A Safety Analysis on the Structural Rupture of Cylindrical Shell by Finite Difference Method

Chi-Kyung Kim* · Hwa-Yong Park**
*University of Incheon & Department of Safety Engineering · **POSCO E&C

차분법에 의한 실린더형 쉘구조의 안정성 연구

김 치 경* · 박 화 용**
*인천대학교 안전공학과 · **포스코 건설

Abstract

본 연구에서는 실린더 형 쉘 구조물의 구조적 안정성에 대하여 해석 하였다. 임계하중은 하중을 점차적으로 증가하여 구조물이 파괴가 발생 할 때의 상태에서 가장 작은 하중을 의미한다. 셸 구조의 안정성을 임계하중의 크기로 기초를 두고 해석 하였다. 실린더 형 쉘의 차분해석은 일차적 원통형 판구조와 같으므로 최근에 많은 연구의 대상이 되어왔다. 차분법은 복잡한 구조물에서도 물론, 다양한 경계조건을 포함하는 문제에 이르기까지 효과적인 수치방법이다. 본 연구에서는 기본 쉘의 지배방정식을 유도하고 차분화 하여 직접적으로 접근하였다. 등분포 하중의 내압을 받고 있는 갇힌 실린더 형 쉘의 처짐 및 응력을 해석 하였다. 수치해석 결과를 해석해와 비교 검토하였으며 안정성에 대하여 임계 하중강도의 범위를 산출하였다.

Keywords : Cylindrical shell, Stability, Finite difference method, Critical load

1. Introduction

A shell is a thin, three-dimensional structural form enclosured in a volume bounded by a curved surface. Loads applied to shell surfaces are carried to the ground by the developed by the development of compressive, tensile and shear stresses acting in the in-plane direction of the surface. The thickness of the surface does not allow the development of appreciable bending resistance. Thin shell structures are uniquely suited to carrying distributed loads and find wide application as roofs of buildings. The three-dimensional shapes, such as masonry domes, are considerably thicker relative to their span and can not be exactly characterized as carrying loads by in-plane axial or shear stresses, because more bending exists and final stresses are not uniform. However, an approximation of this type is good for conceptualizing the behavior of such structures. In a dome, for example, the in-plane forces developed in the surface normally have outwardly directed components that must be absorbed by either a series of closely spaced buttresses or a tension ring. Analysis of thin shells subjected to lateral uniform loads are most commonly accomplished by using a linear theory in which one assumes that the lateral displacements or deflections due to the loads are small[2,5,6]. The finite difference method essentially consists of large sets of algebraic equations in terms of discrete values of the functions at discrete points.

✝ 교신저자: 김치경, 인천시 연수구 송도동 인천대학교 안전공학과
Tel: 032-835-82945, E-mail: kimchi@incheon.ac.kr

2011년 7월 20일 접수; 2011년 9월 21일 수정본 접수; 2011년 9월 21일 게재확정

Most recently, Lijuan, Wenqia and Qiuyan proposed the fractional weighted average finite difference method for space-fractional advection dispersion equation which is based on shifted formula[9]. Mochnacki and Machrzak solved the system of casting-mould by finite difference method[7]. The thermal processes proceeding in a casting subdomain are described using the one domain approach. Boot and Moore solved stiffened plates subjected to transverse loadings using the generalized finite difference method[8]. These proposed methodologys work considerably well at improving solution accuracy. However, these methods are not computationally highly efficient for the shell structure problems. The primary objective of this research is to develop a mathematical model which can analyze thin cylindrical shell and subjected to lateral pressure representing wind or snow loading conditions. Complete derivation of the shell equations is involved. This study also deals with approximate techniques of the finite difference method which are used to determine the deflections and the in-plane stress functions of shells with both fixed boundary conditions[1,3,4]. A brief account of the finite difference technique is presented for ordinary differential equations, and the technique is extended to partial differential equations.

2. Governing Equations for Shell

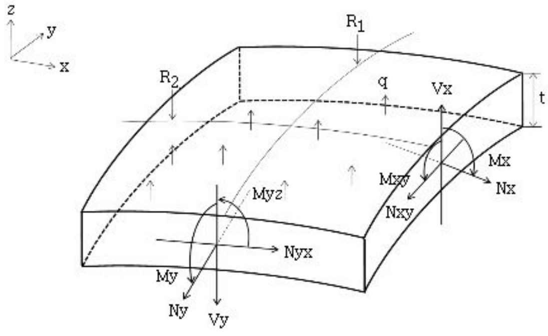
The equilibrium equation in the radial direction z of a shell with curvature in two directions is

$$\frac{t^2}{12}\nabla^4 w+\frac{1}{R_1}\left(\frac{w}{R_1}-\frac{\partial v}{\partial y}-\eta\frac{\partial u}{\partial x}\right)+\frac{1}{R_2}\left(\frac{w}{R_2}-\frac{\partial u}{\partial x}-\eta\frac{\partial v}{\partial y}\right)-t^2\frac{q}{12D}=0 \quad (1)$$

where $\nabla^4=\frac{\partial^4}{\partial x^4}+2\frac{\partial^4}{\partial x^2\partial y^2}+\frac{\partial^4}{\partial y^4}$; t is the thickness of the shell ; u, v, w are the deflections in the x, y, and z directions ; R_1 and R_2 are principal radii, with R_1 in the y direction ; q is the intensity of radially distributed load in the z direction ; η is the Poisson's ratio ; and D is the stiffness of the shell and equals $E\,t^3/12(1-\eta^2)$. The equilibrium equations in the tangential directions x and y are

$$\frac{\partial^2 u}{\partial x^2}+\frac{1-\eta}{2}\frac{\partial^2 u}{\partial y^2}+\frac{1+\eta}{2}\frac{\partial^2 v}{\partial x\partial y}-\frac{\eta}{R_1}\frac{\partial w}{\partial x}-\frac{1}{R_2}\frac{\partial w}{\partial x}=0 \quad (2)$$

$$\frac{\partial^2 v}{\partial y^2}+\frac{1-\eta}{2}\frac{\partial^2 v}{\partial x^2}+\frac{1+\eta}{2}\frac{\partial^2 u}{\partial x\partial y}-\frac{1}{R_1}\frac{\partial w}{\partial y}-\frac{\eta}{R_2}\frac{\partial w}{\partial y}=0 \quad (3)$$



<Figure 1> Shell model and equilibrium of forces and moments

<Figure 1> shows a typical element of the shell with all the moments, direct forces, and shears per unit width, M_x, M_y, M_{xy}, N_x, N_y, N_{xy}, V_x, and V_y acting in the positive direction. The bending moments M_x and M_y, twisting moments M_{xy} and direct forces S_x, N_y and N_{xy}, shears V_x and V_y per unit length are related to the deflections as follows:

$$M_x=D\left[\frac{1}{R_2}\frac{\partial u}{\partial x}+\frac{\partial^2 w}{\partial x^2}+\eta\left(\frac{1}{R_1}\frac{\partial v}{\partial y}+\frac{\partial^2 w}{\partial y^2}\right)\right] \quad (4)$$

$$M_y=D\left[\frac{1}{R_1}\frac{\partial v}{\partial y}+\frac{\partial^2 w}{\partial y^2}+\eta\left(\frac{1}{R_2}\frac{\partial u}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)\right] \quad (5)$$

$$M_{xy}=D(1-\eta)\left(\frac{1}{R_2}\frac{\partial u}{\partial y}+\frac{1}{R_1}\frac{\partial v}{\partial x}+\frac{\partial^2 w}{\partial x\partial y}\right) \quad (6)$$

$$N_x=\frac{Et}{1-\eta^2}\left[\frac{\partial u}{\partial x}-\frac{w}{R_2}+\eta\left(\frac{\partial v}{\partial y}-\frac{w}{R_1}\right)\right] \quad (7)$$

$$N_y=\frac{Et}{1-\eta^2}\left[\frac{\partial v}{\partial y}-\frac{w}{R_1}+\eta\left(\frac{\partial u}{\partial x}-\frac{w}{R_2}\right)\right] \quad (8)$$

$$N_{xy}=\frac{E\ t}{2(1+\eta)}(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}) \quad (9)$$

$$V_x=\frac{\partial M_x}{\partial x}+\frac{\partial M_{xy}}{\partial y} \quad (10)$$

$$V_y=\frac{\partial M_y}{\partial y}+\frac{\partial M_{xy}}{\partial x} \quad (11)$$

For a cylindrical shell which has curvature(R) in one direction only, Equations(1)-(3) becomes:

$$\frac{t^2}{12}\nabla^4 w+\frac{1}{R}(\frac{w}{R}-\frac{\partial v}{\partial y}-\eta\frac{\partial u}{\partial x})-t^2\frac{q}{12D}=0 \quad (12)$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{(1-\eta)}{2}\frac{\partial^2 u}{\partial y^2}+\frac{(1+\eta)}{2}\frac{\partial^2 v}{\partial x\partial y}-\frac{\eta}{R}\frac{\partial w}{\partial x}=0 \quad (13)$$

$$\frac{\partial^2 v}{\partial y^2}+\frac{(1-\eta)}{2}\frac{\partial^2 v}{\partial x^2}+\frac{(1+\eta)}{2}\frac{\partial^2 u}{\partial x\partial y}-\frac{1}{R}\frac{\partial w}{\partial y}=0 \quad (14)$$

The moments, direct forces and shears are as follows:

$$M_x=D[\frac{\partial^2 w}{\partial x^2}+\eta(\frac{1}{R}\frac{\partial v}{\partial y}+\frac{\partial^2 w}{\partial y^2})] \quad (15)$$

$$M_y=D\ [\frac{1}{R}\frac{\partial v}{\partial y}+\frac{\partial^2 w}{\partial y^2}+\eta\frac{\partial^2 w}{\partial x^2}] \quad (16)$$

$$M_{xy}=D(1-\eta)[\frac{1}{R}\frac{\partial v}{\partial x}+\frac{\partial^2 w}{\partial x\partial y}] \quad (17)$$

$$N_x=\frac{Et}{1-\eta^2}[\frac{\partial u}{\partial x}+\eta(\frac{\partial v}{\partial y}-\frac{w}{R})] \quad (18)$$

$$N_y=\frac{Et}{1-\eta^2}[\frac{\partial v}{\partial y}-\frac{w}{R}+\eta\frac{\partial u}{\partial x}] \quad (19)$$

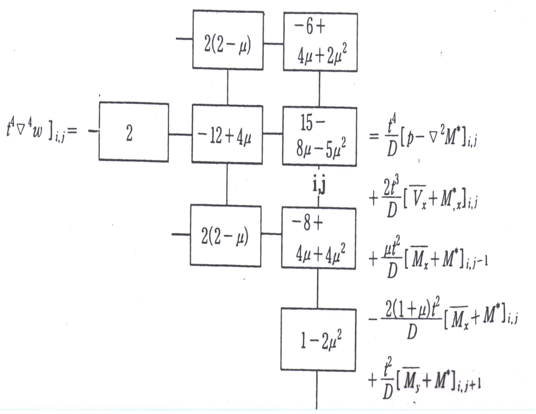
$$N_{xy}=\frac{E\ t}{2(1+\eta)}[\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}] \quad (20)$$

$$V_x=\frac{\partial M_x}{\partial x}+\frac{\partial M_{xy}}{\partial y} \quad (21)$$

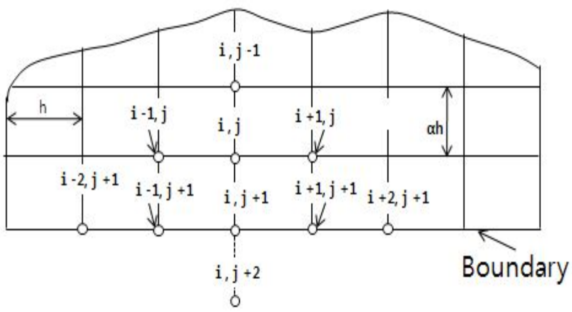
$$V_y=\frac{\partial M_y}{\partial y}+\frac{\partial M_{xy}}{\partial x} \quad (22)$$

3. Finite Difference Representations

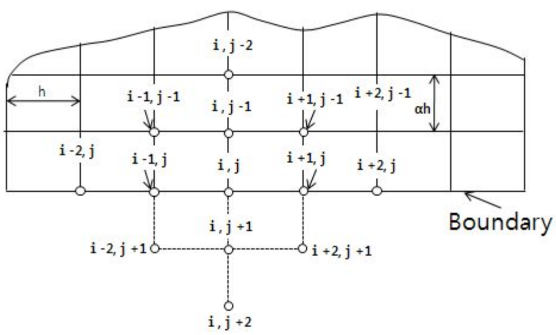
Solutions of the governing differential equations as shown in Equations(12)-(22) by the finite difference method also need proper finite difference approximations for the boundary conditions. The method of finite differences replaces the differential equation and the expressions defining the boundary conditions with equivalent difference equation of a set of algebraic equations, written for every nodal point within the shell. Since the central finite difference equations are used, some fictitious points outside the domain of the shell are required. If the pivotal point i,j is located one point inside the edge, we must introduce one fictitious point outside domain of the shell as shown in <Figure 3>. Whereas, if the pivotal point i,j is on the boundary, four more fictitious points are required as shown in <Figure 4>. Deflections of these fictitious points can be expressed in terms of deflections of the nearby points located on the shell using the boundary conditions. In the case where the pivotal point i,j is located at the edge $x=h$, we specify that the edge moment is zero. We get the finite difference equation for a node on the boundary; the model is shown in <Figure 2>. The finite difference representations will be different at points along or near the boundary. Similar procedures are used to derive them; thus, we have seven more difference equations for the models.



<Figure 2> Model for Biharmonic equation at the corner of the shell



<Figure 3> Fictitious points near the boundary for fixed end



<Figure 4> Fictitious points near the boundary for free end

By using the notations in <Figure 3> and <Figure 4>, the finite difference expressions for Equations(12)-(22) are follows:

$$\frac{6(h^4+k^4)+8h^2k^2}{h^4k^4}w_{i,j}-4\frac{(h^2+k^2)}{h^4k^2}(w_{i-1,j}+w_{i+1,j})-4\frac{(h^2+k^2)}{h^2k^4}(w_{i,j+1}+w_{i,j-1})-4\frac{(h^2+k^2)}{h^4k^2}(w_{i+1,j}+w_{i-1,j})-4\frac{(h^2+k^2)}{h^2k^4}(w_{i,j+1}+w_{i,j-1})+\frac{2}{h^2k^2}(w_{i+1,j+1}+w_{i+1,j-1}+w_{i-1,j+1}+w_{i-1,j-1})+\frac{1}{h^4}(w_{i+2,j}+w_{i-2,j}+w_{i,j+2}+w_{i,j-2})=\frac{q}{D}$$

$$\left(\frac{2}{h^2}+\frac{1-\eta}{k^2}\right)u_{i,j}-\frac{1}{h^2}(u_{i+1,j}+u_{i-1,j})-\frac{1-\eta}{2k^2}(u_{i,j+1}+u_{i,j-1})-\frac{1-\eta}{8hk}(v_{i+1,j+1}-v_{i-1,j+1}-v_{i+1,j-1}+v_{i-1,j-1})+\frac{\eta}{2hR}(w_{i+1,j}+w_{i-1,j})=0$$

$$\left(\frac{2}{k^2}+\frac{1-\eta}{h^2}\right)v_{i,j}-\frac{1}{k^2}(u_{i,j+1}+u_{i,j-1})+\frac{1-\eta}{2h^2}(u_{i+1,j}+u_{i-1,j})-\frac{1+\eta}{8hk}(u_{i+1,j+1}-u_{i-1,j+1}-u_{i+1,j-1}+u_{i-1,j-1})+\frac{1}{2kR}(w_{i,j+1}-w_{i,j-1})=0$$

where h and k are the grid spacings in both x and y directions.

$$(M_x)_{i,j}=D[-2\left(\frac{1}{h^2}+\frac{\eta}{k^2}\right)w_{i,j}+\frac{1}{h^2}(w_{i+1,j}+w_{i-1,j})+\frac{\eta}{k^2}(w_{i,j+1}+w_{i,j-1})+\frac{\eta}{2kR}(v_{i,j+1}+v_{i,j-1})]$$

$$(M_y)_{i,j}=D[-2\left(\frac{1}{k^2}+\frac{\eta}{h^2}\right)w_{i,j}+\frac{1}{k^2}(w_{i+1,j}+w_{i-1,j})+\frac{\eta}{h^2}(w_{i,j+1}+w_{i,j-1})+\frac{1}{2kR}(v_{i,j+1}+v_{i,j-1})]$$

$$(M_{xy})_{i,j}=D(1-\eta)[\frac{1}{2hR}(v_{i+1,j}-v_{i-1,j}) + \frac{1}{4hk}(w_{i+1,j+1}-w_{i+1,j-1} - w_{i-1,j+1}+w_{i-1,j-1})]$$

$$(N_x)_{i,j}=\frac{Et}{1-\eta^2}[\frac{1}{2h}(u_{i+1,j}-u_{i-1,j})+\frac{\eta}{2k}(v_{i,j+1} - v_{i,j-1})-\frac{\eta}{R}w]$$

$$(N_y)_{i,j}=\frac{Et}{1-\eta^2}[\frac{1}{2k}(u_{i+1,j}-u_{i-1,j})+\frac{\eta}{2h}(v_{i,j+1} - v_{i,j-1})-\frac{1}{R}w]$$

$$(N_{xy})_{i,j}=\frac{Et}{2(1+\eta)}[\frac{1}{2k}(u_{i,j+1}-u_{i,j-1}) + \frac{1}{2h}(v_{i+1,j}-v_{i-1,j})]$$

$$(V_x)_{i,j}=D[\frac{1}{2h^3}(w_{i+2,j}-w_{i-2,j})-(\frac{1}{h^3}+\frac{\eta}{hk^2})(w_{i+1,j}-w_{i-1,j})+\frac{\eta}{4hkR}(v_{i+1,j+1} - v_{i+1,j-1}-v_{i-1,j+1}+v_{i-1,j-1})+\frac{\eta}{2hk^2}(w_{i+1,j+1}-w_{i-1,j+1}-w_{i-1,j-1})] + D(1-\eta)[\frac{1}{4hkR}(v_{i+1,j+1}-v_{i+1,j-1} - v_{i-1,j+1}+v_{i-1,j-1})+\frac{1}{2hk^2}(w_{i+1,j+1} - 2w_{i+1,j}+w_{i+1,j-1}-w_{i-1,j+1} + 2w_{i-1,j}-w_{i-1,j-1})]$$

$$(V_y)_{i,j}=D[\frac{1}{k^2R}(v_{i,j+1}-2v_{i,j}+v_{i,j-1})+\frac{1}{2k^3R}(w_{i,j+2}-2w_{i,j+1}-2w_{i,j-1}-w_{i,j-2}) + \frac{\eta}{2h^2k}(w_{i+1,j+1}-2w_{i,j+1}+w_{i-1,j+1} - w_{i+1,j-1}-2w_{i,j-1}-w_{i-1,j-1})] + \frac{D(1-\eta)}{h^2R}(v_{i+1,j}-2v_{i,j}+v_{i-1,j}) + \frac{1}{2h}(w_{i+1,j+1}+w_{i-1,j+1}-w_{i+1,j-1} - w_{i-1,j-1})]$$

Together with the appropriate boundary conditions, a set of simultaneous equations will be obtained to solve the deflections at the grid points. By performing a static condensation for a domain with n points with simple mathematical manipulations, we may get with the equation as a set of linear algebraic equations.

4. Numerical Implementation

To investigate the validity of the proposed equations and calculation procedure, a complete analysis of closed circular cylindrical shell under axisymmetric loads is considered. In this case the loads are restricted so that they are symmetrical about the x -axis of rotation of the cylinder, the stress resultants N_{xy}, N_{yx}, M_{xy} and M_{yx} are considerably vanished due to symmetry. Because of axisymmetry for any particular section normal to the axis of rotation, the stress resultants at this section will not vary with y and will be functions of x only.

Consider a cylindrical pressure vessel of radius R =0.75m, length L=1.2m, elastic modulus E=200GPa, Poisson's ratio η=0.3 and uniform thickness t=0.025 m which is rigidly welded at its ends to thick end plates. The vessel is subjected to an internal pressure q=4000kN/m^2. In order to investigate the effect of grid-spacing on the results, runs have been made for 12 grids which are uniformly discretized. The number of equal horizontal segments into which the span is divided are 11. We will verify numerical solutions of this computational model against analytical solutions. Since we will verify numerical solutions of this computational model against analytical solutions, only deflection in the direction z was considered. The analytical deflection solution w may be obtained as

$$w=c_1\cos\beta x\sinh\beta x+c_4\cos\beta x\cosh\beta x-qR^2/Et$$

where $\beta^4=\frac{3(1-\eta^2)}{R^2t^2}$,

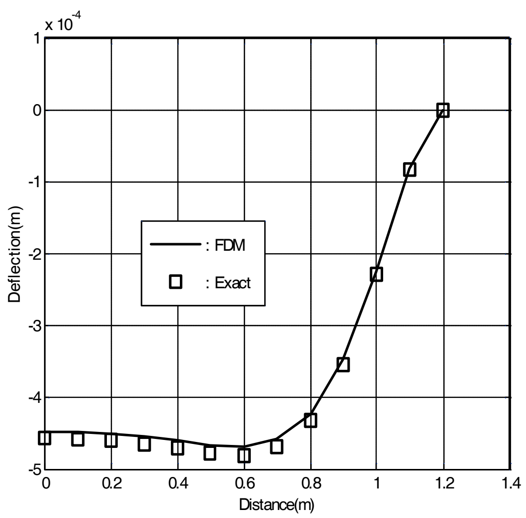
$$c_1=\frac{2qR^2}{Et}[\frac{\sin\alpha\cosh\alpha-\cos\alpha\sinh\alpha}{\sin2\alpha+\sinh2\alpha}],$$

$$c_4=\frac{2qR^2}{Et}[\frac{\cos\alpha\sinh\alpha+\sin\alpha\cosh}{\sin2\alpha+\sinh2\alpha}]$$

<Figure 5> shows numerical results of deflection for the one-fixed case compared to the analytical solutions. The case of one-fixed boundary gives close agreement to the analytical solution, except at the around center where deflection is maximum. It shows good agreements with the analytical solutions. Comparing the results with the analytical solution, maximum deflection at the around center of shell is found to be within 2.5 percentage of the analytical solution. The difference in the deflections at the other of the shell varies from 1.2% to 2.5%. However, if one examines the deflections obtained by finite difference method, it can seen that agreement between the results is fair to good. The numerical results of various stress resultants. from the finite difference method are shown in <Table 1>. <Table 1> shows numerical moments and forces at the finite difference point(node number). The finite difference method produces results that are much more accurate for the solution of shells.

<Table 1> Distribution of various stress resultants

Node No.	Moment $M_x(N\cdot m)$	Moment $M_y(N\cdot m)$	Normal Force $N_x(N)$	Shear Force $V_x(N)$
1	.11830E+03	.35491E+02	.29892E+07	.00000E+00
2	.13807E+03	.41422E+02	.29934E+07	.68017E+03
3	.18068E+03	.54204E+02	.30071E+07	.72823E+03
4	.19147E+03	.57441E+02	.30330E+07	-.64170E+03
5	.68873E+03	.20662E+02	.30713E+07	-.43471E+04
6	-.33395E+03	-.10018E+03	.31127E+07	-.11030E+05
7	-.11698E+04	-.35093E+03	.31287E+07	-.20036E+05
8	-.24828E+04	-.74483E+03	.30620E+07	-.27658E+05
9	-.39829E+04	-.11949E+04	.28244E+07	-.24891E+05
10	-.47077E+04	-.14123E+04	.23186E+07	.42064E+04
11	-.26639E+04	-.79916E+03	.15065E+07	.81060E+05
12	.52778E+04	.15833E+04	.54565E+06	.22384E+06
13	.22696E+05	.68089E+04	.45511E-01	.42611E+06



<Figure 5> Deflection distributions along the cylindrical shell

5. Conclusions

The finite difference method has been developed to solve finite stress and deflection of closed circular cylindrical shell subjected to a uniform axisymmetric load. The complete bending theory is mathematically involved. The displacements are smaller than those from the analytical solution. The maximum values of bending moments and thrust forces in the shell are slightly larger than those obtained from the analytical solution. Furthermore, The critical load in this case is that uniformly distributed live pressure will be in the range of $5000kN/m^2$ to $6000kN/m^2$ for the stability of this shell. An Gauss elimination scheme has been employed to solve these linear algebraic equations. The results of deflection from the model developed agree well with analytical solutions. The differences of deflection compared to the analytical solution are highest at the around middle of cylindrical shell. Generally the values of deflections in the shell are slightly larger than those obtained from the analytical solutions. The variable cross section for general shell is also worth effort in computer programming for solving more practical problems. Furthermore, numerical results have shown that finite difference method is effective for analyzing general boundary value problems including multi-dimension.

6. References

[1] Ferziger, J.H., Numerical Methods for Engineering Application, New York : John Wiley & Sons, (1981)

[2] Gibson, J.E., Thin Shells, Oxford : Pergamon Press, (1980)

[3] Kim, C.K., "A Finite Difference Large Displacement Analysis of Rectangular Thin Glass Plate", Journal of KIIS, Vol. 10 (1995) : 129-133.

[4] Schmit, L.A., and Monforton, G. R., "Finite Deflection Discrete Element Analysis of Sandwich Plates and Cylindrical Shell with Laminated Faces", AIAA, Vol. 8 (1970) : 1454-1461.

[5] Timoshenko, S., and Krieger, S.W., Theory of Plates and Shells, New York: McGraw-Hill Book Co., Inc., (1959)

[6] Ugural, A.C., Stresses in Plates and Shells, New York : "McGraw-Hill Book Co., Inc., (1981)

[7] Mochnachi, B., and Majchrzak, E., "Numerical Modelling of Casting Solidification using Generalized Fini Diference Method", Materials Scienece Forum ,Vol. 638 (2010): 2676-2681

[8] Boot, J.C., and Moore, D.B.. "Stiffened Plates Subjected to Transverse Loadings", International Journal of Solids Structures, Vol.24 (1988): 89-104

[9] Lijuan, S., Wenqia, W., and Qiuyan, X., "Finite Difference Methods for Fractional Dispersion Equations", Applied Mathematics and Computation, ISSN: 009600963003 (2010):1016

저 자 소 개

김 치 경

연세대학교를 토목공학과 학사를 졸업하고 부산대학교 토목공학과에서 석사, University of Alabama에서 박사를 취득하였다. 현재 인천대학교 안전공학과 교수로 재직 중이며 관심분야는 건설안전이다.

주소: 인천시 연수구 송도동 12-1 인천대학교 공과대학 안전공학과

박 화 용

숭실대학교 기계공학과 학사를 졸업 하고 연세대학교 기계공학과에서 석사, 중앙대학교에서 박사과정을 수료하였다. 현재는 포스코건설 이사로 재직중이다. 관심 분야는 건설 환경분야이다.

주소: 인천시 연수구 송도동 16-6 포스코건설 주식회사