

## Inference Based on Generalized Doubly Type-II Hybrid Censored Sample from a Half Logistic Distribution

Kyeongjun Lee<sup>a</sup>, Chankeun Park<sup>b</sup>, Youngseuk Cho<sup>1,a</sup>

<sup>a</sup>Department of Statistics, Pusan National University

<sup>b</sup>Department of Data Information, Korea Maritime University

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### Abstract

Chandrasekar *et al.* (2004) introduced a generalized Type-II hybrid censoring. In this paper, we propose generalized doubly Type-II hybrid censoring. In addition, this paper presents the statistical inference on the scale parameter  $\sigma$  for the half logistic distribution when samples are generalized doubly Type-II hybrid censoring. The approximate maximum likelihood(AMLE) method is developed to estimate the unknown parameter. The scale parameter  $\sigma$  is estimated by the AMLE method using two different Taylor series expansion types. We compare the AMLEs in the sense of the mean square error(MSE). The simulation procedure is repeated 10,000 times for the sample size  $n = 20, 30, 40$  and various censored samples. The  $\text{AMLE}_I$  is better than  $\text{AMLE}_{II}$  in the sense of the MSE.

**Keywords:** Approximate maximum likelihood estimator, generalized doubly Type-II hybrid censored sample, half logistic distribution.

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### 1. Introduction

Consider a life testing experiment in which  $n$  units are put on test. Assume that the life times of  $n$  units are independent and identically distributed(i.i.d) as half logistic distribution with probability density function(pdf)

$$f_X(x; \sigma) = \frac{2\exp\left(-\frac{x}{\sigma}\right)}{\sigma\left\{1 + \exp\left(-\frac{x}{\sigma}\right)\right\}} \quad (1.1)$$

and cumulative distribution function(cdf)

$$F_X(x; \sigma) = \frac{1 - \exp\left(-\frac{x}{\sigma}\right)}{1 + \exp\left(-\frac{x}{\sigma}\right)}. \quad (1.2)$$

Balakrishnan (1985) introduced the half logistic distribution as a life testing model and Balakrishnan and Puthenpura (1986) obtained the best linear unbiased estimators(BLUEs) of the location and scale parameters. Balakrishnan and Wong (1991) obtained the approximate maximum likelihood estimators(AMLEs) for the location and scale parameters. Recently, Kang *et al.* (2009) obtained the AMLEs of the scale parameter in a half logistic distribution based on double hybrid censored samples.

Let us assume that the ordered life times of these items be denoted by  $X_{1:n}, X_{2:n}, \dots, X_{n:n}$ . Epstein (1954) introduced a hybrid censoring scheme in which the test is terminated at a random time  $T_1^* =$

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<sup>1</sup> Corresponding author: Associate Professor, Department of Statistics, Pusan National University, Busan 609-735, Korea.  
E-mail: choys@pusan.ac.kr

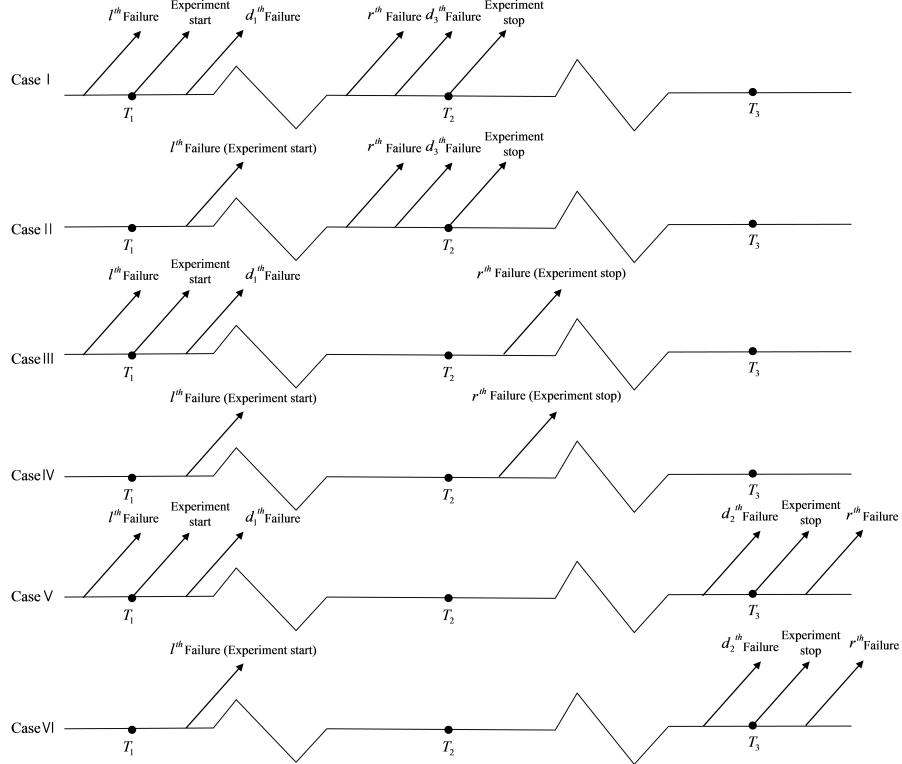


Figure 1: The generalized doubly Type-II hybrid censoring schemes

$\min \{X_{r:n}, T\}$ , where  $r \in \{1, 2, \dots, n\}$  and  $T \in (0, \infty)$  are pre-fixed. Following Childs *et al.* (2003) introduced a Type-I hybrid censoring scheme and Type-II hybrid censoring scheme. The disadvantage of the Type-I hybrid censoring scheme is that there is a possibility that very few failures may occur before time  $T$ ; however, the Type-II hybrid censoring scheme can guarantee a pre-fixed number of failures. In this case, the termination point is  $T_2^* = \max \{X_{r:n}, T\}$ , where  $r \in \{1, 2, \dots, n\}$  and  $T \in (0, \infty)$  are pre-fixed. Though the Type-II hybrid censored scheme guarantees a pre-fixed number of failures, it might take a long time to observe  $r$  failures. In order to provide a guarantee in terms of the number of failures observed as well as time to complete the test, Chandrasekar *et al.* (2004) introduced generalized Type-II hybrid censoring scheme.

The rest of the paper is organized as follows. In Section 2, we propose the generalized doubly Type-II hybrid censoring scheme. In Section 3, we derive some AMLEs of the scale parameter  $\sigma$  for the half logistic distribution under the proposed generalized doubly Type-II hybrid censoring samples. The scale parameter is estimated by the AMLE method using two different Taylor series expansion types.

## 2. Generalized Doubly Type-II Hybrid Censoring Scheme

Consider a life testing experiment in which  $n$  units are put on test. The generalized doubly Type-II hybrid censoring scheme described as follows. Fix  $1 \leq r \leq n$ , and  $T_1, T_2, T_3 \in (0, \infty)$  such that  $T_1 < T_2 < T_3$ . If the  $l^{th}$  failure occurs before time  $T_1$ , start the experiment at  $T_1$ ; if the  $l^{th}$  failure

occurs after time  $T_1$ , start at  $X_{l:n}$ . If the  $r^{th}$  failure occurs before time  $T_2$ , terminate the experiment at  $T_2$ ; if the  $r^{th}$  failure occurs between  $T_2$  and  $T_3$ , terminate at  $X_{r:n}$ ; otherwise, terminate the test at  $T_3$ . Therefore,  $T_1$  represents the start time that the researcher observes the experiment.  $T_2$  represents the least time that the researcher test the experiment.  $T_3$  represents the longest time that the researcher is willing to allow the experiment to continue.

Assuming that the failure times of the units are the half logistic distribution with pdf (1.1). The log-likelihood function (without the constant term) of the censored sample for case I is

$$\ln L_I = -A_I \ln \sigma + (d_1 - 1) \ln F(T_1) + (n - d_3) \ln [1 - F(T_2)] + \sum_{i=d_1}^{d_3} \ln f(x_{i:n}), \quad (2.1)$$

for case II, it is

$$\ln L_{II} = -A_{II} \ln \sigma + (l - 1) \ln F(x_{l:n}) + (n - d_3) \ln [1 - F(T_2)] + \sum_{i=l}^{d_3} \ln f(x_{i:n}), \quad (2.2)$$

for case III, it is

$$\ln L_{III} = -A_{III} \ln \sigma + (d_1 - 1) \ln F(T_l) + (n - r) \ln [1 - F(x_{r:n})] + \sum_{i=d_1}^r \ln f(x_{i:n}), \quad (2.3)$$

for case IV, it is

$$\ln L_{IV} = -A_{IV} \ln \sigma + (l - 1) \ln F(x_{l:n}) + (n - r) \ln [1 - F(x_{r:n})] + \sum_{i=l}^r \ln f(x_{i:n}), \quad (2.4)$$

for case V, it is

$$\ln L_V = -A_V \ln \sigma + (d_1 - 1) \ln F(T_l) + (n - d_2) \ln [1 - F(T_3)] + \sum_{i=d_1}^{d_2} \ln f(x_{i:n}), \quad (2.5)$$

for case VI, it is

$$\ln L_{VI} = -A_{VI} \ln \sigma + (l - 1) \ln F(x_{l:n}) + (n - d_2) \ln [1 - F(T_3)] + \sum_{i=l}^{d_2} \ln f(x_{i:n}), \quad (2.6)$$

where  $A_I = d_3 - d_1 - 1$ ,  $A_{II} = d_3 - l - 1$ ,  $A_{III} = r - d_1 - 1$ ,  $A_{IV} = r - l - 1$ ,  $A_V = d_2 - d_1 - 1$  and  $A_{VI} = d_2 - l - 1$ .

Let  $Z_{i:n} = X_{i:n}/\sigma$ ,  $Z_{i:n}$  has a standard half logistic distribution with pdf and cdf as follow,

$$f(z_{i:n}) = \frac{2e^{-z_{i:n}}}{(1 + e^{-z_{i:n}})^2}, \quad F(z_{i:n}) = \frac{1 - e^{-z_{i:n}}}{1 + e^{-z_{i:n}}}.$$

The  $f'(z_{i:n})$  and  $f(z_{i:n})$  satisfy as

$$f'(z_{i:n}) = -F(z_{i:n})f(z_{i:n}), \quad f(z_{i:n}) = \frac{[1 - F(z_{i:n})][1 + F(z_{i:n})]}{2}.$$

On differentiating the log-likelihood functions with respect to  $\sigma$  of (2.1) ~ (2.6) and equation to zero, we obtain the estimating equations as

$$\begin{aligned} \frac{\partial \ln L_I}{\partial \sigma} = & -\frac{1}{2\sigma} \left[ 2A_I + (d_1 - 1) \left\{ \frac{1}{F(z_{T_1})} - F(z_{T_1}) \right\} z_{T_1} \right. \\ & \left. - (n - d_3) \{1 + F(z_{T_2})\} z_{T_2} - 2 \sum_{i=d_1}^{d_3} F(z_{i:n}) z_{i:n} \right] = 0, \end{aligned} \quad (2.7)$$

$$\begin{aligned} \frac{\partial \ln L_{II}}{\partial \sigma} = & -\frac{1}{2\sigma} \left[ 2A_{II} + (l - 1) \left\{ \frac{1}{F(z_{l:n})} - F(z_{l:n}) \right\} z_{l:n} \right. \\ & \left. - (n - d_3) \{1 + F(z_{T_2})\} z_{T_2} - 2 \sum_{i=l}^{d_3} F(z_{i:n}) z_{i:n} \right] = 0, \end{aligned} \quad (2.8)$$

$$\begin{aligned} \frac{\partial \ln L_{III}}{\partial \sigma} = & -\frac{1}{2\sigma} \left[ 2A_{III} + (d_1 - 1) \left\{ \frac{1}{F(z_{T_1})} - F(z_{T_1}) \right\} z_{T_1} \right. \\ & \left. - (n - r) \{1 + F(z_{r:n})\} z_{r:n} - 2 \sum_{i=d_1}^r F(z_{i:n}) z_{i:n} \right] = 0, \end{aligned} \quad (2.9)$$

$$\begin{aligned} \frac{\partial \ln L_{IV}}{\partial \sigma} = & -\frac{1}{2\sigma} \left[ 2A_{IV} + (l - 1) \left\{ \frac{1}{F(z_{l:n})} - F(z_{l:n}) \right\} z_{l:n} \right. \\ & \left. - (n - r) \{1 + F(z_{r:n})\} z_{r:n} - 2 \sum_{i=l}^r F(z_{i:n}) z_{i:n} \right] = 0, \end{aligned} \quad (2.10)$$

$$\begin{aligned} \frac{\partial \ln L_V}{\partial \sigma} = & -\frac{1}{2\sigma} \left[ 2A_V + (d_1 - 1) \left\{ \frac{1}{F(z_{T_1})} - F(z_{T_1}) \right\} z_{T_1} \right. \\ & \left. - (n - d_2) \{1 + F(z_{T_3})\} z_{T_3} - 2 \sum_{i=d_1}^{d_2} F(z_{i:n}) z_{i:n} \right] = 0, \end{aligned} \quad (2.11)$$

$$\begin{aligned} \frac{\partial \ln L_{VI}}{\partial \sigma} = & -\frac{1}{2\sigma} \left[ 2A_{VI} + (l - 1) \left\{ \frac{1}{F(z_{l:n})} - F(z_{l:n}) \right\} z_{l:n} \right. \\ & \left. - (n - d_2) \{1 + F(z_{T_3})\} z_{T_3} - 2 \sum_{i=l}^{d_2} F(z_{i:n}) z_{i:n} \right] = 0, \end{aligned} \quad (2.12)$$

respectively, where  $z_{T_1} = T_1/\sigma$ ,  $z_{T_2} = T_2/\sigma$  and  $z_{T_3} = T_3/\sigma$ .

### 3. Approximate Maximum Likelihood Estimators

Because the log-likelihood equations cannot be solved explicitly, it will be desirable to consider an approximation to the likelihood equations that provide us with explicit estimators for the scale parameter.

We expand the functions  $z_{i:n}/F(z_{i:n})$  and  $F(z_{i:n})z_{i:n}$  in Taylor series around the points  $\xi_i$ , where  $\xi_i = F^{-1}(p_i) = -\ln\{q_i/(1+p_i)\}$ ,  $p_i = i/(n+1)$ .

First, we can approximate the functions by

$$\frac{z_{i:n}}{F(z_{i:n})} \approx \alpha_{1i} + \beta_{1i} z_{i:n}, \quad (3.1)$$

$$F(z_{i:n}) z_{i:n} \simeq \gamma_{1i} + \delta_{1i} z_{i:n}, \quad (3.2)$$

where

$$\begin{aligned} \alpha_{1i} &= \frac{1 - p_i^2}{2} \left( \frac{\xi_i}{p_i} \right)^2, & \beta_{1i} &= \frac{1}{p_i} \left( 1 - \frac{1 - p_i^2}{2p_i} \xi_i \right), \\ \gamma_{1i} &= -\frac{1 - p_i^2}{2} \xi_i^2, & \delta_{1i} &= \frac{1 - p_i^2}{2} \xi_i + p_i. \end{aligned}$$

By substituting the equations (3.1) and (3.2) into the equation (2.7)~(2.12), we may approximate the equation in (2.7)~(2.12) by

$$\begin{aligned} \frac{\partial \ln L_I}{\partial \sigma} \simeq & -\frac{1}{2\sigma} \left[ 2A_I + (d_1 - 1) (\alpha_{1d_1^*} + \beta_{1d_1^*} z_{T_1}) - (d_1 - 1) (\gamma_{1d_1^*} + \delta_{1d_1^*} z_{T_1}) \right. \\ & \left. - (n - d_3) z_{T_2} - (n - d_3) (\gamma_{1d_3^*} + \delta_{1d_3^*} z_{T_2}) - 2 \sum_{i=d_1}^{d_3} (\gamma_{1i} + \delta_{1i} z_{i:n}) \right] = 0, \end{aligned} \quad (3.3)$$

$$\begin{aligned} \frac{\partial \ln L_{II}}{\partial \sigma} \simeq & -\frac{1}{2\sigma} \left[ 2A_{II} + (l - 1) (\alpha_{1l} + \beta_{1l} z_{l:n}) - (l - 1) (\gamma_{1l} + \delta_{1l} z_{l:n}) \right. \\ & \left. - (n - d_3) z_{T_2} - (n - d_3) (\gamma_{1d_3^*} + \delta_{1d_3^*} z_{T_2}) - 2 \sum_{i=l}^{d_3} (\gamma_{1i} + \delta_{1i} z_{i:n}) \right] = 0, \end{aligned} \quad (3.4)$$

$$\begin{aligned} \frac{\partial \ln L_{III}}{\partial \sigma} \simeq & -\frac{1}{2\sigma} \left[ 2A_{III} + (d_1 - 1) (\alpha_{1d_1^*} + \beta_{1d_1^*} z_{T_1}) - (d_1 - 1) (\gamma_{1d_1^*} + \delta_{1d_1^*} z_{T_1}) \right. \\ & \left. - (n - r) z_{r:n} - (n - r) (\gamma_{1r} + \delta_{1r} z_{r:n}) - 2 \sum_{i=d_1}^r (\gamma_{1i} + \delta_{1i} z_{i:n}) \right] = 0, \end{aligned} \quad (3.5)$$

$$\begin{aligned} \frac{\partial \ln L_{IV}}{\partial \sigma} \simeq & -\frac{1}{2\sigma} \left[ 2A_{IV} + (l - 1) (\alpha_{1l} + \beta_{1l} z_{l:n}) - (l - 1) (\gamma_{1l} + \delta_{1l} z_{l:n}) \right. \\ & \left. - (n - r) z_{r:n} - (n - r) (\gamma_{1r} + \delta_{1r} z_{r:n}) - 2 \sum_{i=l}^r (\gamma_{1i} + \delta_{1i} z_{i:n}) \right] = 0, \end{aligned} \quad (3.6)$$

$$\begin{aligned} \frac{\partial \ln L_V}{\partial \sigma} \simeq & -\frac{1}{2\sigma} \left[ 2A_V + (d_1 - 1) (\alpha_{1d_1^*} + \beta_{1d_1^*} z_{T_1}) - (d_1 - 1) (\gamma_{1d_1^*} + \delta_{1d_1^*} z_{T_1}) \right. \\ & \left. - (n - d_2) z_{T_3} - (n - d_2) (\gamma_{1d_2^*} + \delta_{1d_2^*} z_{T_3}) - 2 \sum_{i=d_1}^{d_2} (\gamma_{1i} + \delta_{1i} z_{i:n}) \right] = 0, \end{aligned} \quad (3.7)$$

$$\begin{aligned} \frac{\partial \ln L_{VI}}{\partial \sigma} \simeq & -\frac{1}{2\sigma} \left[ 2A_{VI} + (l - 1) (\alpha_{1l} + \beta_{1l} z_{l:n}) - (l - 1) (\gamma_{1l} + \delta_{1l} z_{l:n}) \right. \\ & \left. - (n - d_2) z_{T_3} - (n - d_2) (\gamma_{1d_2^*} + \delta_{1d_2^*} z_{T_3}) - 2 \sum_{i=l}^{d_2} (\gamma_{1i} + \delta_{1i} z_{i:n}) \right] = 0. \end{aligned} \quad (3.8)$$

We can derive AMLE as follows;

$$\hat{\sigma}_{1I} = -\frac{B_{1I}}{C_{1I}}, \quad (3.9)$$

$$\hat{\sigma}_{1II} = -\frac{B_{1II}}{C_{1II}}, \quad (3.10)$$

$$\hat{\sigma}_{1III} = -\frac{B_{1III}}{C_{1III}}, \quad (3.11)$$

$$\hat{\sigma}_{1IV} = -\frac{B_{1IV}}{C_{1IV}}, \quad (3.12)$$

$$\hat{\sigma}_{1V} = -\frac{B_{1V}}{C_{1V}}, \quad (3.13)$$

$$\hat{\sigma}_{1VI} = -\frac{B_{1VI}}{C_{1VI}}, \quad (3.14)$$

where

$$B_{1I} = (d_1 - 1) (\beta_{1d_1^*} - \delta_{1d_1^*}) T_1 - (n - d_3) (1 + \delta_{1d_3^*}) T_2 - 2 \sum_{i=d_1}^{d_3} \delta_{1i} x_{i:n},$$

$$C_{1I} = 2A_I + (d_1 - 1) (\alpha_{1d_1^*} - \gamma_{1d_1^*}) - (n - d_3) \gamma_{1d_3^*} - 2 \sum_{i=d_1}^{d_3} \gamma_{1i},$$

$$B_{1II} = (l - 1) (\beta_{1l} - \delta_{1l}) x_{l:n} - (n - d_3) (1 + \delta_{1d_3^*}) T_2 - 2 \sum_{i=l}^{d_3} \delta_{1i} x_{i:n},$$

$$C_{1II} = 2A_{II} + (l - 1) (\alpha_{1l} - \gamma_{1l}) - (n - d_3) \gamma_{1d_3^*} - 2 \sum_{i=l}^{d_3} \gamma_{1i},$$

$$B_{1III} = (d_1 - 1) (\beta_{1d_1^*} - \delta_{1d_1^*}) T_1 - (n - r) (1 + \delta_{1r}) x_{r:n} - 2 \sum_{i=d_1}^r \delta_{1i} x_{i:n},$$

$$C_{1III} = 2A_{III} + (d_1 - 1) (\alpha_{1d_1^*} - \gamma_{1d_1^*}) - (n - r) \gamma_{1r} - 2 \sum_{i=d_1}^r \gamma_{1i},$$

$$B_{1IV} = (l - 1) (\beta_{1l} - \delta_{1l}) x_{l:n} - (n - r) (1 + \delta_{1r}) x_{r:n} - 2 \sum_{i=l}^r \delta_{1i} x_{i:n},$$

$$C_{1IV} = 2A_{IV} + (l - 1) (\alpha_{1l} - \gamma_{1l}) - (n - r) \gamma_{1r} - 2 \sum_{i=l}^r \gamma_{1i},$$

$$B_{1V} = (d_1 - 1) (\beta_{1d_1^*} - \delta_{1d_1^*}) T_1 - (n - d_2) (1 + \delta_{1d_2^*}) T_3 - 2 \sum_{i=d_1}^{d_2} \delta_{1i} x_{i:n},$$

$$C_{1V} = 2A_V + (d_1 - 1) (\alpha_{1d_1^*} - \gamma_{1d_1^*}) - (n - d_2) \gamma_{1d_2^*} - 2 \sum_{i=d_1}^{d_2} \gamma_{1i},$$

$$B_{1VI} = (l - 1) (\beta_{1l} - \delta_{1l}) x_{l:n} - (n - d_2) (1 + \delta_{1d_2^*}) T_3 - 2 \sum_{i=l}^{d_2} \delta_{1i} x_{i:n},$$

$$C_{1VI} = 2A_{VI} + (l - 1) (\alpha_{1l} - \gamma_{1l}) - (n - d_2) \gamma_{1d_2^*} - 2 \sum_{i=l}^{d_2} \gamma_{1i}.$$

Since  $\alpha_{1i} > 0$ ,  $\gamma_{1i} > 0$ ,  $\beta_{1i} - \gamma_{1i} < 0$  and  $\delta_{1i} > 0$ , the estimator  $\hat{\sigma}_I(\hat{\sigma}_{1I}, \hat{\sigma}_{1II}, \hat{\sigma}_{1III}, \hat{\sigma}_{1IV}, \hat{\sigma}_{1V}, \hat{\sigma}_{1VI})$  is always positive. Second, we can approximate the functions by the equations,

$$\frac{1}{F(z_{i:n})} \simeq \alpha_{2i} + \beta_{2i} z_{i:n}, \quad (3.15)$$

$$F(z_{i:n}) \simeq \gamma_{2i} + \delta_{2i} z_{i:n}, \quad (3.16)$$

where

$$\begin{aligned} \alpha_{2i} &= \frac{1}{p_i} \left( 1 + \frac{1 - p_i^2}{2p_i} \xi_i \right), & \beta_{2i} &= -\frac{1 - p_i^2}{2p_i^2}, \\ \gamma_{2i} &= p_i - \frac{1 - p_i^2}{2} \xi_i, & \delta_{2i} &= \frac{1 - p_i^2}{2}. \end{aligned}$$

By substituting the equations (3.15) and (3.16) into the equation (2.7) ~ (2.12), we may approximate the equations in (2.7) ~ (2.12) by

$$\begin{aligned} \frac{\partial \ln L_I}{\partial \sigma} &\simeq -\frac{1}{2\sigma} \left[ 2A_I + (d_1 - 1) \{ (\alpha_{2d_1^*} + \beta_{2d_1^*} z_{T_1}) - (\gamma_{2d_1^*} + \delta_{2d_1^*} z_{T_1}) \} z_{T_1} - (n - d_3) z_{T_2} \right. \\ &\quad \left. - (n - d_3) (\gamma_{2d_3^*} + \delta_{2d_3^*} z_{T_2}) z_{T_2} - 2 \sum_{i=d_1}^{d_3} (\gamma_{2i} + \delta_{2i} z_{i:n}) z_{i:n} \right] = 0, \end{aligned} \quad (3.17)$$

$$\begin{aligned} \frac{\partial \ln L_{II}}{\partial \sigma} &\simeq -\frac{1}{2\sigma} \left[ 2A_{II} + (l - 1) \{ (\alpha_{2l} + \beta_{2l} z_{l:n}) - (\gamma_{2l} + \delta_{2l} z_{l:n}) \} z_{l:n} - (n - d_3) z_{T_2} \right. \\ &\quad \left. - (n - d_3) (\gamma_{2d_3^*} + \delta_{2d_3^*} z_{T_2}) z_{T_2} - 2 \sum_{i=l}^{d_3} (\gamma_{2i} + \delta_{2i} z_{i:n}) z_{i:n} \right] = 0, \end{aligned} \quad (3.18)$$

$$\begin{aligned} \frac{\partial \ln L_{III}}{\partial \sigma} &\simeq -\frac{1}{2\sigma} \left[ 2A_{III} + (d_1 - 1) \{ (\alpha_{2d_1^*} + \beta_{2d_1^*} z_{T_1}) - (\gamma_{2d_1^*} + \delta_{2d_1^*} z_{T_1}) \} z_{T_1} - (n - r) z_{r:n} \right. \\ &\quad \left. - (n - r) (\gamma_{2r} + \delta_{2r} z_{r:n}) z_{r:n} - 2 \sum_{i=d_1}^r (\gamma_{2i} + \delta_{2i} z_{i:n}) z_{i:n} \right] = 0, \end{aligned} \quad (3.19)$$

$$\begin{aligned} \frac{\partial \ln L_{IV}}{\partial \sigma} &\simeq -\frac{1}{2\sigma} \left[ 2A_{IV} + (l - 1) \{ (\alpha_{2l} + \beta_{2l} z_{l:n}) - (\gamma_{2l} + \delta_{2l} z_{l:n}) \} z_{l:n} - (n - r) z_{r:n} \right. \\ &\quad \left. - (n - r) (\gamma_{2r} + \delta_{2r} z_{r:n}) z_{r:n} - 2 \sum_{i=l}^r (\gamma_{2i} + \delta_{2i} z_{i:n}) z_{i:n} \right] = 0, \end{aligned} \quad (3.20)$$

$$\begin{aligned} \frac{\partial \ln L_V}{\partial \sigma} &\simeq -\frac{1}{2\sigma} \left[ 2A_V + (d_1 - 1) \{ (\alpha_{2d_1^*} + \beta_{2d_1^*} z_{T_1}) - (\gamma_{2d_1^*} + \delta_{2d_1^*} z_{T_1}) \} z_{T_1} - (n - d_2) z_{T_3} \right. \\ &\quad \left. - (n - d_2) (\gamma_{2d_2^*} + \delta_{2d_2^*} z_{T_3}) z_{T_3} - 2 \sum_{i=d_1}^{d_2} (\gamma_{2i} + \delta_{2i} z_{i:n}) z_{i:n} \right] = 0, \end{aligned} \quad (3.21)$$

$$\begin{aligned} \frac{\partial \ln L_{VI}}{\partial \sigma} &\simeq -\frac{1}{2\sigma} \left[ 2A_{VI} + (l - 1) \{ (\alpha_{2l} + \beta_{2l} z_{l:n}) - (\gamma_{2l} + \delta_{2l} z_{l:n}) \} z_{l:n} - (n - d_2) z_{T_3} \right. \\ &\quad \left. - (n - d_2) (\gamma_{2d_2^*} + \delta_{2d_2^*} z_{T_3}) z_{T_3} - 2 \sum_{i=l}^{d_2} (\gamma_{2i} + \delta_{2i} z_{i:n}) z_{i:n} \right] = 0. \end{aligned} \quad (3.22)$$

The equations (3.17) ~ (3.22) are a quadratic equation in  $\sigma$ , with its roots given by

$$\hat{\sigma}_{2I} = \frac{-B_{2I} + \sqrt{B_{2I}^2 - 8A_I C_{2I}}}{4A_I}, \quad (3.23)$$

$$\hat{\sigma}_{2II} = \frac{-B_{2II} + \sqrt{B_{2II}^2 - 8A_{II} C_{2II}}}{4A_{II}}, \quad (3.24)$$

$$\hat{\sigma}_{2III} = \frac{-B_{2III} + \sqrt{B_{2III}^2 - 8A_{III} C_{2III}}}{4A_{III}}, \quad (3.25)$$

$$\hat{\sigma}_{2IV} = \frac{-B_{2IV} + \sqrt{B_{2IV}^2 - 8A_{IV} C_{2IV}}}{4A_{IV}}, \quad (3.26)$$

$$\hat{\sigma}_{2V} = \frac{-B_{2V} + \sqrt{B_{2V}^2 - 8A_V C_{2V}}}{4A_V}, \quad (3.27)$$

$$\hat{\sigma}_{2VI} = \frac{-B_{2VI} + \sqrt{B_{2VI}^2 - 8A_{VI} C_{2VI}}}{4A_{VI}}, \quad (3.28)$$

where

$$B_{2I} = (d_1 - 1)(\alpha_{2d_1^*} - \gamma_{2d_1^*})T_1 - (n - d_3)(1 + \gamma_{2d_3^*})T_2 - 2 \sum_{i=d_1}^{d_3} \gamma_{2i} x_{i:n},$$

$$C_{2I} = (d_1 - 1)(\beta_{2d_1^*} - \delta_{2d_1^*})T_1^2 - (n - d_3)\delta_{2d_3^*}T_2^2 - 2 \sum_{i=d_1}^{d_3} \delta_{2i} x_{i:n}^2,$$

$$B_{2II} = (l - 1)(\alpha_{2l} - \gamma_{2l})x_{l:n} - (n - d_3)(1 + \gamma_{2d_3^*})T_2 - 2 \sum_{i=l}^{d_3} \gamma_{2i} x_{i:n},$$

$$C_{2II} = (l - 1)(\beta_{2l} - \delta_{2l})x_{l:n}^2 - (n - d_3)\delta_{2d_3^*}T_2^2 - 2 \sum_{i=l}^{d_3} \delta_{2i} x_{i:n}^2,$$

$$B_{2III} = (d_1 - 1)(\alpha_{2d_1^*} - \gamma_{2d_1^*})T_1 - (n - r)(1 + \gamma_{2r})x_{r:n} - 2 \sum_{i=d_1}^r \gamma_{2i} x_{i:n},$$

$$C_{2III} = (d_1 - 1)(\beta_{2d_1^*} - \delta_{2d_1^*})T_1^2 - (n - r)\delta_{2r}x_{r:n}^2 - 2 \sum_{i=d_1}^r \delta_{2i} x_{i:n}^2,$$

$$B_{2IV} = (l - 1)(\alpha_{2l} - \gamma_{2l})x_{l:n} - (n - r)(1 + \gamma_{2r})x_{r:n} - 2 \sum_{i=l}^r \gamma_{2i} x_{i:n},$$

$$C_{2IV} = (l - 1)(\beta_{2l} - \delta_{2l})x_{l:n}^2 - (n - r)\delta_{2r}x_{r:n}^2 - 2 \sum_{i=l}^r \delta_{2i} x_{i:n}^2,$$

$$B_{2V} = (d_1 - 1)(\alpha_{2d_1^*} - \gamma_{2d_1^*})T_1 - (n - d_2)(1 + \gamma_{2d_2^*})T_3 - 2 \sum_{i=d_1}^{d_2} \gamma_{2i} x_{i:n},$$

**Table 1:** The relative mean squared errors and biases for the estimators of the scale parameter  $\sigma$ 

$T_1$	$T_2$	$T_3$	$n$	$l$	$r$	MSE(Bias)	
						AMLE <sub>I</sub>	AMLE <sub>II</sub>
0.3	1.0	3.0	20	1	17	0.054352(0.073802)	0.064904(0.108680)
				3	18	0.054181(0.074089)	0.066555(0.110364)
				3	17	0.054379(0.073875)	0.066376(0.109779)
				3	16	0.056418(0.075457)	0.068473(0.111714)
				5	18	0.054306(0.074472)	0.064857(0.107612)
			30	6	18	0.054396(0.074721)	0.064678(0.106242)
				1	27	0.031426(0.045903)	0.036569(0.070705)
				3	27	0.031430(0.045923)	0.036345(0.069890)
				3	26	0.031491(0.044833)	0.036145(0.068772)
				6	26	0.031538(0.045023)	0.035959(0.067344)
0.4	1.5	3.0	40	6	25	0.032462(0.045069)	0.037028(0.067532)
				8	23	0.034834(0.049993)	0.039452(0.071845)
				3	37	0.022456(0.032356)	0.025404(0.051850)
				3	35	0.022553(0.030478)	0.025343(0.049969)
				3	30	0.024873(0.033360)	0.028173(0.054516)
			30	6	37	0.022460(0.032384)	0.025071(0.050512)
				7	35	0.022558(0.030527)	0.024904(0.048077)
				7	30	0.024882(0.033416)	0.027656(0.052234)
				1	17	0.054283(0.074216)	0.066025(0.111866)
				3	18	0.054185(0.074372)	0.065943(0.110207)
0.5	2.0	3.0	20	3	17	0.054305(0.074264)	0.065869(0.109913)
				3	16	0.055951(0.076615)	0.067418(0.112641)
				5	18	0.054274(0.074701)	0.065429(0.108899)
				6	18	0.054376(0.074904)	0.064909(0.107315)
			30	1	27	0.031426(0.046062)	0.036881(0.070562)
				3	27	0.031430(0.046072)	0.036540(0.069538)
				3	26	0.031369(0.045002)	0.036450(0.068507)
				6	26	0.031401(0.045157)	0.036345(0.067636)
				6	25	0.032365(0.045314)	0.037391(0.067900)
0.6	2.5	3.0	40	8	23	0.034196(0.051259)	0.039216(0.07339)
				3	37	0.022456(0.032446)	0.025720(0.051656)
				3	35	0.022459(0.030569)	0.025651(0.049720)
				3	30	0.024507(0.034242)	0.028286(0.054920)
				6	37	0.022460(0.032468)	0.025220(0.050317)
			30	7	35	0.022465(0.030606)	0.025032(0.047921)
				7	30	0.024515(0.034284)	0.027538(0.052830)

$$C_{2V} = (d_1 - 1) (\beta_{2d_1^*} - \delta_{2d_1^*}) T_1^2 - (n - d_2) \delta_{2d_2^*} T_3^2 - 2 \sum_{i=d_1}^{d_2} \delta_{2i} x_{i:n}^2,$$

$$B_{2VI} = (l - 1) (\alpha_{2l} - \gamma_{2l}) x_{l:n} - (n - d_2) (1 + \gamma_{2d_2^*}) T_3 - 2 \sum_{i=l}^{d_2} \gamma_{2i} x_{i:n},$$

$$C_{2VI} = (l - 1) (\beta_{2l} - \delta_{2l}) x_{l:n}^2 - (n - d_2) \delta_{2d_2^*} T_3^2 - 2 \sum_{i=l}^{d_2} \delta_{2i} x_{i:n}^2.$$

Since  $C_2 < 0$ , only one root is admissible.

#### 4. Simulated Results

From the Section 3, the mean squared errors of the estimators are simulated by the Monte Carlo method (based on 10,000 Monte Carlo runs) for sample size  $n = 20, 30, 40$ , and different  $l, r, T_1, T_2$

**Table 2:** The relative mean squared errors and biases for the estimators of the scale parameter  $\sigma$ 

$T_1$	$T_2$	$T_3$	$n$	$l$	$r$	MSE(Bias)	
						AMLE <sub>I</sub>	AMLE <sub>II</sub>
0.3	1.0	4.0	20	1	17	0.051010(0.070345)	0.060259(0.104234)
				3	18	0.049103(0.067431)	0.059438(0.101882)
				3	17	0.051030(0.070417)	0.061809(0.105413)
				3	16	0.054883(0.074083)	0.066393(0.110018)
				5	18	0.049220(0.067806)	0.057989(0.099284)
				6	18	0.049268(0.068038)	0.057942(0.097974)
			30	1	27	0.028877(0.041533)	0.033100(0.065467)
				3	27	0.028882(0.041553)	0.032911(0.064682)
				3	26	0.029580(0.042165)	0.033654(0.065538)
				6	26	0.029621(0.042350)	0.033530(0.064200)
			40	6	25	0.031384(0.043726)	0.035549(0.065912)
				8	23	0.034624(0.049804)	0.039118(0.071583)
				3	37	0.021070(0.027957)	0.023581(0.046827)
				3	35	0.021811(0.028801)	0.024459(0.047955)
				3	30	0.024873(0.033360)	0.028173(0.054516)
0.4	1.5	4.0	20	6	37	0.021073(0.027985)	0.023307(0.045540)
				7	35	0.021816(0.028850)	0.024034(0.046073)
				7	30	0.024882(0.033416)	0.027656(0.052234)
				1	17	0.050931(0.070755)	0.06162(0.107514)
				3	18	0.049093(0.067700)	0.059198(0.101858)
			30	3	17	0.050949(0.070802)	0.061498(0.105593)
				3	16	0.054414(0.075240)	0.065430(0.110973)
				5	18	0.049183(0.068026)	0.058577(0.100499)
				6	18	0.049245(0.068215)	0.058282(0.099046)
				1	27	0.028873(0.041686)	0.033528(0.065389)
			40	3	27	0.028876(0.041696)	0.033190(0.064376)
				3	26	0.029553(0.042331)	0.034003(0.065302)
				6	26	0.029581(0.042485)	0.033913(0.064471)
				6	25	0.031285(0.043970)	0.035892(0.066259)
				8	23	0.033986(0.051069)	0.038882(0.073128)
			30	3	37	0.021070(0.028043)	0.023865(0.046571)
				3	35	0.021806(0.028890)	0.024737(0.047688)
				3	30	0.024507(0.034242)	0.028286(0.054920)
				6	37	0.021073(0.028064)	0.023416(0.045257)
				7	35	0.021813(0.028927)	0.024131(0.045890)
				7	30	0.024515(0.034284)	0.027538(0.052830)

and  $T_3$  values. we mainly compare the performances of the proposed estimators of the scale parameter  $\sigma$ , in terms of their biases and mean squared errors for different censoring schemes.

From Table 1 and Table 2, the following general observations can be made. For all methods, for fixed  $l, r, T_1, T_2$  and  $T_3$ , the MSEs decrease as  $n$  increases from 20 to 40, for fixed  $l, r, T_1, T_2$  and  $T_3$ , the MSEs decrease as  $r$  increases.

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