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A GENERAL MULTIPLE-TIME-SCALE METHOD FOR SOLVING AN n-TH ORDER WEAKLY NONLINEAR DIFFERENTIAL EQUATION WITH DAMPING

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Abstract. Based on the multiple-time-scale (MTS) method, a general formula has been presented for solving an n-th, $n = 2, 3, \ldots$, order ordinary differential equation with strong linear damping forces. Like the solution of the unified Krylov-Bogoliubov-Mitropolskii (KBM) method or the general Struble's method, the new solution covers the un-damped, under-damped and over-damped cases. The solutions are identical to those obtained by the unified KBM method and the general Struble's method. The technique is a new form of the classical MTS method. The formulation as well as the determination of the solution from the derived formula is very simple. The method is illustrated by several examples. The general MTS solution reduces to its classical form when the real parts of eigen-values of the unperturbed equation vanish.

1. Introduction

In [5] Hassan has proved that the Krylov-Bogoliubov-Mitropolskii (KBM) method [4, 7] is equivalent to the multiple-time-scale (MTS) method [10, 11] for any order of approximation. Hassan has limited his investigation to second approximation of some second-order ordinary differential equations with small damping effect. But some authors extended these methods to similar second-order differential equation with strong damping effects as well as to higher order differential equations. Popov [12] extended the KBM method to second-order equations with a strong linear damping force. Then utilizing Popov's technique, Bojadziev [1] investigated a damped forced vibration. On the other hand, Bojadziev [2] extended the two-time-scale method to second-order systems with strong damping. Murty *et al.* [9] extended the KBM method to second- and fourth-order over-damped systems. Murty [8] presented a unified KBM method for solving a second-order differential equation, which covers the three cases, *i.e.*, un-damped, under-damped and over-damped. Recently, Shamsul [14] has

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generalized Murty's technique [8] for solving an n-th, $n = 2, 3, \ldots$, order equation

$$x^{(n)} + c_1 x^{(n-1)} + \cdots + c_n x = \varepsilon f(x, \dot{x}, \ldots), \tag{1}$$

where $x^{(j)},\ j \geq 4$ represents a j-th derivative of x, over-dot is used for the first-, second- and third-derivatives, ϵ is a small parameter, $c_j,\ j = 1, 2, \ldots, n$ are constants and f is a nonlinear function.

Shamsul [15] further extended the unified KBM method to study some nonlinear differential equations with slowly varying coefficients. In another recent paper, Shamsul *et al.* [17] have generalized the Struble's technique for solving Eq.(1) and show that the solutions obtained for various damping effect of second- and third-order nonlinear equations are identical to those determined by the unified KBM method [14, 15]. In this paper a general MTS method is presented and it is shown that the solutions are identical to those obtained by the unified KBM [14, 15] method and the general Struble's method [17].

2. The method

To solve Eq.(1), an approximate solution is chosen in the form [15]

$$x(t, \varepsilon) = \sum_{j=1}^{n} a_j(t) + \varepsilon u_1(a_1,\, a_2, \ldots,\, a_n) + \varepsilon^2 u_2(a_1,\, a_2, \ldots,\, a_n) + \cdots . \tag{2}$$

In this paper a set of variables $a_j,\ j = 1, 2, \ldots, n$, have been considered rather than the amplitude and phase variables. Recently these variables are used to present a general formula for solving an n-th, $n = 2, 3, \ldots$, order differential equation with slowly varying coefficients according to the unified KBM method (see [15] for details). Under a suitable transformation, the variables, $a_j,\ j = 1, 2, \ldots, n$, are transformed to the amplitude and phase variables. The choice of the new variables (*i.e.*, $a_j,\ j = 1, 2, \ldots, n$) is important for the formulation of the method as well as determination of an approximate solution from the derived formula. Generally, all the variables, $a_j,\ j = 1, 2, \ldots, n$, depend on several time $t_0, t_1, t_2, \ldots$, where $t = t_0 + \varepsilon t_1 + \varepsilon^2 t_2 + \cdots$. Herein we denote some notations and a relation as

$$D_k() = \partial()/\partial t_k, \quad k = 0,\ 1,\ 2,\ \ldots,\ D_0 a_j = \lambda_j a_j. \tag{3}$$

Now we can rewrite Eq.(1) as

$$\prod_{j=1}^{n} (D - \lambda_j)\, x = \varepsilon f, \tag{4}$$

where $\lambda_j,\ j = 1,\ 2, \ldots, n$, are eigen-values of the unperturbed equation of Eq.(1). Substituting Eq.(2) into Eq.(4) and equating the coefficients of $\varepsilon^1,\ \varepsilon^2$, we obtain

$$
\begin{aligned}
&\sum_{j=1}^{n}\left(\prod_{k=1,\,k\neq j}^{n}(D_0-\lambda_k)(D_1 a_j)\right)+\prod_{j=1}^{n}(D_0-\lambda_j)u_1\\
&= f(\Sigma_{j=1}^{n} a_j,\ \Sigma_{j=1}^{n} D_0 a_j,\ldots)\sum_{j=1}^{n}\left(\prod_{k=1,\,k\neq j}^{n}(D_0-\lambda_k)(D_2 a_j)\right)\\
&\quad+[D_1(D_0^{n-1}+c_1 D_0^{n-2}+c_2 D_0^{n-3}+\cdots)\\
&\quad+D_0 D_1(D_0^{n-2}+c_1 D_0^{n-3}+\cdots)+D_0^2 D_1(D_0^{n-3}+\cdots)+\cdots]\,u_1 \qquad (5)\\
&\quad+\sum_{j=1}^{n}[D_1(D_0^{n-2}+c_1^{(j)}D_0^{n-3}+\cdots)+D_0 D_1(D_0^{n-3}+c_1^{(j)}D_0^{n-4}+\cdots)\\
&\quad+D_0^2 D_1(D_0^{n-4}+\cdots)+\cdots](D_1 a_j)+\prod_{j=1}^{n}(D_0-\lambda_j)u_2\\
&= u_1 f_x(\Sigma_{j=1}^{n} a_j,\ \Sigma_{j=1}^{n} D_0 a_j,\ldots)\\
&\quad+(D_0 u_1+\Sigma_{j=1}^{n} D_0 a_j)\times f_{\dot{x}}(\Sigma_{j=1}^{n} a_j e^{\lambda_j t},\ \Sigma_{j=1}^{n} D_0 a_j,\ldots)+\cdots,
\end{aligned}
$$

where $c_1^{(j)},\ c_2^{(j)},\ldots,c_{n-1}^{(j)}$ are the coefficients of the algebraic equation

$$\prod_{j^{/}=1,\ j^{/}\neq j}^{n}(\lambda-\lambda_{j^{/}})=0. \qquad (6)$$

We can easily find a second approximate solution of Eq. (1) utilizing formula Eq.(5). To avoid the secular terms in solution of Eq.(2), it has been proposed in [15] that $u_1,\ u_2,\ldots$ exclude the terms $\cdots a_{2l-1}^{m_{2l-1}}a_{2l}^{m_{2l}}\cdots$, $m_{2l-1}-m_{2l-1}=\pm 1$, $l=1,\ 2,\ \ldots,\ n/2$ or $(n-1)/2$ according to n is even or odd. This assumption assures that $u_1,\ u_2,\ldots$ as well as solution Eq.(2) will be free from secular terms. When n is an odd number, an additional restriction is imposed that $u_1,\ u_2,\ldots$ exclude all the terms involving a_n (see [3, 15] for details).

3. Example

3.1. A second-order equation

Let us consider the *Duffing* equation with linear damping

$$\ddot{x}+2k\dot{x}+w^2x=-\varepsilon x^3,\ 0<\varepsilon<1. \qquad (7)$$

This equation represents the un-damped, under-damped and over-damped cases depending on the value of damping coefficient, k. If k vanishes, the motion becomes un-damped and periodic. On the other hand, the motion will be under-damped or over-damped if $0<k<w$ or $w<k$. The unperturbed equation has two eigen-values $\lambda_1=-k+i\omega$, $\lambda_2=-k-i\omega$ when $0\leq k<w$, $\omega^2=w^2-k^2$. On the contrary, they become $\lambda_1=-k+\omega$, $\lambda_2=-k-\omega$ when

$w < k$. Since $f = -x^3$, we obtain $f = -(a_1 + a_2)^3$, $f_x = -3(a_1 + a_2)^2$, and formula Eq.(5) becomes

$$(8)\quad \begin{aligned} &(D_0 - \lambda_2)(D_1 a_1) + (D_0 - \lambda_1)(D_1 a_2) + (D_0 - \lambda_1)(D_0 - \lambda_2)u_1 \\ = &-a_1^3 - 3a_1^2 a_2 - 3a_1 a_2^2 - a_2^3, \end{aligned}$$

$$(9)\quad \begin{aligned} &(D_0 - \lambda_2)(D_2 a_1) + (D_0 - \lambda_1)(D_2 a_2) + (D_0 - \lambda_1)(D_0 - \lambda_2)u_2 \\ &+ D_1^2(a_1 + a_2) + [D_0 D_1 + D_1 D_0 - (\lambda_1 + \lambda_2)D_1]u_1 \\ = &-3(a_1^2 + 2a_1 a_2 + a_2^2)u_1. \end{aligned}$$

To solve Eq.(8), it has already been restricted that u_1 excludes terms $a_1^2 a_2$ and $a_1 a_2^2$ (see Section 2). Therefore, Eq.(8) can be separated into three parts for $D_1 a_1$, $D_1 a_2$ and u_1 as

$$(10)\quad (D_0 - \lambda_2)(D_1 a_1) = -3a_1^2 a_2,$$

$$(11)\quad (D_0 - \lambda_1)(D_1 a_2) = -3a_1 a_2^2,$$

$$(12)\quad (D_0 - \lambda_1)(D_0 - \lambda_2)u_1 = -(a_1^3 + a_2^3).$$

Solving the above three equations (see Appendix A and also [15] for solution technique), we obtain

$$(13)\quad D_1 a_1 = l_1 a_1^2 a_2, \quad D_1 a_2 = l_1^* a_1 a_2^2, \quad l_1 = -3/(2\lambda_1), \quad l_1^* = -3/(2\lambda_2),$$

and

$$(14)\quad u_1 = C_1 a_1^3 + C_1^* a_2^3,\ C_1 = -1/[2\lambda_1(3\lambda_1 - \lambda_2)],\ C_1^* = -1/[2\lambda_2(3\lambda_2 - \lambda_1)].$$

Substituting the values of u_1 from Eq.(14) into Eq.(9), utilizing the relations of Eq.(13) and then imposing the restriction that u_2 excludes the terms $a_1^3 a_2^2$ and $a_1^2 a_2^3$, equations for $D_2 a_1$, $D_2 a_2$ and u_2 can be separated into three parts as

$$(15)\quad (D_0 - \lambda_2)(D_2 a_1) = -[l_1(2l_1 + l_1^*) + 3C_1]a_1^3 a_2^2,$$

$$(16)\quad (D_0 - \lambda_1)(D_2 a_2) = -[l_1^*(l_1 + 2l_1^*) + 3C_1^*]a_1^2 a_2^3,$$

$$(17)\quad \begin{aligned} &(D_0 - \lambda_1)(D_0 - \lambda_2)u_2 \\ = &-6[C_1(3\lambda_1 l_1 + 1)a_1^4 a_2 + C_1^*(3\lambda_2 l_1^* + 1)a_1 a_2^4] - 3(C_1 a_1^5 + C_1^* a_2^5). \end{aligned}$$

Solving Eqs.(15)-(17), we obtain

$$(18)\quad \begin{aligned} D_2 a_1 &= l_2 a_1^3 a_2^2, \\ D_2 a_2 &= l_2^* a_1^2 a_2^3, \\ l_2 &= -[l_1(2l_1 + l_1^*) + 3C_1]/(3\lambda_1 + \lambda_2), \\ l_2^* &= -[l_1^*(2l_1^* + l_1) + 3C_1^*]/(3\lambda_2 + \lambda_1), \end{aligned}$$

$$
\begin{aligned}
u_2 &= c_2 a_1^4 a_2 + c_2^* a_1 a_2^4 + e_2 a_1^5 + e_2^* a_2^5,\\
c_2 &= -3c_1(3\lambda_1 l_1 + 1)/[2\lambda_1(3\lambda_1 + \lambda_2)],\\
c_2^* &= -3c_1^*(3\lambda_2 l_1^* + 1)/[2\lambda_2(3\lambda_2 + \lambda_1)],\\
e_2 &= -3c_1/[4\lambda_1(5\lambda_1 - \lambda_2)],\\
e_2^* &= -3c_1^*/[2\lambda_2(3\lambda_2 - \lambda_1)].
\end{aligned} \tag{19}
$$

All these results obtained in Eqs.(13)-(14), (18)-(19) give the second approximate solution of Eq.(7). Now we write the variational equations as follows

$$
\begin{aligned}
\dot a_1 &= D\,a_1 = (D_0 + \varepsilon D_1 + \cdots)\,a_1 = \lambda_1 a_1 + \varepsilon l_1 a_1^2 a_2 + \varepsilon^2 l_2 a_1^3 a_2^2 + \mathrm{O}(\varepsilon^3),\\
\dot a_2 &= D\,a_2 = (D_0 + \varepsilon D_1 + \cdots)\,a_2 = \lambda_2 a_2 + \varepsilon l_1^* a_1 a_2^2 + \varepsilon^2 l_2^* a_1^2 a_2^3 + \mathrm{O}(\varepsilon^3).
\end{aligned} \tag{20}
$$

Using the unified KBM method [14] and general Struble's technique [17], we will have the same result as obtained in Eq.(20). As a verification of this result, we may choose a known problem. Rink [13] found a third approximate solution of $\ddot x + 3\dot x + 2x = \mu x^3$, $\mu << 1$, based on the KBM method. We shall compare our solution to that of Rink. Clearly, this equation is similar to equation, $\ddot x + 2k\dot x + \omega^2 x = -\varepsilon x^3$, where $2k = 3$, $\omega^2 = 2$ and $\varepsilon = -\mu$. In this case $\lambda_1 = -1$ and $\lambda_2 = -2$. Therefore, $l_1 = 3/2$, $l_1^* = 3/4$, $C_1 = -1/2$, $C_1^* = -1/20$; $l_2 = 33/40$, $l_2^* = 3/10$, $C_2 = -21/40$, $C_2^* = -3/160$; $E_2 = 1/8$, $E_2^* = 1/480$. Substituting the values of l_1, l_1^*, l_2, l_2^* into Eq.(20) and replacing the variables a_1, a_2 by $\frac{1}{2}ae^{\varphi}$, $\frac{1}{2}ae^{-\varphi}$, we obtain

$$
\begin{aligned}
(\dot a + a\dot\varphi)e^{\varphi}/2 &= -ae^{\varphi}/2 - 3\varepsilon(a/2)^3 e^{\varphi}/2 + 33\varepsilon^2(a/2)^5 e^{\varphi}/40 + \mathrm{O}(\varepsilon^3),\\
(\dot a - a\dot\varphi)e^{-\varphi}/2 &= -ae^{-\varphi} - 3\varepsilon(a/2)^3 e^{-\varphi}/4 + 3\varepsilon^2(a/2)^5 e^{-\varphi}/10 + \mathrm{O}(\varepsilon^3).
\end{aligned} \tag{21}
$$

By adding and subtracting, we can easily obtain the values of $\dot a$ and $\dot\varphi$ as follows:

$$
\begin{aligned}
\dot a &= -3a/2 + 9\varepsilon a^3/32 + 9\varepsilon^2 a^5/256 + \mathrm{O}(\varepsilon^3),\\
\dot\varphi &= 1/2 + 3\varepsilon a^2/32 + 21\varepsilon^2 a^4/1280 + \mathrm{O}(\varepsilon^3).
\end{aligned} \tag{22}
$$

Now substituting the values of C_1, C_1^*; C_2, C_2^*, E_2, E_2^* into Eqs.(14), (19), and replacing the variables a_1, a_2 by $\frac{1}{2}ae^{\varphi}$, $\frac{1}{2}ae^{-\varphi}$, we obtain

$$
\begin{aligned}
u_1 &= -a^3 e^{-9t/2}[11\cosh 3(t/2+\varphi) + 9\sinh 3(t/2+\varphi)]/160,\\
u_2 &= a^5 e^{-15t/2}[-261\cosh 3(t/2+\varphi) - 243\sinh 3(t/2+\varphi)\\
&\quad + 61\cosh 5(t/2+\varphi) + 59\sinh 5(t/2+\varphi)]/15360.
\end{aligned} \tag{23}
$$

All these results of Eqs.(22) and (23) are similar to those obtained by Rink [13]. Equations (22) and (23) can be brought to exact form of Rink if we substitute $t/2 + \varphi = \psi$ and replace ϵ by $-\mu$ (in [17] Struble's general solution compared to Rink's solution and had the same result).

If the damping force is absent, the motion becomes un-damped and periodic. In this case the eigen-values become $\lambda_1 = iw$, $\lambda_2 = -iw$ and $l_1 = 3i/(2w)$, $l_1^* = -3i/(2w)$, $C_1 = C_1^* = 1/(8w^2)$; $l_2 = -15i/(16w^3)$, $l_2^* = 15i/(16w^3)$;

$C_2 = C_2^* = -21/(64w^4)$; $E_2 = E_2^* = 1/(64w^4)$. Substituting these results into Eqs.(20), (14), (19); then transforming the variables a_1, a_2 by $\frac{1}{2}ae^{i\varphi}$, $\frac{1}{2}ae^{-i\varphi}$ and simplifying, we obtain

$$\dot{a} = 0, \quad \dot{\varphi} = w + 3\varepsilon a^2/(8w) - 15\varepsilon^2 a^4/(256w^3) + \mathrm{O}(\varepsilon^3); \tag{24}$$

$$u_1 = a^3 \cos 3\varphi/32, \quad u_2 = a^5(21\cos 3\varphi - \cos 5\varphi)/1024. \tag{25}$$

All the results of Eqs.(24)-(25) are similar to those obtained by the original KBM method (see [10] for details).

4. A fourth-order equation

In this subsection, we solve the following fourth order equation utilizing formula Eq.(5).

$$(D^2 + 2k_1 D + w_1^2)(D^2 + 2k_2 D + w_2^2)x = \varepsilon\, x^3. \tag{26}$$

For this equation

$$\begin{aligned} f &= (a_1 + \cdots + a_4 + \varepsilon\, u_1 + \varepsilon^2 \cdots)^3 \\ &= 3a_1^2 a_2 + 6a_1 a_3 a_4 + 3a_1 a_2^2 + 6a_2 a_3 a_4 + 3a_3^2 a_4 + 6a_1 a_2 a_3 \\ &\quad + 3a_3 a_4^3 + 6a_1 a_2 a_4 + a_1^3 + a_2^3 \\ &\quad + 3(a_1^2 a_3 + a_2^2 a_4 + a_1^2 a_4 + a_2^2 a_3 + a_1 a_3^2 + a_2 a_4^2 + a_1 a_4^2 + a_2 a_3^2) \\ &\quad + a_3^3 + a_4^3 + \mathrm{O}(\varepsilon). \end{aligned}$$

Now substituting this value of f into Eq.(5), we obtain

$$\begin{aligned} &(D_0 - \lambda_3)(D_0 - \lambda_4)(D_0 - \lambda_2)(D_1 a_1) \\ &+ (D_0 - \lambda_3)(D_0 - \lambda_4)(D_0 - \lambda_1)(D_1 a_2) \\ &+ (D_0 - \lambda_1)(D_0 - \lambda_2)(D_0 - \lambda_4)(D_1 a_3) \\ &+ (D_0 - \lambda_1)(D_0 - \lambda_2)(D_0 - \lambda_3)(D_1 a_4) \\ &+ (D_0 - \lambda_3)(D_0 - \lambda_4)(D_0 - \lambda_1)(D_0 - \lambda_2)u_1 \\ = {}& 3a_1^2 a_2 + 6a_1 a_3 a_4 + 3a_1 a_2^2 + 6a_2 a_3 a_4 + 3a_3^2 a_4 + 6a_1 a_2 a_3 \\ &+ 3a_3 a_4^3 + 6a_1 a_2 a_4 + a_1^3 + a_2^3 + 3(a_1^2 a_3 + a_2^2 a_4 + \cdots + a_2 a_3^2) + a_3^3 + a_4^3 \end{aligned} \tag{27}$$

$$\begin{aligned} &(D_0 - \lambda_3)(D_0 - \lambda_4)(D_0 - \lambda_2)(D_2 a_1) \\ &+ (D_0 - \lambda_3)(D_0 - \lambda_4)(D_0 - \lambda_1)(D_2 a_2) \\ &+ (D_0 - \lambda_1)(D_0 - \lambda_2)(D_0 - \lambda_4)(D_2 a_3) \\ &+ (D_0 - \lambda_1)(D_0 - \lambda_2)(D_0 - \lambda_3)(D_2 a_4) \\ &+ (D_0 - \lambda_3)(D_0 - \lambda_4)(D_0 - \lambda_1)(D_0 - \lambda_2)u_2 \\ &+ [D_1\{D_0^2 - (\lambda_2 + \lambda_3 + \lambda_4)D_0 + \lambda_2\lambda_3 + \lambda_3\lambda_4 + \lambda_4\lambda_2\} \\ &+ D_0 D_1(D_0 - \lambda_2 - \lambda_3 - \lambda_4) + D_0^2 D_1](D_1 a_1) \end{aligned} \tag{28}$$

$$
\begin{aligned}
&+ [D_1\{D_0^2 - (\lambda_1 + \lambda_3 + \lambda_4)D_0 + \cdots](D_1 a_2) \\
&+ [D_1(D_0^3 + c_1 D_0^2 + c_2 D_0 + c_3) \\
&+ D_0 D_1(D_0^2 + c_1 D_0 + c_2) + D_0^2 D_1(D_0 + c_1) + D_0^3 D_1]u_1 \\
= {}& 3(a_1 + a_2 + a_3 + a_4)^2 u_1.
\end{aligned}
$$

The functions related to the first approximation are found from Eq.(27) (subject to the similar imposed conditions; see Subsection 3.1) as

$$
\begin{aligned}
D_1 a_1 &= L_1 a_1^2 a_2 + L_2 a_1 a_3 a_4, \quad D_1 a_2 = L_1^* a_1 a_2^2 + L_2^* a_2 a_3 a_4, \\
D_1 a_3 &= L_3 a_3^2 a_4 + L_4 a_1 a_2 a_3, \quad D_1 a_4 = L_3^* a_3 a_4^3 + L_4^* a_1 a_2 a_4,
\end{aligned} \tag{29}
$$

and

$$
\begin{aligned}
u_1 = {}& C_1 a_1^3 + C_1^* a_2^3 + C_2 a_1^2 a_3 + C_2^* a_2^2 a_4 + C_3 a_1^2 a_4 + C_3^* a_2^2 a_3 \\
&+ C_4 a_1 a_3^2 + C_4^* a_2 a_4^2 + C_5 a_1 a_4^2 + C_5^* a_2 a_3^2 + C_6 a_3^3 + C_6^* a_4^3,
\end{aligned} \tag{30}
$$

where

$$
\begin{aligned}
L_1 &= 3(2\lambda_1(2\lambda_1 + \lambda_2 - \lambda_3)(2\lambda_1 + \lambda_2 - \lambda_4))^{-1}, \\
L_1^* &= 3(2\lambda_2(\lambda_1 + 2\lambda_2 - \lambda_3)(\lambda_1 + 2\lambda_2 - \lambda_4))^{-1}, \\
L_2 &= 6((\lambda_1 + \lambda_3)(\lambda_1 + \lambda_4)(\lambda_1 + \lambda_3 + \lambda_4 - \lambda_2))^{-1}, \\
L_2^* &= 6((\lambda_2 + \lambda_3)(\lambda_2 + \lambda_4)(\lambda_2 + \lambda_3 + \lambda_4 - \lambda_1))^{-1}, \\
L_3 &= 3(2\lambda_3(2\lambda_3 + \lambda_4 - \lambda_1)(2\lambda_3 + \lambda_4 - \lambda_2))^{-1}, \\
L_3^* &= 3(2\lambda_4(\lambda_3 + 2\lambda_2 - \lambda_1)(\lambda_3 + 2\lambda_4 - \lambda_2))^{-1}, \\
L_4 &= 6((\lambda_1 + \lambda_4)(\lambda_2 + \lambda_4)(\lambda_1 + \lambda_2 + \lambda_4 - \lambda_3)), \\
L_4^* &= 6((\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)(\lambda_1 + \lambda_2 + \lambda_3 - \lambda_4))^{-1},
\end{aligned} \tag{31}
$$

and

$$
\begin{aligned}
C_1 &= (2\lambda_1(3\lambda_1 - \lambda_2)(3\lambda_1 - \lambda_3)(3\lambda_1 - \lambda_4))^{-1}, \\
C_1^* &= (2\lambda_2(3\lambda_2 - \lambda_1)(3\lambda_2 - \lambda_3)(3\lambda_2 - \lambda_4))^{-1}, \\
C_2 &= 3(2\lambda_1(\lambda_1 + \lambda_3)(2\lambda_1 + \lambda_3 - \lambda_2)(2\lambda_1 + \lambda_3 - \lambda_4))^{-1}, \\
C_2^* &= 3(2\lambda_2(2\lambda_2 + \lambda_4 - \lambda_1)(\lambda_2 + \lambda_4)(2\lambda_2 + \lambda_4 - \lambda_3))^{-1}, \\
C_3 &= 3(2\lambda_1(\lambda_1 + \lambda_4)(2\lambda_1 + \lambda_4 - \lambda_2)(2\lambda_1 + \lambda_4 - \lambda_3))^{-1}, \\
C_3^* &= 3(2\lambda_2(2\lambda_2 + \lambda_3 - \lambda_1)(\lambda_2 + \lambda_3)(2\lambda_2 + \lambda_3 - \lambda_4))^{-1}, \\
C_4 &= 3(2\lambda_3(\lambda_1 + 2\lambda_3 - \lambda_2)(\lambda_1 + \lambda_3)(\lambda_1 + 2\lambda_3 - \lambda_4))^{-1}, \\
C_4^* &= 3(2\lambda_4(\lambda_2 + 2\lambda_4 - \lambda_1)(\lambda_2 + 2\lambda_4 - \lambda_3)(\lambda_2 + \lambda_4))^{-1}, \\
C_5 &= 3(2\lambda_4(\lambda_1 + 2\lambda_4 - \lambda_2)(\lambda_1 + 2\lambda_4 - \lambda_3)(\lambda_1 + \lambda_4))^{-1}, \\
C_5^* &= 3(2\lambda_3(\lambda_2 + 2\lambda_3 - \lambda_1)(\lambda_2 + \lambda_3)(\lambda_2 + 2\lambda_3 - \lambda_4))^{-1}, \\
C_6 &= (2\lambda_3(3\lambda_3 - \lambda_1)(3\lambda_3 - \lambda_2)(3\lambda_3 - \lambda_4))^{-1}, \\
C_6^* &= (2\lambda_4(3\lambda_4 - \lambda_1)(3\lambda_4 - \lambda_2)(3\lambda_4 - \lambda_3))^{-1}.
\end{aligned} \tag{32}
$$

To find the second approximation, we have to calculate

(33)
$$
\begin{aligned}
&[D_1\{D_0^2-(\lambda_2+\lambda_3+\lambda_4)D_0+\lambda_2\lambda_3+\lambda_3\lambda_4+\lambda_4\lambda_2\}\\
&+D_0D_1(D_0-\lambda_2-\lambda_3-\lambda_4)+D_0^2D_1](L_1a_1^2a_2+L_2a_1a_3a_4)\\
=&(19\lambda_1^2+18\lambda_1\lambda_2+4\lambda_2^2-5\lambda_1\lambda_3-5\lambda_1\lambda_4-2\lambda_2\lambda_3-2\lambda_2\lambda_4+\lambda_3\lambda_4)\\
&\times L_1(2L_1+L_1^*)a_1^3a_2^2\\
&+(12\lambda_1^2+8\lambda_1\lambda_2+\lambda_2^2+2\lambda_1\lambda_3+2\lambda_1\lambda_4+\lambda_2\lambda_3+\lambda_2\lambda_4+\lambda_3\lambda_4)\\
&\times L_1(2L_2+L_2^*)a_1^2a_2a_3a_4\\
&+(7\lambda_1^2+2\lambda_1\lambda_2+6\lambda_1\lambda_3+6\lambda_1\lambda_4+\lambda_2\lambda_3+\lambda_2\lambda_4+\lambda_3^2+\lambda_4^2+3\lambda_3\lambda_4)\\
&\times L_2(L_1+L_3+L_3^*)a_1^2a_2a_3a_4\\
&+(3\lambda_1^2-2\lambda_1\lambda_2+7\lambda_1\lambda_3+7\lambda_1\lambda_4-2\lambda_2\lambda_3-2\lambda_2\lambda_4+4\lambda_3^2+4\lambda_4^2+9\lambda_3\lambda_4)\\
&\times L_2(L_2+L_4+L_4^*)a_1^2a_2a_3a_4,
\end{aligned}
$$

(34)
$$
\begin{aligned}
&3(a_1+a_2+a_3+a_4)^2u_1\\
&=C_1a_1^3a_2^2+2(C_2+C_3)a_1^2a_2a_3a_4+(C_4+C_5)a_1a_3^2a_4^2\\
&\quad+C_1^*a_1^2a_2^3+2(C_2^*+C_3^*)a_1a_2^2a_3a_4+(C_4^*+C_5^*)a_2a_3^2a_4^2\\
&\quad+(C_2+C_3^*)a_1^2a_2^2a_3+2(C_4+C_5^*)a_1a_2a_3^2a_4+C_6a_3^3a_4^2+\cdots.
\end{aligned}
$$

Herein we find only one equation for D_2a_1 as

(35)
$$
\begin{aligned}
&(D_0-\lambda_2)(D_0-\lambda_3)(D_0-\lambda_4)(D_2a_1)\\
=&[-(19\lambda_1^2+18\lambda_1\lambda_2+4\lambda_2^2-5\lambda_1\lambda_3-5\lambda_1\lambda_4-2\lambda_2\lambda_3-2\lambda_2\lambda_4+\lambda_3\lambda_4)\\
&\times L_1(2L_1+L_1^*)+3C_1]a_1^3a_2^2\\
&+[-(12\lambda_1^2+8\lambda_1\lambda_2+\lambda_2^2+2\lambda_1\lambda_3+2\lambda_1\lambda_4+\lambda_2\lambda_3+\lambda_2\lambda_4+\lambda_3\lambda_4)\\
&L_1(2L_2+L_2^*)\\
&-(7\lambda_1^2+2\lambda_1\lambda_2+6\lambda_1\lambda_3+6\lambda_1\lambda_4+\lambda_2\lambda_3+\lambda_2\lambda_4+\lambda_3^2+\lambda_4^2+3\lambda_3\lambda_4)\\
&\times L_2(L_1+L_3+L_3^*)+6(C_2+C_3)]a_1^2a_2a_3a_4\\
&+[-(3\lambda_1^2-2\lambda_1\lambda_2+7\lambda_1\lambda_3+7\lambda_1\lambda_4-2\lambda_2\lambda_3-2\lambda_2\lambda_4+4\lambda_3^2+4\lambda_4^2+9\lambda_3\lambda_4)\\
&\times L_2(L_2+L_4+L_4^*)+3(C_4+C_5)]a_1^2a_2a_3a_4.
\end{aligned}
$$

Solving Eq.(35), we obtain

(36)
$$D_2a_1=p_1a_1^3a_2^2+p_2a_1^2a_2a_3a_4+p_3a_1a_3^2a_4^2,$$

where

(37)
$$p_1=[-(19\lambda_1^2+18\lambda_1\lambda_2+4\lambda_2^2-5\lambda_1\lambda_3-5\lambda_1\lambda_4-2\lambda_2\lambda_3-2\lambda_2\lambda_4+\lambda_3\lambda_4)$$

$$\times L_1(2L_1+L_1^*)+3C_1]$$
$$\times((3\lambda_1+\lambda_2)(3\lambda_1+2\lambda_2-\lambda_3)(3\lambda_1+2\lambda_2-\lambda_4))^{-1},$$
$$p_2=[-(12\lambda_1^2+8\lambda_1\lambda_2+\lambda_2^2+2\lambda_1\lambda_3+2\lambda_1\lambda_4+\lambda_2\lambda_3+\lambda_2\lambda_4+\lambda_3\lambda_4)$$
$$\times L_1(2L_2+L_2^*)-(7\lambda_1^2+2\lambda_1\lambda_2+6\lambda_1\lambda_3+6\lambda_1\lambda_4+\lambda_2\lambda_3+\lambda_2\lambda_4$$
$$+\lambda_3^2+\lambda_4^2+3\lambda_3\lambda_4)L_2(L_1+L_3+L_3^*)+6(C_2+C_3)]$$
$$\times((2\lambda_1+\lambda_2+\lambda_3)(2\lambda_1+\lambda_2+\lambda_4)(2\lambda_1+\lambda_3+\lambda_4))^{-1},$$
$$p_3=[-(3\lambda_1^2-2\lambda_1\lambda_2+7\lambda_1\lambda_3+7\lambda_1\lambda_4-2\lambda_2\lambda_3-2\lambda_2\lambda_4$$
$$+4\lambda_3^2+4\lambda_4^2+9\lambda_3\lambda_4)L_2(L_2+L_4+L_4^*)+3(C_4+C_5)]$$
$$\times((\lambda_1-\lambda_2+2\lambda_3+2\lambda_4)(\lambda_1+\lambda_3+2\lambda_4)(\lambda_1+2\lambda_3+\lambda_4))^{-1}.$$

It is a laborious task to separate the unknown coefficients in real and imaginary part; but not difficult to compute them when the eigen-values are specified. However, the calculation is very easy when all the eigen-values are real or purely imaginary. Let us consider the later case, *i.e.*, the un-damped case. In this case $k_1=k_2=0$ and $\lambda_1=iw_1,\ \lambda_2=-iw_1,\ \lambda_3=iw_2,\ \lambda_4=-iw_2$. Therefore, we obtain the following results

$$\begin{aligned}2L_1&=-2L_1^*=L_2=-L_2^*=3i[w_1(w_1^2-w_2^2)]^{-1},\\2L_3&=-2L_3^*=L_4=-L_4^*=3i[w_2(w_2^2-w_1^2)]^{-1},\end{aligned}\tag{38}$$

$$\begin{aligned}C_1&=C_1^*=[8w_1^2(9w_1^2-w_2^2)]^{-1},\\C_2&=C_2^*=3[2w_1(w_1+w_2)^2(3w_1+w_2)]^{-1},\\C_3&=C_3^*=3[2w_1(w_1-w_2)^2(3w_1-w_2)]^{-1},\\C_4&=C_4^*=3[2w_2(w_2+w_1)^2(3w_2+w_1)]^{-1},\\C_5&=C_5^*=3[2w_2(w_2-w_1)^2(3w_2-w_1)]^{-1},\\C_6&=C_6^*=[8w_2^2(9w_2^2-w_1^2)]^{-1}\end{aligned}\tag{39}$$

and

$$\begin{aligned}p_1&=-3i(269w_1^4-82w_1^2w_2^2+5w_2^4)[8w_1^3(w_1^2-w_2^2)^3(9w_1^2-w_2^2)]^{-1},\\p_2&=-9i(42w_1^4-19w_1^2w_2^2+w_2^4)[2w_1^3(w_1^2-w_2^2)^3(9w_1^2-w_2^2)]^{-1},\\p_3&=9i(15w_1^4-89w_1^2w_2^2+18w_2^4)[4w_1^3(w_1^2-w_2^2)^3(9w_2^2-w_1^2)]^{-1}.\end{aligned}\tag{40}$$

Now substituting the values of $D_0a_1,\ D_1a_1,\ D_2a_1$ into $Da_1=\dot{a}_1$, utilizing the results of Eqs.(38) and (40), transforming $a_1=\frac{1}{2}ae^{i\varphi}$, $a_2=\frac{1}{2}ae^{i\varphi}$, $a_3=\frac{1}{2}be^{i\psi}$, $a_4=\frac{1}{2}be^{-i\psi}$ and then separating into real and imaginary parts, we obtain

$$\dot{a}=0,\quad \dot{\varphi}=w_1+\frac{3\varepsilon(a^2+2b^2)}{8w_1(w_1^2-w_2^2)}+\varepsilon^2(Q_1a^4+Q_2a^2b^2+Q_3b^4),\tag{41}$$

where

$$
\begin{aligned}
&Q_1 = -3(269w_1^4 - 82w_1^2w_2^2 + 5w_2^4)[128w_1^3(w_1^2 - w_2^2)^3(9w_1^2 - w_2^2)]^{-1},\\
(42)\qquad &Q_2 = -9(42w_1^4 - 19w_1^2w_2^2 + w_2^4)[32w_1^3(w_1^2 - w_2^2)^3(9w_1^2 - w_2^2)]^{-1},\\
&Q_3 = 9(15w_1^4 - 89w_1^2w_2^2 + 18w_2^4)[64w_1^3(w_1^2 - w_2^2)^3(9w_2^2 - w_1^2)]^{-1}.
\end{aligned}
$$

It is interesting to note that we obtain $\dot{b}$, $\dot{\psi}$ by replacing a by b, b by a, w_1 by w_2 and w_2 by w_1 in Eqs.(41)-(42). Therefore,

$$
(43)\qquad \dot{b} = 0, \quad \dot{\psi} = w_2 + \frac{3\varepsilon(2a^2 + b^2)}{8w_2(w_2^2 - w_1^2)} + \varepsilon^2(Q_3'a^4 + Q_2'a^2b^2 + Q_1'b^4),
$$

where

$$
\begin{aligned}
&Q_1' = -3(269w_2^4 - 82w_1^2w_2^2 + 5w_1^4)[128w_2^3(w_2^2 - w_1^2)^3(9w_2^2 - w_1^2)]^{-1},\\
(44)\qquad &Q_2' = -9(42w_2^4 - 19w_1^2w_2^2 + w_1^4)[32w_2^3(w_2^2 - w_1^2)^3(9w_2^2 - w_1^2)]^{-1},\\
&Q_3' = 9(15w_2^4 - 89w_1^2w_2^2 + 18w_1^4)[64w_2^3(w_2^2 - w_1^2)^3(9w_1^2 - w_2^2)]^{-1}.
\end{aligned}
$$

Substituting the values of $C_1,\ C_1^*,\ \ldots,\ C_6^*$ from Eq.(39) into Eq.(30), we obtain

$$
\begin{aligned}
u_1 = &\frac{a^3\cos 3\varphi}{128w_1^2(9w_1^2 - w_2^2)} + \frac{3a^2b\cos(2\varphi + \psi)}{32w_1(w_1 + w_2)^2(3w_1 + w_2)}\\
(45)\qquad &+ \frac{3a^2b\cos(2\varphi - \psi)}{32w_1(w_1 - w_2)^2(3w_1 - w_2)} + \frac{3ab^2\cos(\varphi + 2\psi)}{32w_2(w_2 + w_1)^2(3w_2 + w_1)}\\
&+ \frac{3ab^2\cos(\varphi - 2\psi)}{32w_2(w_2 - w_1)^2(3w_2 - w_1)} + \frac{b^3\cos 3\psi}{128w_2^2(9w_2^2 - w_1^2)}.
\end{aligned}
$$

In general, the variational equations of $\dot{a}$, $\dot{\varphi}$ are

$$
\begin{aligned}
\dot{a} = &-k_1a + \varepsilon[\mathrm{Re}(L_1)a^3 + \mathrm{Re}(L_2)ab^2)]/4\\
&+ \varepsilon^2[\mathrm{Re}(p_1)a^5 + \mathrm{Re}(p_2)a^3b^2 + \mathrm{Re}(p_3)ab^4]/16,\\
(46)\qquad \dot{\varphi} = &\ w_1 + \varepsilon[\mathrm{Im}(L_1)a^2 + \mathrm{Im}(L_2)b^2)]/4\\
&+ \varepsilon^2[\mathrm{Im}(p_1)a^4 + \mathrm{Im}(p_2)a^2b^2 + \mathrm{Im}(p_3)b^4]/16,
\end{aligned}
$$

and the variational equations of $\dot{b}$, $\dot{\psi}$ can be similarly found by replacing a by $b, \ldots,$ w_2 by w_1 and k_2 by k_1. Thus it is no need to calculate functions related to $\dot{b}$, $\dot{\psi}$.

5. Results and discussion

Most of the perturbation methods were originally formulated to investigate periodic motion. Then small damping effect was discussed in few articles. The main reason of negligence of studying strongly damped nonlinear problems was the difficulty of formulation of the method. Moreover, the determination of solution from the derived formula is a laborious task especially when the system possesses more than the second derivative. Usually, a first approximate solution was found for the strong damping effect. Recently, Shamsul *et al.* [17] have found the second approximate solution of a third-order nonlinear differential equation with small damping utilizing the general Struble's technique. It is noted that the terms with ε of the variational equations of amplitude and phase vanish while those of ε^2 only contribute to the oscillating process. In this article we have found a second approximation of a fourth-order nonlinear differential equation with strong damping effect utilizing the MTS method. The first approximate solution of the same problem was early investigated by the modified KBM method [16]. We can find the same result utilizing the KBM method or the Struble's technique; but the determination of the solution is more laborious. The first and second approximate solutions give desired results for a short time interval and it slowly deviates from the numerical solution as t is increased. On the contrary, the second approximate solution shows a good agreement with the numerical solution even if t is large (see Figs. 1 and 2). This statement is certainly true for the case of un-damped solution (see Figs. 3-4).

In this paper a set of new variables is considered which quickly communicates among varies perturbation methods. The noted variables are complex for the oscillatory or damped oscillatory systems and real for the non-oscillatory systems. The complex form solution is being considered for simplification (see [1, 6]), but the new technique is entirely different. The complex form solution was early chosen (by several authors) including amplitude and phase variables, which relates to the real form directly. On the contrary, the new complex form solution is transformed to a usual form by a variable transformation (see [15] for details). The set of new variables greatly speeds up all the noted perturbation methods.

6. Conclusion

The MTS has been modified and applied to investigate certain nonlinear problems possessing more than the second derivative. The first and second approximate solutions are derived for the strong liner damming effects. Moreover, MTS method is compared to KBM method. Such a comparison study is not new at all. Earlier the comparison study of these methods was limited to a second-order nonlinear differential equation with a small damping effect. In this paper these methods are again compared to one another choosing some known problems possessing more than the second derivatives. Moreover, it has been shown that the methods cover both oscillatory and non-oscillatory processes.

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Appendix A

Let us consider $Da_1 = l_1 a_1^2 a_2$ be the particular solution of Eq.(19). Since $D_0 a_1 = \lambda_1 a_1$, $D_0 a_2 = \lambda_2 a_2$, we obtain $(D_0 - \lambda_2)(D_1 a_1) = (D_0 - \lambda_2)(l_1 a_1^2 a_2) = 2\lambda_1 l_1 a_1^2 a_2$. Substituting this value in to left hand side of Eq.(19), we obtain $2\lambda_1 l_1 a_1^2 a_2 = -3a_1^2 a_2$, or $l_1 = -3/(2\lambda_1)$. Thus the value of l_1 is found. In a similar way the values of $l_1^*, \ldots, l_3, l_3^*$ as well as the solutions of u_1, u_2 can be found.

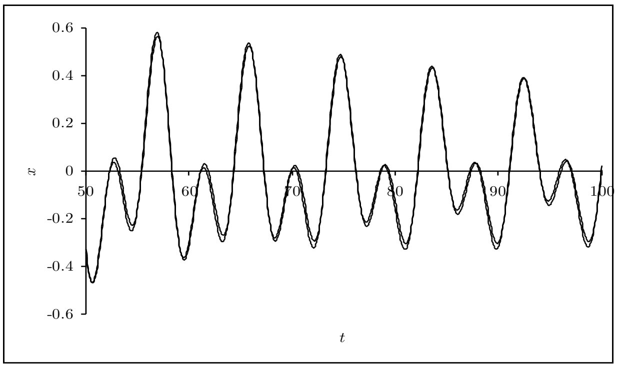

FIGURE 1. First approximate solution (here line) with corresponding numerical solution (solid line) are plotted when damping coefficients $k_1 = k_2 = 0.01$, frequencies $\omega_1 = 1/\sqrt{2}$, $\omega_2 = \sqrt{2}$, with initial conditions $[x(0) = 1.01053,\ \dot{x}(0) = -0.00923,\ \ddot{x}(0) = -1.25985,\ \dddot{x}(0) = 0.03274]$.

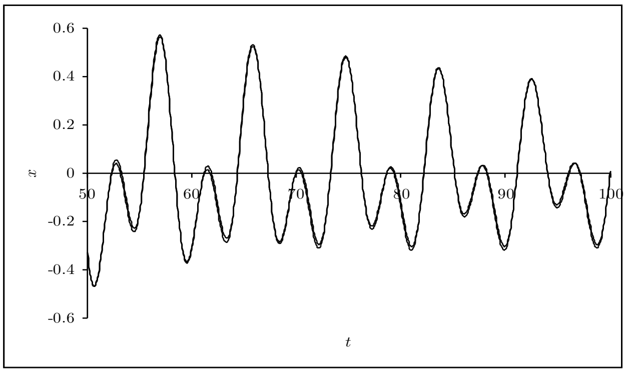

FIGURE 2. Second approximate solution (here line) with corresponding numerical solution (solid line) are plotted when damping coefficients $k_1 = k_2 = 0.01$, frequencies $\omega_1 = 1/\sqrt{2}$, $\omega_2 = \sqrt{2}$, with initial conditions $[x(0) = 1.01053,\ \dot{x}(0) = -0.00965,\ \ddot{x}(0) = -1.25979,\ \dddot{x}(0) = 0.03389]$.

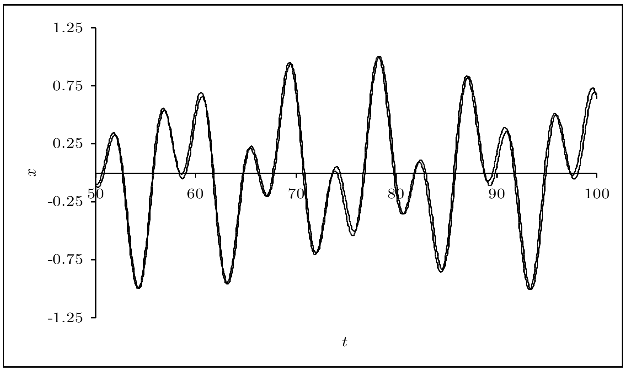

FIGURE 3. First approximate solution (here line) with corresponding numerical solution (solid line) are plotted when $k_1 = k_2 = 0$ frequencies $\omega_1 = 1/\sqrt{2}$, $\omega_2 = \sqrt{2}$, with initial conditions $[x(0) = 1.01054,\ \dot{x}(0) = 0,\ \ddot{x}(0) = -1.25997,\ \dddot{x}(0) = 0]$.

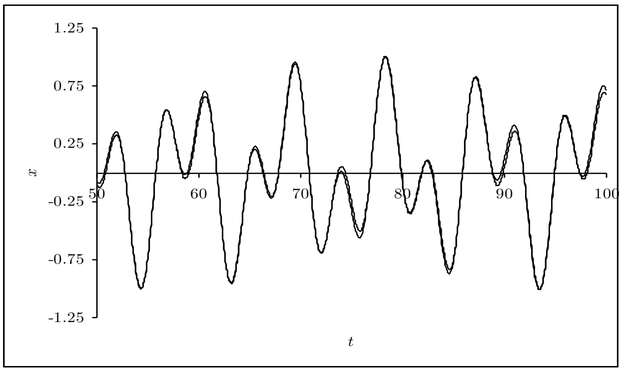

FIGURE 4. Second approximate solution (here line) with corresponding numerical solution (solid line) are plotted when $k_1 = k_2 = 0$, frequencies $\omega_1 = 1/\sqrt{2}$, $\omega_2 = \sqrt{2}$, with initial conditions $[x(0) = 1.01054,\ \dot{x}(0) = 0,\ \ddot{x}(0) = -1.25991,\ \dddot{x}(0) = 0]$.

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