

Default Bayesian testing for the bivariate normal correlation coefficient

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Abstract

This article deals with the problem of testing for the correlation coefficient in the bivariate normal distribution. We propose Bayesian hypothesis testing procedures for the bivariate normal correlation coefficient under the noninformative prior. The noninformative priors are usually improper which yields a calibration problem that makes the Bayes factor to be defined up to a multiplicative constant. So we propose the default Bayesian hypothesis testing procedures based on the fractional Bayes factor and the intrinsic Bayes factors under the reference priors. A simulation study and an example are provided.

Keywords: Bivariate normal distribution, correlation coefficient, fractional Bayes factor, intrinsic Bayes factor, reference prior.

1. Introduction

Let (X, Y) be a random vector distributed as a bivariate normal distribution $\mathcal{BN}(\mu_1, \mu_2, \sigma_1, \sigma_2, \rho)$ with means μ_1 and μ_2 , variances σ_1^2 and σ_2^2 , and correlation coefficient ρ . The probability density function of (X, Y) is given by

$$f(x, y | \mu_1, \mu_2, \sigma_1, \sigma_2, \rho) = \frac{1}{2\pi} \sigma_1^{-1} \sigma_2^{-1} (1 - \rho^2)^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2(1 - \rho^2)} \right. \\ \left. \times \left[\frac{(x - \mu_1)^2}{\sigma_1^2} - 2\rho \frac{(x - \mu_1)(y - \mu_2)}{\sigma_1 \sigma_2} + \frac{(y - \mu_2)^2}{\sigma_2^2} \right] \right\}. \quad (1.1)$$

respectively. The present paper focuses on Bayesian hypothesis testing for the correlation coefficient in the bivariate normal distribution.

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Inference concerning the correlation coefficient of two random variables from the bivariate normal distribution has been investigated by some authors (Fisher, 1915; Hotelling, 1953; Ruben, 1966). Recently, Sun and Wong (2007) derived the confidence intervals for the correlation coefficient based the likelihood-based higher-order asymptotic method, and showed that the confidence interval based on the modified signed log-likelihood ratio method yields very good results in terms of coverage probabilities.

When the null hypothesis is $H_0 : \rho = 0$, one can use t statistic for testing the null hypothesis. However, when the null hypothesis is $H_0 : \rho = \rho_0$, where $\rho_0 \neq 0$, one can test H_0 by using the statistic

$$W = \frac{1}{2} \log \left\{ \frac{1+R}{1-R} \right\},$$

where R is a sample correlation coefficient. We know that W converges standard normal distribution when the sample size is large. This test is an approximate hypothesis test. When the sample size is small, it may cause a problem in test or interval estimation.

Bayesian hypothesis testing provides the posterior probabilities of hypotheses under consideration. This gives exact posterior probabilities of hypotheses of being true when the prior probabilities of hypotheses of being true are given. Even in small sample, exact posterior probabilities can be calculated in Bayesian hypothesis testing based on Bayes factor. So, Bayesian hypothesis testing based on Bayes factor has merits over the test using W .

In Bayesian model selection or testing problem, the Bayes factors under proper priors or informative priors have been very successful. However, limited information and time constraints often require the use of noninformative priors. Since noninformative priors such as Jeffreys' prior or reference prior (Berger and Bernardo, 1989, 1992) are typically improper so that such priors are only defined up to arbitrary constants which affect the values of Bayes factors. Spiegelhalter and Smith (1982), O'Hagan (1995) and Berger and Pericchi (1996) have made efforts to compensate for that arbitrariness.

Spiegelhalter and Smith (1982) used the device of imaginary training sample in the context of linear model comparisons to choose the arbitrary constants. But the choice of imaginary training sample depends on the models under comparison, and so there is no guarantee that the Bayes factor of Spiegelhalter and Smith (1982) is coherent for multiple model comparisons. Berger and Pericchi (1996) introduced the intrinsic Bayes factor using a data-splitting idea, which would eliminate the arbitrariness of improper prior. O'Hagan (1995) proposed the fractional Bayes factor. For removing the arbitrariness he utilized a portion of the likelihood with a so-called the fraction b ($0 < b < 1$). These approaches have shown to be quite useful in many statistical areas (Kang *et al.*, 2006, 2008, 2010). An excellent exposition of the objective Bayesian method to model selection is Berger and Pericchi (2001).

For the noninformative priors of the parameters of bivariate normal distribution, Berger and Sun (2008) have considered reference priors and quantile matching priors for a variety of parameters. One important example of theirs is the bivariate normal correlation coefficient. However, these authors have not considered other matching criterion. So Ghosh *et al.* (2010) derived the matching priors for the correlation coefficient based various matching criteria, namely, quantile matching, highest posterior density matching, and matching via inversion of test statistics. However the problem of Bayesian hypothesis testing for the correlation coefficient was not considered. Therefore there is a necessity for developing objective Bayesian hypothesis testing procedure under noninformative priors.

In this paper, we propose the objective Bayesian hypothesis testing procedures for the correlation coefficient in bivariate normal distribution based on the Bayes factors. The outline of the remaining sections is as follows. In Section 2, we introduce the Bayesian hypothesis testing based on the Bayes factors. In Section 3, under the reference priors, we provide the Bayesian hypothesis testing procedures based on the fractional Bayes factor and the intrinsic Bayes factors. In Section 4, simulation study and an example are given.

2. Intrinsic and fractional Bayes factors

Suppose that hypotheses H_1, H_2, \dots, H_q are under consideration, with the data $\mathbf{x} = (x_1, x_2, \dots, x_n)$ having probability density function $f_i(\mathbf{x}|\theta_i)$ under hypothesis H_i . The parameter vector θ_i is unknown. Let $\pi_i(\theta_i)$ be the prior distributions of hypothesis H_i , and let p_i be the prior probability of hypothesis $H_i, i = 1, 2, \dots, q$. Then the posterior probability that the hypothesis H_i is true is

$$P(H_i|\mathbf{x}) = \left(\sum_{j=1}^q \frac{p_j}{p_i} \cdot B_{ji} \right)^{-1}, \tag{2.1}$$

where B_{ji} is the Bayes factor of hypothesis H_j to hypothesis H_i defined by

$$B_{ji} = \frac{\int f_j(\mathbf{x}|\theta_j)\pi_j(\theta_j)d\theta_j}{\int f_i(\mathbf{x}|\theta_i)\pi_i(\theta_i)d\theta_i} = \frac{m_j(\mathbf{x})}{m_i(\mathbf{x})}. \tag{2.2}$$

The B_{ji} is interpreted as the comparative support of the data for H_j versus H_i . The computation of B_{ji} needs specification of the prior distribution $\pi_i(\theta_i)$ and $\pi_j(\theta_j)$. Often in Bayesian analysis, one can use noninformative prior π_i^N . Common choices for this are the uniform prior, Jeffreys' prior and the reference prior. The noninformative prior π_i^N is typically improper. Hence the use of noninformative prior π_i^N in (2.2) causes the B_{ji} to contain unspecified constants. To solve this problem, Berger and Pericchi (1996) proposed the intrinsic Bayes factor, and O'Hagan (1995) proposed the fractional Bayes factor.

One solution to this indeterminacy problem is to use part of the data as a training sample. Let $\mathbf{x}(l)$ denote the part of the data to be so used and let $\mathbf{x}(-l)$ be the remainder of the data, such that

$$0 < m_i^N(\mathbf{x}(l)) < \infty, i = 1, \dots, q. \tag{2.3}$$

In view of (2.3), the posterior $\pi_i^N(\theta_i|\mathbf{x}(l))$ is well defined. Now, consider the Bayes factor $B_{ji}(l)$ using $\pi_i^N(\theta_i|\mathbf{x}(l))$ as a prior based on the remainder of the data $\mathbf{x}(-l)$:

$$B_{ji}(l) = \frac{\int f(\mathbf{x}(-l)|\theta_j, \mathbf{x}(l))\pi_j^N(\theta_j|\mathbf{x}(l))d\theta_j}{\int f(\mathbf{x}(-l)|\theta_i, \mathbf{x}(l))\pi_i^N(\theta_i|\mathbf{x}(l))d\theta_i} = B_{ji}^N \cdot B_{ij}^N(\mathbf{x}(l)) \tag{2.4}$$

where

$$B_{ji}^N = B_{ji}^N(\mathbf{x}) = \frac{m_j^N(\mathbf{x})}{m_i^N(\mathbf{x})}$$

and

$$B_{ij}^N(\mathbf{x}(l)) = \frac{m_i^N(\mathbf{x}(l))}{m_j^N(\mathbf{x}(l))}$$

are the Bayes factors that would be obtained for the full data \mathbf{x} and training samples $\mathbf{x}(l)$, respectively.

Berger and Pericchi (1996) proposed the use of a minimal training sample to compute $B_{ij}^N(\mathbf{x}(l))$. Then, an average over all the possible minimal training samples contained in the sample is computed. Thus the arithmetic intrinsic Bayes factor (AIBF) of H_j to H_i is

$$B_{ji}^{AI} = B_{ji}^N \times \frac{1}{L} \sum_{l=1}^L B_{ij}^N(\mathbf{x}(l)), \quad (2.5)$$

where L is the number of all possible minimal training samples. Also the median intrinsic Bayes factor (MIBF) by Berger and Pericchi (1998) of H_j to H_i is

$$B_{ji}^{MI} = B_{ji}^N \times ME[B_{ij}^N(\mathbf{x}(l))], \quad (2.6)$$

where ME indicates the median for all the training sample Bayes factors.

Therefore we can also calculate the posterior probability of H_i using (2.1), where B_{ji} is replaced by B_{ji}^{AI} and B_{ji}^{MI} from (2.5) and (2.6), respectively.

The fractional Bayes factor (O'Hagan, 1995) is based on a similar intuition to that behind the intrinsic Bayes factor but, instead of using part of the data to turn noninformative priors into proper priors, it uses a fraction, b , of each likelihood function, $L(\theta_i) = f_i(\mathbf{x}|\theta_i)$, with the remaining $1 - b$ fraction of the likelihood used for model discrimination. Then the fractional Bayes factor (FBF) of hypothesis H_j versus hypothesis H_i is

$$B_{ji}^F = B_{ji}^N \cdot \frac{\int L^b(\mathbf{x}|\theta_i)\pi_i^N(\theta_i)d\theta_i}{\int L^b(\mathbf{x}|\theta_j)\pi_j^N(\theta_j)d\theta_j} = B_{ji}^N \cdot \frac{m_i^b(\mathbf{x})}{m_j^b(\mathbf{x})}. \quad (2.7)$$

O'Hagan (1995) proposed three ways for the choice of the fraction b . One common choice of b is $b = m/n$, where m is the size of the minimal training sample, assuming that this number is uniquely defined.

3. Bayesian hypothesis testing procedures

Let $(X_i, Y_i), i = 1, \dots, n$ denote observations from the bivariate normal distribution $\mathcal{BN}(\mu_1, \mu_2, \sigma_1, \sigma_2, \rho)$ with means μ_1 and μ_2 , variances σ_1^2 and σ_2^2 , and correlation coefficient ρ . Then the joint probability density function of $(X_i, Y_i), i = 1, \dots, n$ is given by

$$f(\mathbf{x}, \mathbf{y}|\mu_1, \mu_2, \sigma_1, \sigma_2, \rho) = \left(\frac{1}{2\pi}\right)^n \sigma_1^{-n} \sigma_2^{-n} (1 - \rho^2)^{-\frac{n}{2}} \exp \left\{ -\frac{1}{2(1 - \rho^2)} \left(\sum_{i=1}^n \frac{(x_i - \mu_1)^2}{\sigma_1^2} - 2\rho \sum_{i=1}^n \frac{(x_i - \mu_1)(y_i - \mu_2)}{\sigma_1 \sigma_2} + \sum_{i=1}^n \frac{(y_i - \mu_2)^2}{\sigma_2^2} \right) \right\}, \quad (3.1)$$

where $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{y} = (y_1, \dots, y_n)$, $-\infty < \mu_1 < \infty$, $-\infty < \mu_2 < \infty$, $\sigma_1 > 0$, $\sigma_2 > 0$ and $|\rho| < 1$. We are interested in testing the hypotheses $H_1 : \rho = \rho_0$ versus $H_2 : \rho \neq \rho_0$ based on the fractional Bayes factor and the intrinsic Bayes factors.

3.1. Bayesian hypothesis testing procedure based on the fractional Bayes factor

From (3.1) the likelihood function under the hypothesis $H_1 : \rho = \rho_0$ is

$$L_1(\mu_1, \mu_2, \sigma_1, \sigma_2, \rho | \mathbf{x}, \mathbf{y}) = (2\pi)^{-n} \sigma_1^{-n} \sigma_2^{-n} (1 - \rho_0^2)^{-\frac{n}{2}} \exp \left\{ -\frac{1}{2(1 - \rho_0^2)} \left(\sum_{i=1}^n \frac{(x_i - \mu_1)^2}{\sigma_1^2} - 2\rho_0 \sum_{i=1}^n \frac{(x_i - \mu_1)(y_i - \mu_2)}{\sigma_1 \sigma_2} + \sum_{i=1}^n \frac{(y_i - \mu_2)^2}{\sigma_2^2} \right) \right\}. \quad (3.2)$$

And under the hypothesis H_1 , the reference prior for $(\mu_1, \mu_2, \sigma_1, \sigma_2)$ is

$$\pi_1^N(\mu_1, \mu_2, \sigma_1, \sigma_2) \propto \sigma_1^{-1} \sigma_2^{-1}. \quad (3.3)$$

Then from the likelihood (3.2) and the reference prior (3.3), the element $m_1^b(\mathbf{x}, \mathbf{y})$ of the FBF under H_1 is given by

$$\begin{aligned} m_1^b(\mathbf{x}, \mathbf{y}) &= \int_0^\infty \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty L_1^b(\mu_1, \mu_2, \sigma_1, \sigma_2 | \mathbf{x}, \mathbf{y}) \pi_1^N(\mu_1, \mu_2, \sigma_1, \sigma_2) d\mu_1 d\mu_2 d\sigma_1 d\sigma_2 \\ &= \int_0^\infty \int_0^\infty \frac{b}{n} \left(\frac{1}{2\pi} \right)^{bn-1} \sigma_1^{-bn} \sigma_2^{-bn} (1 - \rho_0^2)^{-\frac{bn-1}{2}} \\ &\quad \times \exp \left\{ -\frac{b}{2(1 - \rho_0^2)} \left(\frac{S_1}{\sigma_1^2} - 2\frac{\rho_0}{\sigma_1 \sigma_2} S_{12} + \frac{S_2}{\sigma_2^2} \right) \right\} d\sigma_1 d\sigma_2, \end{aligned} \quad (3.4)$$

where $\bar{x} = \sum_{i=1}^n x_i/n$, $S_1 = \sum_{i=1}^n (x_i - \bar{x})^2$, $\bar{y} = \sum_{i=1}^n y_i/n$, $S_2 = \sum_{i=1}^n (y_i - \bar{y})^2$ and $S_{12} = \sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})$. Let $U = \frac{\sigma_1^2 \sigma_2^2}{S_1 S_2}$ and $V = \frac{\sigma_1^2 / S_1}{\sigma_2^2 / S_2}$. Then integrating with respect to U in (3.4), we can get

$$\begin{aligned} m_1^b(\mathbf{x}, \mathbf{y}) &= \int_0^\infty \frac{b\Gamma[bn-1]}{4n} \left(\frac{1}{2\pi} \right)^{bn-1} S_1^{-\frac{bn-1}{2}} S_2^{-\frac{bn-1}{2}} (1 - \rho_0^2)^{-\frac{bn-1}{2}} V^{-1} \\ &\quad \times \left[\frac{b(V^{-\frac{1}{2}} + V^{\frac{1}{2}} - 2\rho_0 r)}{2(1 - \rho_0^2)} \right]^{-(bn-1)} dV, \end{aligned} \quad (3.5)$$

where $r = S_{12}/(S_1^{\frac{1}{2}} S_2^{\frac{1}{2}})$. For the hypothesis $H_2 : \rho \neq \rho_0$, the reference prior for $(\rho, \mu_1, \mu_2, \sigma_1, \sigma_2)$ is

$$\pi^N(\rho, \mu_1, \mu_2, \sigma_1, \sigma_2) \propto (1 - \rho^2)^{-1} \sigma_1^{-1} \sigma_2^{-1}. \quad (3.6)$$

This reference prior derived by Sun and Berger (2008), and satisfy the second order matching criterion (Ghosh *et al.*, 2010). The likelihood function under the hypothesis H_2 is

$$L_2(\rho, \mu_1, \mu_2, \sigma_1, \sigma_2 | \mathbf{x}, \mathbf{y}) = (2\pi)^{-n} \sigma_1^{-n} \sigma_2^{-n} (1 - \rho^2)^{-\frac{n}{2}} \exp \left\{ -\frac{1}{2(1 - \rho^2)} \left(\sum_{i=1}^n \frac{(x_i - \mu_1)^2}{\sigma_1^2} - 2\rho \sum_{i=1}^n \frac{(x_i - \mu_1)(y_i - \mu_2)}{\sigma_1 \sigma_2} + \sum_{i=1}^n \frac{(y_i - \mu_2)^2}{\sigma_2^2} \right) \right\}. \quad (3.7)$$

Thus from the likelihood (3.7) and the reference prior (3.6), the element $m_2^b(\mathbf{x}, \mathbf{y})$ of FBF under H_2 is given as follows.

$$\begin{aligned}
 & m_2^b(\mathbf{x}, \mathbf{y}) \\
 &= \int_{-1}^1 \int_0^\infty \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty L_2^b(\rho, \mu_1, \mu_2, \sigma_1, \sigma_2 | \mathbf{x}, \mathbf{y}) \pi_2^N(\rho, \mu_1, \mu_2, \sigma_1, \sigma_2) d\mu_1 d\mu_2 d\sigma_1 d\sigma_2 d\rho \\
 &= \int_{-1}^1 \int_0^\infty \int_0^\infty \frac{b}{n} \left(\frac{1}{2\pi}\right)^{bn-1} \sigma_1^{-bn} \sigma_2^{-bn} (1-\rho^2)^{-\frac{bn+1}{2}} \\
 &\times \exp\left\{-\frac{b}{2(1-\rho^2)}\left(\frac{S_1}{\sigma_1^2} - 2\frac{\rho}{\sigma_1\sigma_2}S_{12} + \frac{S_2}{\sigma_2^2}\right)\right\} d\sigma_1 d\sigma_2 d\rho. \tag{3.8}
 \end{aligned}$$

Let $U = \frac{\sigma_1^2\sigma_2^2}{S_1S_2}$ and $V = \frac{\sigma_1^2/S_1}{\sigma_2^2/S_2}$. Then integrating with respect to U in (3.8), we can get

$$\begin{aligned}
 m_2^b(\mathbf{x}, \mathbf{y}) &= \int_{-1}^1 \int_0^\infty \frac{b\Gamma[bn-1]}{4n} \left(\frac{1}{2\pi}\right)^{bn-1} S_1^{-\frac{bn-1}{2}} S_2^{-\frac{bn-1}{2}} (1-\rho^2)^{-\frac{bn+1}{2}} V^{-1} \\
 &\times \left[\frac{b(V^{-\frac{1}{2}} + V^{\frac{1}{2}} - 2r\rho)}{2(1-\rho^2)}\right]^{-(bn-1)} dV d\rho. \tag{3.9}
 \end{aligned}$$

Combining (3.2) and (3.3), $m_1^N(\mathbf{x}, \mathbf{y})$ is $m_1^b(\mathbf{x}, \mathbf{y})$ in (3.5) with $b = 1$. That is

$$m_1^N(\mathbf{x}, \mathbf{y}) = m_1^1(\mathbf{x}, \mathbf{y}).$$

Similarly, $m_2^N(\mathbf{x}, \mathbf{y})$ is $m_2^b(\mathbf{x}, \mathbf{y})$ in (3.8) with $b = 1$. Therefore, the element B_{21}^N of FBF is given by

$$B_{21}^N = \frac{m_2^N(\mathbf{x}, \mathbf{y})}{m_1^N(\mathbf{x}, \mathbf{y})} = \frac{S_2(\mathbf{x}, \mathbf{y})}{S_1(\mathbf{x}, \mathbf{y})}, \tag{3.10}$$

where

$$S_1(\mathbf{x}, \mathbf{y}) = \int_0^\infty (1-\rho_0^2)^{\frac{n-1}{2}} V^{-1} \left[V^{-\frac{1}{2}} + V^{\frac{1}{2}} - 2\rho_0r\right]^{-(n-1)} dV$$

and

$$S_2(\mathbf{x}, \mathbf{y}) = \int_{-1}^1 \int_0^\infty (1-\rho^2)^{\frac{n-3}{2}} V^{-1} \left[V^{-\frac{1}{2}} + V^{\frac{1}{2}} - 2r\rho\right]^{-(n-1)} dV d\rho.$$

And the ratio of marginal densities with fraction b is

$$\frac{m_1^b(\mathbf{x}, \mathbf{y})}{m_2^b(\mathbf{x}, \mathbf{y})} = \frac{S_1(\mathbf{x}, \mathbf{y}; b)}{S_2(\mathbf{x}, \mathbf{y}; b)}, \tag{3.11}$$

where

$$S_1(\mathbf{x}, \mathbf{y}; b) = \int_0^\infty (1-\rho_0^2)^{\frac{bn-1}{2}} V^{-1} \left[V^{-\frac{1}{2}} + V^{\frac{1}{2}} - 2\rho_0r\right]^{-(bn-1)} dV$$

and

$$S_2(\mathbf{x}, \mathbf{y}; b) = \int_{-1}^1 \int_0^\infty (1-\rho^2)^{\frac{bn-3}{2}} V^{-1} \left[V^{-\frac{1}{2}} + V^{\frac{1}{2}} - 2r\rho\right]^{-(bn-1)} dV d\rho.$$

Thus the FBF of H_2 versus H_1 is given by

$$B_{21}^F = \frac{S_2(\mathbf{x}, \mathbf{y})}{S_1(\mathbf{x}, \mathbf{y})} \cdot \frac{S_1(\mathbf{x}, \mathbf{y}; b)}{S_2(\mathbf{x}, \mathbf{y}; b)}. \tag{3.12}$$

Note that the calculations of the FBF of H_2 versus H_1 requires only two dimensional integration.

3.2. Bayesian hypothesis testing procedure based on the intrinsic Bayes factor

The element B_{21}^N of the intrinsic Bayes factor is computed in the fractional Bayes factor. So under minimal training sample, we only calculate the marginal densities for the hypotheses H_1 and H_2 , respectively. The marginal density of $(X_{j_1}, Y_{j_1}), (X_{j_2}, Y_{j_2})$ and (X_{j_3}, Y_{j_3}) is finite for all $1 \leq j_1 < j_2 < j_3 \leq n$ under each hypothesis. Thus we conclude that any training sample of size 3 is a minimal training sample.

The marginal density $m_1^N((x_{j_1}, y_{j_1}), (x_{j_2}, y_{j_2}), (x_{j_3}, y_{j_3}))$ under H_1 is given by

$$\begin{aligned} & m_1^N((x_{j_1}, y_{j_1}), (x_{j_2}, y_{j_2}), (x_{j_3}, y_{j_3})) \\ &= \int_0^\infty \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty f((x_{j_1}, y_{j_1}), (x_{j_2}, y_{j_2}), (x_{j_3}, y_{j_3}) | \mu_1, \mu_2, \sigma_1, \sigma_2) \\ & \times \pi_1^N(\mu_1, \mu_2, \sigma_1, \sigma_2) d\mu_1 d\mu_2 d\sigma_1 d\sigma_2 \\ &= \int_0^\infty \frac{1}{3} \left(\frac{1}{2\pi}\right)^2 (S_1^*)^{-1} (S_2^*)^{-1} (1 - \rho_0^2) V^{-1} \left[V^{-\frac{1}{2}} + V^{\frac{1}{2}} - 2\rho_0 r^* \right]^{-2} dV. \end{aligned}$$

where $\bar{x}^* = \sum_{i=1}^3 x_{j_i}/3, S_1^* = \sum_{i=1}^3 (x_{j_i} - \bar{x}^*)^2, \bar{y}^* = \sum_{i=1}^3 y_{j_i}/3, S_2^* = \sum_{i=1}^3 (y_{j_i} - \bar{y}^*)^2$ and $S_{12}^* = \sum_{i=1}^3 (x_{j_i} - \bar{x}^*)(y_{j_i} - \bar{y}^*)$ and $r^* = S_{12}^*/[(S_1^*)^{\frac{1}{2}}(S_2^*)^{\frac{1}{2}}]$. And the marginal density $m_2^N((x_{j_1}, y_{j_1}), (x_{j_2}, y_{j_2}), (x_{j_3}, y_{j_3}))$ under H_2 is given by

$$\begin{aligned} & m_2^N((x_{j_1}, y_{j_1}), (x_{j_2}, y_{j_2}), (x_{j_3}, y_{j_3})) \\ &= \int_{-1}^1 \int_0^\infty \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty f((x_{j_1}, y_{j_1}), (x_{j_2}, y_{j_2}), (x_{j_3}, y_{j_3}) | \rho, \mu_1, \mu_2, \sigma_1, \sigma_2) \\ & \times \pi_2^N(\rho, \mu_1, \mu_2, \sigma_1, \sigma_2) d\mu_1 d\mu_2 d\sigma_1 d\sigma_2 d\rho \\ &= \int_{-1}^1 \int_0^\infty \frac{1}{3} \left(\frac{1}{2\pi}\right)^2 (S_1^*)^{-1} (S_2^*)^{-1} V^{-1} \left[V^{-\frac{1}{2}} + V^{\frac{1}{2}} - 2r^* \rho \right]^{-2} dV d\rho \\ &= \int_0^\infty \frac{1}{3} \left(\frac{1}{2\pi}\right)^2 (S_1^*)^{-1} (S_2^*)^{-1} \frac{2}{(1+V)^2 - 4(r^*)^2 V} dV. \end{aligned}$$

Therefore the AIBF of H_2 versus H_1 is given by

$$B_{21}^{AI} = \frac{S_2(\mathbf{x}, \mathbf{y})}{S_1(\mathbf{x}, \mathbf{y})} \left[\frac{1}{L} \sum_{j_1, j_2, j_3}^n \frac{T_1(x_{j_1}, x_{j_2}, x_{j_3}, y_{j_1}, y_{j_2}, y_{j_3})}{T_2(x_{j_1}, x_{j_2}, x_{j_3}, y_{j_1}, y_{j_2}, y_{j_3})} \right], \tag{3.13}$$

where $L = [n(n-1)(n-2)]/6,$

$$T_1(x_{j_1}, x_{j_2}, x_{j_3}, y_{j_1}, y_{j_2}, y_{j_3}) = \int_0^\infty (1 - \rho_0^2) V^{-1} \left[V^{-\frac{1}{2}} + V^{\frac{1}{2}} - 2\rho_0 r^* \right]^{-2} dV$$

and

$$T_2(x_{j_1}, x_{j_2}, x_{j_3}, y_{j_1}, y_{j_2}, y_{j_3}) = \int_0^\infty \frac{2}{(1+V)^2 - 4(r^*)^2 V} dV.$$

Also the MIBF of H_2 versus H_1 is given by

$$B_{21}^{MI} = \frac{S_2(\mathbf{x}, \mathbf{y})}{S_1(\mathbf{x}, \mathbf{y})} ME \left[\frac{T_1(x_{j_1}, x_{j_2}, x_{j_3}, y_{j_1}, y_{j_2}, y_{j_3})}{T_2(x_{j_1}, x_{j_2}, x_{j_3}, y_{j_1}, y_{j_2}, y_{j_3})} \right]. \quad (3.14)$$

Note that the calculations of the AIBF and the MIBF of H_2 versus H_1 require only two dimensional integration.

4. Numerical studies

In order to assess the Bayesian hypothesis testing procedures, we evaluate the posterior probability for several configurations of $(\rho, \mu_1, \mu_2, \sigma_1, \sigma_2)$ and n . In particular, for fixed $(\rho, \mu_1, \mu_2, \sigma_1, \sigma_2)$, we take 1,000 independent random samples of (X_i, Y_i) with sample size n from the models (1.1). We want to test the hypotheses $H_1 : \rho = 0$ versus $H_2 : \rho \neq 0$. The posterior probabilities of H_1 being true are computed assuming equal prior probabilities. For the choice of b for FBF, we use $b = m_0/n$, where m_0 is the number of minimal training sample. Table 4.1 shows the results of the averages and the standard deviations in parentheses of posterior probabilities. In Table 4.1, $P^F(\cdot), P^{AI}(\cdot)$ and $P^{MI}(\cdot)$ are the posterior probabilities of the hypothesis H_1 being true based on FBF, AIBF and MIBF, respectively. From the results of Table 4.1, the FBF, the AIBF and the MIBF give fairly reasonable answers for all configurations. Also the AIBF and the MIBF give a similar behavior for all sample sizes. However the FBF slightly favors the hypothesis H_2 than the AIBF and the MIBF.

Example 4.1 This example taken from Sun and Wong (2007). Levine *et al.* (1999) examines the role of nonexercise activity thermogenesis in resistance to fat gain in humans. Let x be the increase in energy use (in cal) from activity other than deliberate exercise, and y be the fat gain (in kg). The data obtained by Levine *et al.* (1999) in $(x_i, y_i), i = 1, \dots, 16$ pairs are

$$\begin{aligned} &(-94,4.2), (-57,3.0), (-29,3.7), (135,2.7), (143,3.2), (151,3.6), (245,2.4), (355,1.3) \\ &(392,3.8), (473,1.7), (486,1.6), (535,2.2), (571,1.0), (580,0.4), (620,2.3), (690,1.1). \end{aligned}$$

For this data sets, the maximum likelihood estimate (MLE) of ρ is -0.779. We want to test the hypotheses $H_1 : \rho = \rho_0$ versus $H_2 : \rho \neq \rho_0$. The p -values based on the t -statistic and z -statistic, the values of the Bayes factors and the posterior probabilities of H_1 are given in Table 4.2. The results in Table 4.2 indicate that for values of ρ_0 that are far from the MLE they select the hypothesis H_2 . Also from the results of Table 4.2, the FBF favors the hypothesis H_2 than the AIBF and the MIBF. The AIBF and the MIBF give almost the same result.

5. Concluding remarks

In this paper, we developed the objective Bayesian hypothesis testing procedures based on the fractional Bayes factor and the intrinsic Bayes factors for the correlation coefficient in the bivariate normal distribution under the reference priors.

Table 4.1 The averages and the standard deviations in parentheses of posterior probabilities

n	σ_1	σ_2	ρ	$P^F(H_1 \mathbf{x}, \mathbf{y})$	$P^{AI}(H_1 \mathbf{x}, \mathbf{y})$	$P^{MI}(H_1 \mathbf{x}, \mathbf{y})$
10	1	1	0.1	0.596(0.137)	0.690 (0.145)	0.679 (0.139)
			0.3	0.523(0.187)	0.611 (0.203)	0.603 (0.192)
			0.5	0.396(0.216)	0.473 (0.242)	0.474 (0.231)
			0.7	0.220(0.201)	0.269 (0.236)	0.276 (0.229)
	1	3	0.1	0.597(0.134)	0.690 (0.142)	0.679 (0.136)
			0.3	0.529(0.180)	0.618 (0.196)	0.610 (0.186)
			0.5	0.401(0.218)	0.479 (0.247)	0.480 (0.237)
			0.7	0.218(0.200)	0.267 (0.235)	0.275 (0.228)
	1	5	0.1	0.593(0.145)	0.686 (0.155)	0.676 (0.148)
			0.3	0.536(0.176)	0.626 (0.191)	0.618 (0.182)
			0.5	0.410(0.214)	0.487 (0.240)	0.486 (0.229)
			0.7	0.219(0.198)	0.268 (0.232)	0.276 (0.225)
20	1	1	0.1	0.657(0.158)	0.746 (0.156)	0.735 (0.153)
			0.3	0.529(0.228)	0.618 (0.240)	0.609 (0.235)
			0.5	0.272(0.240)	0.336 (0.275)	0.333 (0.269)
			0.7	0.054(0.108)	0.073 (0.137)	0.074 (0.135)
	1	3	0.1	0.654(0.161)	0.743 (0.160)	0.731 (0.158)
			0.3	0.517(0.235)	0.605 (0.248)	0.596 (0.242)
			0.5	0.272(0.245)	0.336 (0.281)	0.333 (0.275)
			0.7	0.063(0.122)	0.084 (0.152)	0.085 (0.150)
	1	5	0.1	0.650(0.166)	0.740 (0.165)	0.728 (0.162)
			0.3	0.522(0.236)	0.610 (0.250)	0.602 (0.245)
			0.5	0.265(0.240)	0.328 (0.274)	0.325 (0.268)
			0.7	0.062(0.125)	0.082 (0.155)	0.083 (0.153)
30	1	1	0.1	0.692(0.162)	0.777 (0.156)	0.766 (0.155)
			0.3	0.476(0.265)	0.560 (0.280)	0.551 (0.276)
			0.5	0.177(0.217)	0.226 (0.254)	0.223 (0.249)
			0.7	0.016(0.058)	0.022 (0.076)	0.022 (0.075)
	1	3	0.1	0.693(0.163)	0.777 (0.155)	0.767 (0.155)
			0.3	0.489(0.269)	0.573 (0.283)	0.565 (0.279)
			0.5	0.167(0.210)	0.213 (0.249)	0.209 (0.243)
			0.7	0.012(0.044)	0.018 (0.059)	0.018 (0.058)
	1	5	0.1	0.688(0.172)	0.771 (0.167)	0.761 (0.166)
			0.3	0.484(0.262)	0.570 (0.276)	0.561 (0.273)
			0.5	0.180(0.215)	0.230 (0.252)	0.227 (0.247)
			0.7	0.013(0.050)	0.018 (0.066)	0.018 (0.065)

Table 4.2 Bayes factors and posterior probabilities of $H_1 : \rho = \rho_0$

ρ_0	p -value	B_{21}^F	$P^F(H_1 \mathbf{x}, \mathbf{y})$	B_{21}^{AI}	$P^{AI}(H_1 \mathbf{x}, \mathbf{y})$	B_{21}^{MI}	$P^{MI}(H_1 \mathbf{x}, \mathbf{y})$
0.0	0.00038	125.08192	0.00793	93.16346	0.01062	76.69393	0.01287
-0.1	0.00069	46.78417	0.02093	33.49975	0.02899	28.84963	0.03350
-0.3	0.00829	7.38451	0.11927	4.92974	0.16864	4.68627	0.17586
-0.4607	0.05004	1.87655	0.34764	1.19902	0.45475	1.24431	0.44557
-0.5413	0.11620	0.99882	0.50029	0.62919	0.61380	0.67104	0.59843
-0.7	0.52949	0.37129	0.72924	0.23539	0.80946	0.26102	0.79301

From our numerical results, we found that the developed hypothesis testing procedures work well irrespective of parameter values. In Table 1, the AIBF and MIBF have higher posterior probabilities in favor of the hypothesis H_1 than that of FBF. They choose the same hypothesis consistently when the sample size is 10 or 20. But when the sample size is 30 and $\rho = 0.3$, AIBF and MIBF favor $H_1 : \rho = 0$ but FBF favors $H_2 : \rho \neq 0$. Intuitively, $\rho = 0.3$ is far from $\rho = 0$, it is reasonable to reject $H_1 : \rho = 0$. When $\rho = 0.5, 0.7$, they select $H_2 : \rho \neq 0$.

Finally, we recommend the use of the FBF than the AIBF or MIBF for practical application in view of its simplicity and ease of implementation.

References

- Berger, J. O. and Bernardo, J. M. (1989). Estimating a product of means : Bayesian analysis with reference priors. *Journal of the American Statistical Association*, **84**, 200-207.
- Berger, J. O. and Bernardo, J. M. (1992). On the development of reference priors (with discussion). In *Bayesian Statistics IV*, edited by J. M. Bernardo *et al.*, Oxford University Press, Oxford, 35-60.
- Berger, J. O. and Pericchi, L. R. (1996). The intrinsic Bayes factor for model selection and prediction. *Journal of the American Statistical Association*, **91**, 109-122.
- Berger, J. O. and Pericchi, L. R. (1998). Accurate and stable Bayesian model selection: The median intrinsic Bayes factor. *Sankya B*, **60**, 1-18.
- Berger, J. O. and Pericchi, L. R. (2001). Objective Bayesian methods for model selection: Introduction and comparison (with discussion). In *Model Selection, Institute of Mathematical Statistics Lecture Notes-Monograph Series*, **38**, edited by P. Lahiri, 135-207, Beachwood, Ohio.
- Berger, J. O. and Sun, D. (2008). Objective priors for a bivariate normal model. *The Annals of Statistics*, **36**, 963-982.
- Fisher, R. A. (1915). Frequency distribution of the values of the correlation coefficient in samples from an indefinitely large population. *Biometrika*, **10**, 507-521.
- Ghosh, M., Mukherjee, B., Santra, U. and Kim, D. (2010). Bayesian and likelihood-based inference for the bivariate normal correlation coefficient. *Journal of Statistical Planning and Inference*, **140**, 1410-1416.
- Hotelling, H. (1953). New light on the correlation coefficient and its transform. *Journal of Royal Statistical Society B*, **15**, 193-232.
- Kang, S. G., Kim, D. H. and Lee, W. D. (2006). Bayesian one-sided testing for the ratio of Poisson means. *Journal of the Korean Data & Information Science Society*, **17**, 619-631.
- Kang, S. G., Kim, D. H. and Lee, W. D. (2008). Bayesian model selection for inverse Gaussian populations with heterogeneity. *Journal of the Korean Data & Information Science Society*, **19**, 621-634.
- Kang, S. G., Kim, D. H. and Lee, W. D. (2010). Default Bayesian testing for normal mean with known coefficient of variation. *Journal of the Korean Data & Information Science Society*, **21**, 297-308.
- Levine, J. A., Eberhardt, N. L. and Jensen, M. D. (1999). Role of nonexercise activity thermogenesis in resistance to fat gain in human. *Science*, **283**, 212-214.
- O'Hagan, A. (1995). Fractional Bayes factors for model comparison (with discussion). *Journal of Royal Statistical Society B*, **57**, 99-118.
- O'Hagan, A. (1997). Properties of intrinsic and fractional Bayes factors. *Test*, **6**, 101-118.
- Rubén, H. (1966). Some new results on the distribution of the sample correlation coefficient. *Journal of Royal Statistical Society B*, **28**, 513-525.
- Spiegelhalter, D. J. and Smith, A. F. M. (1982). Bayes factors for linear and log-linear models with vague prior information. *Journal of Royal Statistical Society B*, **44**, 377-387.
- Sun, Y. and Wong, A. C. M. (2007). Interval estimation for the normal correlation coefficient. *Statistics & Probability Letters*, **77**, 1652-1661.