) \wedge \forall z\, (N(x, z) \supset z = y)]$, and so $\vdash_{\text{ZF}} \forall x\, \exists y\, [N(x, y) \wedge \forall z\, (N(x, z) \supset z = y)]$. But all axioms of ZF are true over L. So $\forall x\, \exists y\, [N(x, y) \wedge \forall z\, (N(x, z) \supset z = y)]$ is true over L.

11) If both $\text{Ordinal}(x) \wedge M(x, y)$ and $\text{Ordinal}(x) \wedge M(x, z)$ are true in V, then $L_x = y$ and $L_x = z$, so $y = z$.
And if both $\neg\text{Ordinal}(x) \wedge y = 0$ and $\neg\text{Ordinal}(x) \wedge z = 0$ are true in V, then $y = 0 = z$.

Proof. By theorem 7.5 we must now show that it is absolute downwards over L. Take $a, b \in L$ such that $\vDash_V L(a,b) \Rightarrow b$ is really an ordinal and $a \in L_b \Rightarrow$ Let $c = L_b$. Then $a \in c$ and $c \in L \Rightarrow \vDash_L a \in c$. Also $\vDash_V M(b,c) \Rightarrow$ Since $M(x,y)$ is absolute over L and $b, c \in L$, $\vDash_L M(b,c)$. So $\vDash_L M(b,c) \wedge a \in c \Rightarrow \vDash_L \exists y\, M(b,y) \wedge a \in y \Rightarrow \vDash_L L(a,b)$. □

We now let Const(x) be the formula $\exists y\, L(x,y)$.[12] It obviously defines over V the property of being a constructible set—that is, it defines the class L over V $[L = \{x \in V : \vDash_V \text{Const}(x)\}]$.

Theorem 7.7. *The formula Const(x) is absolute over L. (So the property of being a constructible set is absolute over L.)*

Proof. Since the formula Const(x) is Σ and L is transitive, it is, by theorem 7.1, absolute upwards over L. We must now show that it is absolute downwards over L, i.e., that for any $a \in L$, the sentence Const(a) (which of course is true over V) is true over L.

Well, suppose $a \in L \Rightarrow a \in L\alpha$ for some ordinal $\alpha \Rightarrow \vDash_V L(a,\alpha) \Rightarrow$ since $\alpha \in L$ (C14), by theorem 7.6 $\vDash_L L(a,\alpha) \Rightarrow \vDash_L \exists y\, L(a,y) \Rightarrow$ that is, $\vDash_L$ Const(a). □

Theorem 7.8. *The axiom of constructibility—the sentence $\forall x\,$Const(x)—is true over the first-order universe L.*

Proof. For any $a \in L$, since $\vDash_V$ Const(a), by theorem 7.7 $\vDash_L$ Const(a). So $\vDash_L \forall x$ Const(x). □

Theorem 7.9. *Let T be a transitive class such that the axiom of constructibility is true over T. Then every element of T is constructible and its order is in T.*

Proof. Let a be any element of T. Then $\vDash_T$ Const(a), i.e., $\vDash_T \exists y\, L(a,y)$. Hence for some $b \in T$, $\vDash_T L(a,b)$. But T is transitive and $L(x,y)$ is absolute upwards (theorem 7.5), hence $\vDash_V L(a,b)$, which means that b is an ordinal and $a \in L_b$. Since $a \in L_b$, then by C_9, the order of a is $< b$. And since $b \in T$ and T is transitive, T contains all ordinals $< b$, and so T contains the order of a. □

8 Constructibility and the GCH

For any infinite cardinal c, by $c*$ we mean the next cardinal after c. (If c is the cardinal $\aleph_\alpha$, then $c*$ is the cardinal $\aleph_{\alpha+1}$.)

By Cantor's theorem, we know that $c* \leq |P(c)|$, the cardinality of $P(c)$.

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12) $\vDash_V M(y,z) \Leftrightarrow y$ is an ordinal and $Ly = z$. $L(x,y) \equiv_{\text{df}} \exists z\,(M(y,z) \wedge x \in z)$. So $\vDash_V L(x,y) \Leftrightarrow y$ is an ordinal and $x \in Ly$. Const$(x) \equiv_{\text{df}} \exists y\, L(x,y)$. So $\vDash_V$ Const$(x) \Leftrightarrow x \in Ly$ for some ordinal $y \Leftrightarrow x$ is constructible. $\vDash_V \forall x\,$Const$(x) \Leftrightarrow$ every set is constructible $\Leftrightarrow V = L$.

Cantor's generalized continuum hypothesis(GCH) is that for any infinite cardinal c, $|P(c)| = c*$. The continuum hypothesis is the special case of this where c is the cardinal $\aleph_0 (= \omega)$.

Now we will prove Gödel's remarkable result:

Theorem 8.1. *For any infinite cardinal c, every constructible subset of L_c is an element of L_{c*}.*

Proof. We use the M.S.T.V. theorem. In the theorem, we take K to be the class L and X to be the sentence $\forall x\, \mathrm{Const}(x)$ (the axiom of constructibility). We proved that the axiom of constructibility is true over L (theorem 7.8). $(L, \in)$ is extensional[13] and well founded. And it can be proved that L is well orderable.[14]

Now, let m be any constructible subset of L_c (c an infinite cardinal). The set $L_c \cup \{m\}$ is of cardinality c[15] and is transitive(since $m \subseteq L_c$). Then by the M.S.T.V. theorem, $L_c \cup \{m\}$ is a subset of a transitive set T of cardinality c such that $\forall x\, \mathrm{Const}(x)$ is true over T. Of course m is an element of T and since $\forall x\, \mathrm{Const}(x)$ is true over T, then by theorem 7.9, the order α of m lies in T. But T is transitive, hence $\alpha \subseteq T$. Then, since T has cardinality c, α must have cardinality $\leq c$, hence $\alpha < c*$. And so by C_9, $m \in L_{c*}$. □

Corollary 8.2. $V = L$ *implies GCH. (If all sets are constructible, then the generalized continuum hypothesis is true.)*

Proof. Let c be an infinite cardinal. Since $c \subseteq L_c$, every constructible subset of c is a constructible subset of L_c. Then by theorem 8.1 every constructible subset of c is an element of L_{c*}. Then, since $|L_{c*}| = c*$, there are at most $c*$ constructible subsets of c. So if all sets are constructible, then there are at most $c*$ subsets of c (that is, $|P(c)| \leq c*$). Since $c* \leq |P(c)|$ by Cantor's theorem, it follows that if all sets are constructible then $|P(c)| = c*$. □

We have proved in informal NBG that $V = L$ implies GCH. In fact[16] the sentence $\forall x\, \mathrm{Const}(x) \supset \mathrm{GCH}$ is a theorem of ZF. That is, $\vdash_{\mathrm{ZF}} \forall x\, \mathrm{Const}(x) \supset \mathrm{GCH}$. So, since

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13) For $x, y \in L$, if $\forall z \in L\, (z \in x \equiv z \in y)$, then, since L is transitive, $\forall z \in V\, (z \in x \equiv z \in y)$ and so by P_1 $x = y$.

14) see Jech p. 188–190.

15) We prove that for $\alpha \geq \omega$, $|L_\alpha| = |\alpha|$. (See Devlin p. 59–60.)
If $\beta \in \alpha \Rightarrow$ by C_{14}, the order of $\beta = \beta < \alpha \Rightarrow$ by C_9, $\beta \in L_\alpha$. So we have $|\alpha| \leq |L_\alpha|$ for all α. By induction on $\alpha \geq \omega$ we prove that $|L_\alpha| \leq |\alpha|$ for $\alpha \geq \omega$. For $\alpha = \omega$ this holds, since $|L_\omega| = |R_\omega| = \omega$. Suppose next that for a limit ordinal λ, $|L_\alpha| \leq |\alpha|$ for $\alpha < \lambda$. Then $|L_\lambda| = |\bigcup_{\alpha<\lambda} L_\alpha| \leq \sum_{\alpha<\lambda} |L_\alpha| \leq \sum_{\alpha<\lambda} |\alpha| = |\lambda|$. Finally, suppose that $|L_\alpha| \leq |\alpha|$. We prove that $|L_{\alpha+1}| \leq |\alpha| (= |\alpha + 1|)$. Well, since our basic language is countable, the set of L_α-formulas is easily seen to have cardinality $|L_\alpha|$. But this at once implies that $|L_{\alpha+1}| = |\mathrm{Def}(L_\alpha)| \leq |L_\alpha| \leq |\alpha|$, and we are done.

16) See footnote 9).

L is a first-order universe, $\models_L \forall x\,\mathrm{Const}(x) \supset \mathrm{GCH}$. So, since by theorem 7.8 $\models_L \forall x\,\mathrm{Const}(x)$, $\models_L \mathrm{GCH}$.

Therefore (assuming that there exists an NBG universe, "genuine" Zermelo-Fraenkel universe, (V) in the first place) there exists a first-order universe L in which GCH is true. Now if the sentence ¬GCH were provable in ZF, it would be true in all first-order universe, contrary to the fact that it is *not* true in L. Thus the negation of the generalized continuum hypothesis is not provable in ZF (assuming the existence of an NBG universe).

By Sierpiński's theorem[17)] the generalized continuum hypothesis implies the axiom of choice. The proof of this is formalizable in ZF—i.e.,$\vdash_{\mathrm{ZF}} \mathrm{GCH} \supset \mathrm{AC}$.

Hence if $\vdash_{\mathrm{ZF}} \neg\mathrm{AC}$, then $\vdash_{\mathrm{ZF}} \neg\mathrm{GCH}$, which is not the case. That is, (assuming the existence of an NBG universe) the negation of axiom of choice also is not provable in ZF.[18)]

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17) Sierpiński (1947).

18) The model-theoretic (or semantic) argument above can be given in a purely syntactic form as follows. Let us write $\exists x \in L(\cdots)$ to abbreviate $\exists x\,(\mathrm{Const}(x) \wedge (\cdots))$. Now, given any formula Φ, we define the formula Φ^L (called the *relativization of* Φ *to* L) to be the formula obtained from Φ by replacing every occurrence of "$\exists x$" by "$\exists x \in L$". Thus:

(0) $(x \in y)^L = x \in y$ and $(x = y)^L = (x = y)$.
(1) $(\neg\Phi)^L = \neg(\Phi^L)$.
(2) $(\Phi \wedge \Psi)^L = \Phi^L \wedge \Psi^L$.
(3) $(\exists x\Phi)^L = (\exists x \in L)\Phi^L$ (i.e., $\exists x\,(\mathrm{Const}(x) \wedge \Phi^L)$).

To say that a sentence X is true over L is equivalent to saying that X^L is true over V.
Since $\models_L \forall x\,\mathrm{Const}(x)$, $\models_V (\forall x\,\mathrm{Const}(x))^L$. This proof can be formalized in ZF—i.e., $\vdash_{\mathrm{ZF}} (\forall x\,\mathrm{Const}(x))^L$.
Similarly for every axiom X of ZF, since X is true over the first order universe L, $\vdash_{\mathrm{ZF}} X^L$.
From this it follows that for every theorem X of ZF, $\vdash_{\mathrm{ZF}} X^L$.
We know that $\vdash_{\mathrm{ZF}} \forall x\,\mathrm{Const}(x) \supset \mathrm{GCH}$.
It follows that $\vdash_{\mathrm{ZF}} (\forall x\,\mathrm{Const}(x) \supset \mathrm{GCH})^L \Rightarrow \vdash_{\mathrm{ZF}} (\forall x\,\mathrm{Const}(x))^L \supset \mathrm{GCH}^L \Rightarrow \vdash_{\mathrm{ZF}} \mathrm{GCH}^L$.
Then, if ¬GCH were provable in ZF, the sentence $(\neg\mathrm{GCH})^L$ would be provable in ZF, but this is the sentence $\neg(\mathrm{GCH})^L$, and so ZF would be inconsistent. And if ¬AC were provable in ZF, since $\vdash_{\mathrm{ZF}} \mathrm{GCH} \supset \mathrm{AC}$, it would follow that $\vdash_{\mathrm{ZF}} \neg\mathrm{GCH}$.
Thus we have the following:
GCH and AC are consistent relative to ZF. (A sentence X is called *relatively consistent* with the axioms of a system if it is consistent with the axioms, provided the axiom themselves are consistent. Actually the consistency of AC relative to ZF can be proved by much more elementary means. See Devlin p. 71–76.)
Since NBG is a conservative extension of ZF, it immediately follows that GCH and AC are also consistent relative to NBG.

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