Commun. Korean Math. Soc. **26** (2011), No. 4, pp. 635–647
http://dx.doi.org/10.4134/CKMS.2011.26.4.635

REAL HALF LIGHTLIKE SUBMANIFOLDS OF AN INDEFINITE KAEHLER MANIFOLD

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Abstract. In this paper, we study the geometry of real half lightlike submanifolds of an indefinite Kaehler manifold $\bar{M}$. We provide several new results on such a real half lightlike submanifold M by using the F-structure of M induced by the almost complex structure J of $\bar{M}$.

1. Introduction and preliminaries

It is well known that the radical distribution $Rad(TM) = TM \cap TM^{\perp}$ of half lightlike submanifolds M of a semi-Rimannian manifold $(\bar{M}, \bar{g})$ of codimension 2 is a vector subbundle of the tangent bundle TM and the normal bundle $TM^{\perp}$, of rank 1. Thus there exists complementary non-degenerate distributions $S(TM)$ and $S(TM^{\perp})$ of $Rad(TM)$ in TM and $TM^{\perp}$ respectively, which called the *screen* and *coscreen distribution* on M, such that

$$TM = Rad(TM) \oplus_{orth} S(TM), \; TM^{\perp} = Rad(TM) \oplus_{orth} S(TM^{\perp}), \tag{1.1}$$

where the symbol $\oplus_{orth}$ denotes the orthogonal direct sum. We denote such a half lightlike submanifold by $M = (M, g, S(TM))$. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of any vector bundle E over M. Choose $L \in \Gamma(S(TM^{\perp}))$ as a unit vector field with $\bar{g}(L, L) = \epsilon = \pm 1$. Consider the orthogonal complementary distribution $S(TM)^{\perp}$ to $S(TM)$ in $T\bar{M}$. Certainly ξ and L belong to $\Gamma(S(TM)^{\perp})$. Hence we have the following orthogonal decomposition

$$S(TM)^{\perp} = S(TM^{\perp}) \oplus_{orth} S(TM^{\perp})^{\perp},$$

where $S(TM^{\perp})^{\perp}$ is the orthogonal complementary to $S(TM^{\perp})$ in $S(TM)^{\perp}$. For any null section ξ of $Rad(TM)$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a uniquely defined null vector field $N \in \Gamma(ltr(TM))$ [1] satisfying

$$\bar{g}(\xi, N) = 1, \; \bar{g}(N, N) = \bar{g}(N, X) = \bar{g}(N, L) = 0, \; \forall\, X \in \Gamma(S(TM)). \tag{1.2}$$

Received June 7, 2010.
2010 *Mathematics Subject Classification.* Primary 53C25, 53C40, 53C50.
Key words and phrases. lightlike transversal umbilical, irrotational, indefinite Kaehler manifold.

We call N, $ltr(TM)$ and $tr(TM) = S(TM^{\perp}) \oplus_{orth} ltr(TM)$ the *lightlike transversal vector field, lightlike transversal vector bundle* and *transversal vector bundle* of M with respect to $S(TM)$ respectively. Thus $T\bar{M}$ is decomposed as follows:

$$\begin{aligned} T\bar{M} &= TM \oplus tr(TM) = \{Rad(TM) \oplus tr(TM)\} \oplus_{orth} S(TM) \\ &= \{Rad(TM) \oplus ltr(TM)\} \oplus_{orth} S(TM) \oplus_{orth} S(TM^{\perp}). \end{aligned} \tag{1.3}$$

The objective of this paper is to study the geometry of real half lightlike submanifolds of an indefinite Kaehler manifold $\bar{M}$ or an indefinite complex space form $\bar{M}(c)$. We provide several new results on such a real half lightlike submanifold M by using the F-structure of M induced by the almost complex structure J of $\bar{M}$. In Section 1, we recall some of fundamental formulas in the theory of half lightlike submanifolds. In Section 2, we prove some basic theorems of half lightlike submanifolds which will be used in the sequel. In Section 3, we study real half lightlike submanifolds M of an indefinite Kaehler manifold $\bar{M}$ by using the induced F-structure and some umbilical properties of M. In Section 4, we investigate real half lightlike submanifolds M of an indefinite complex space form $\bar{M}(c)$ such that the screen distribution $S(TM)$ is totally umbilical in M. Recall the following structure equations:

Let $\bar{\nabla}$ be the Levi-Civita connection of $\bar{M}$ and P the projection morphism of $\Gamma(TM)$ on $\Gamma(S(TM))$ with respect to the decomposition (1.1). The local Gauss and Weingarten formulas of M and $S(TM)$ are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X,Y)N + D(X,Y)L, \tag{1.4}$$

$$\bar{\nabla}_X N = -A_N X + \tau(X)N + \rho(X)L, \tag{1.5}$$

$$\bar{\nabla}_X L = -A_L X + \phi(X)N, \tag{1.6}$$

$$\nabla_X PY = \nabla^*_X PY + C(X,PY)\xi, \tag{1.7}$$

$$\nabla_X \xi = -A^*_\xi X - \tau(X)\xi, \tag{1.8}$$

for all $X, Y \in \Gamma(TM)$, where ∇ and ∇^* are induced linear connections on TM and $S(TM)$ respectively, B and D are called the *local second fundamental forms* of M, C is called the *local second fundamental form* on $S(TM)$. A_N, A^*_ξ and A_L are linear operators on TM and τ, ρ and ϕ are 1-forms on TM. We say that $h(X,Y) = B(X,Y)N + D(X,Y)L$ is the *second fundamental tensor* of M. Since $\bar{\nabla}$ is torsion-free, ∇ is also torsion-free, and B and D are symmetric. From the facts $B(X,Y) = \bar{g}(\bar{\nabla}_X Y, \xi)$ and $D(X,Y) = \epsilon\bar{g}(\bar{\nabla}_X Y, L)$, we know that B and D are independent of the choice of $S(TM)$ and satisfy

$$B(X,\xi) = 0, \ D(X,\xi) = -\epsilon\phi(X), \ \forall\, X \in \Gamma(TM). \tag{1.9}$$

The induced connection ∇ of M is not metric and satisfies

$$(\nabla_X g)(Y,Z) = B(X,Y)\,\eta(Z) + B(X,Z)\,\eta(Y), \tag{1.10}$$

for all $X, Y, Z \in \Gamma(TM)$, where η is a 1-form on TM such that

$$\eta(X) = \bar{g}(X,N), \ \forall\, X \in \Gamma(TM). \tag{1.11}$$

But the connection ∇^* on $S(TM)$ is metric. The above three local second fundamental forms are related to their shape operators by

$$B(X,Y) = g(A^*_\xi X, Y), \qquad \bar{g}(A^*_\xi X, N) = 0, \tag{1.12}$$

$$C(X,PY) = g(A_N X, PY), \qquad \bar{g}(A_N X, N) = 0, \tag{1.13}$$

$$\epsilon D(X,PY) = g(A_L X, PY), \qquad \bar{g}(A_L X, N) = \epsilon\rho(X), \tag{1.14}$$

$$\epsilon D(X,Y) = g(A_L X, Y) - \phi(X)\eta(Y), \ \forall\, X,\, Y \in \Gamma(TM). \tag{1.15}$$

By (1.12) and (1.13), we show that A^*_ξ and A_N are $\Gamma(S(TM))$-valued shape operators related to B and C respectively and A^*_ξ is self-adjoint on TM and

$$A^*_\xi \xi = 0. \tag{1.16}$$

But A_N is not self-adjoint on $S(TM)$. We know that A_N is self-adjoint in $S(TM)$ if and only if $S(TM)$ is an integrable distribution [1]. From (1.15), we show that A_L is not self-adjoint on TM. A_L is self-adjoint in TM if and only if $\phi(X) = 0$ for all $X \in \Gamma(S(TM))$ [4]. From (1.4), (1.8) and (1.9), we have

$$\bar{\nabla}_X \xi = -A^*_\xi X - \tau(X)\xi - \epsilon\phi(X)L, \ \forall\, X \in \Gamma(TM). \tag{1.17}$$

Definition 1. A half lightlike submanifold M of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is said to be *irrotational* [8] if $\bar{\nabla}_X \xi \in \Gamma(TM)$ for any $X \in \Gamma(TM)$.

Note 1. From (1.17) we show that the above definition is equivalent to the condition $\phi(X) = 0$, i.e., $D(X, \xi) = 0$ for all $X \in \Gamma(TM)$ due to (1.9).

Denote by $\bar{R}$ and R the curvature tensors of the connections $\bar{\nabla}$ and ∇ respectively. Using the local Gauss-Weingarten formulas (1.4) $\sim$ (1.6) for M, we have the Gauss-Codazzi equations for M, for all $X,\, Y,\, Z \in \Gamma(TM)$:

$$\begin{aligned}\bar{R}(X,Y)Z &= R(X,Y)Z \\ &+ B(X,Z)A_N Y - B(Y,Z)A_N X + D(X,Z)A_L Y - D(Y,Z)A_L X \\ &+ \{(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + \tau(X)B(Y,Z) - \tau(Y)B(X,Z) \\ &\qquad + \phi(X)D(Y,Z) - \phi(Y)D(X,Z)\}N \\ &+ \{(\nabla_X D)(Y,Z) - (\nabla_Y D)(X,Z) + \rho(X)B(Y,Z) - \rho(Y)B(X,Z)\}L,\end{aligned} \tag{1.18}$$

$$\begin{aligned}\bar{R}(X,Y)N &= -\nabla_X(A_N Y) + \nabla_Y(A_N X) + A_N[X,Y] \\ &+ \tau(X)A_N Y - \tau(Y)A_N X + \rho(X)A_L Y - \rho(Y)A_L X \\ &+ \{B(Y, A_N X) - B(X, A_N Y) + 2d\tau(X,Y) + \phi(X)\rho(Y) - \phi(Y)\rho(X)\}N \\ &+ \{D(Y, A_N X) - D(X, A_N Y) + 2d\rho(X,Y) + \rho(X)\tau(Y) - \rho(Y)\tau(X)\}L,\end{aligned} \tag{1.19}$$

$$\begin{aligned}\bar{R}(X,Y)L &= -\nabla_X(A_L Y) + \nabla_Y(A_L X) + A_L[X,Y] \\ &+ \phi(X)A_L Y - \phi(Y)A_L X \\ &+ \{B(Y, A_L X) - B(X, A_L Y) + 2d\phi(X,Y) + \tau(X)\phi(Y) - \tau(Y)\phi(X)\}N \\ &+ \{D(Y, A_L X) - D(X, A_L Y) + \rho(X)\phi(Y) - \rho(Y)\phi(X)\}L.\end{aligned} \tag{1.20}$$

2. Some results of half lightlike submanifolds

Proposition 2.1. *Let M be a half lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. For all X, Y, $Z \in \Gamma(TM)$, we have the following equation*:

$$(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) = \tau(Y)B(X, Z) - \tau(X)B(Y, Z). \tag{2.1}$$

Proof. Replace Z by ξ in (1.18) and use (1.8) and (1.9), we have

$$\begin{aligned}\bar{R}(X, Y)\xi &= R(X, Y)\xi + \epsilon\phi(Y)A_L X - \epsilon\phi(X)A_L Y \\ &\quad + \epsilon\{B(X, A_L Y) - B(Y, A_L X) - 2d\phi(X, Y) + \phi(X)\tau(Y) - \phi(Y)\tau(X)\}L.\end{aligned} \tag{2.2}$$

Taking the scalar product with $Z \in \Gamma(TM)$ to (2.2) and using (1.15) and the facts $\bar{g}(\bar{R}(X, Y)Z,\, \xi) = -\bar{g}(\bar{R}(X, Y)\xi,\, Z)$ and $R(X, Y)Z \in \Gamma(TM)$, we get

$$\bar{g}(\bar{R}(X, Y)Z,\, \xi) = \phi(X)D(Y, Z) - \phi(Y)D(X, Z),\ \forall\, X, Y, Z \in \Gamma(TM). \tag{2.3}$$

Taking the scalar product with ξ to (1.18) and using (2.3), we have (2.1). □

Definition 2. We say that M is *lightlike transversal umbilical* if, on any coordinate neighborhood $\mathcal{U}$, there is a smooth function β such that

$$B(X, Y) = \beta\, g(X, Y), \quad \forall\, X,\, Y \in \Gamma(TM). \tag{2.4}$$

In case $\beta = 0$ on $\mathcal{U}$, we say that M is *lightlike transversal geodesic*.

Theorem 2.2. *Let M be a lightlike transversal umbilical half lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then the function β defined by* (2.4) *satisfies the partial differential equation*

$$\xi(\beta) + \beta\tau(\xi) - \beta^2 = 0. \tag{2.5}$$

Moreover, if rank $(S(TM)) > 1$, then β satisfies the partial differential equation

$$PX(\beta) + \beta\tau(PX) = 0, \quad \forall\, X \in \Gamma(TM).$$

Proof. From (1.10), (2.1) and (2.4), for all X, Y, $Z \in \Gamma(TM)$, we have

$$\{X(\beta) + \beta\tau(X) - \beta^2\eta(X)\}g(Y, Z) = \{Y(\beta) + \beta\tau(Y) - \beta^2\eta(Y)\}g(X, Z).$$

Take $X = \xi$ and $Z = Y$ such that $g(Y, Y) \neq 0$ in this equation, we have (2.5).

Take $X = PX$, $Y = PY$ and $Z = PZ$ in the last equation and by using (1.11) and the fact that $S(TM)$ is non-degenerate, we get

$$\{PX(\beta) + \beta\tau(PX)\}PY = \{PY(\beta) + \beta\tau(PY)\}PX.$$

Now suppose there exist a vector field $X_o \in \Gamma(TM)$ such that $PX_o(\beta) + \beta\tau(PX_o) \neq 0$ at a point $x \in M$. Then from the last equation it follows that all vectors from the fibre $S(TM)_x$ are colinear with $(PX_o)_x$. This is a contradiction as $\dim(S(TM)_x) > 1$. Thus we have $PX(\beta) + \beta\tau(PX) = 0$ for all $X \in \Gamma(TM)$. Thus we have our assertions. □

Definition 3. A vector field X on M is said to be a *conformal Killing* [5] if there exists a non-vanishing smooth function β on M such that $\mathcal{L}_X g = -2\beta g$, where $\mathcal{L}_X$ denotes the Lie derivative with respect to the vector field X. In particular, if $\beta = 0$, then X is called a *Killing* vector field. A distribution D on M is said to be a *conformal Killing* (*Killing*) if each vector field belonging to D is a conformal Killing (Killing) vector field.

Theorem 2.3. *Let M be a half lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then the radical distribution Rad(TM) of M is a conformal Killing distribution if and only if M is lightlike transversal umbilical.*

Proof. By straightforward calculations and use (1.8) and (1.12), we have

$$(\mathcal{L}_\xi g)(X, Y) = \bar{g}(\nabla_X \xi, Y) + g(X, \nabla_Y \xi),$$
$$g(\nabla_X \xi, Y) = -g(A^*_\xi X, Y) = -B(X, Y).$$

Thus we have $\mathcal{L}_\xi g(X, Y) = -2B(X, Y)$ for any $X, Y \in \Gamma(TM)$. Thus we show that $\mathcal{L}_\xi g = -2\beta g \iff B = \beta g$. Therefore we get our assertion. □

As the Riemannian curvature tensor R of M can be considered as an $F(M)$-multilinear function on individual vector fields. The operator

$$R_{XY} : \Gamma(TM) \to \Gamma(TM), \; X, Y \in \Gamma(TM),$$

sending each Z to $R(X, Y)Z$, is called a *curvature operator*.

Theorem 2.4. *Let M be a half lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then the radical vector bundle Rad(TM) is an invariant distribution with respect to the curvature operator R_{XY}, for all $X, Y \in \Gamma(TM)$.*

Proof. Using the local Gauss-Weingarten formulas for $S(TM)$, we obtain

$$R(X, Y)\xi = -\nabla^*_X(A^*_\xi Y) + \nabla^*_Y(A^*_\xi X) + A^*_\xi[X, Y] - \tau(X)A^*_\xi Y + \tau(Y)A^*_\xi X + \{C(Y, A^*_\xi X) - C(X, A^*_\xi Y) - 2d\tau(X, Y)\}\xi. \tag{2.6}$$

Taking the scalar product with any $Z \in \Gamma(TM)$ to (2.6) and using the facts that $S(TM)$ is non-degenerate and $g(R(X, Y)Z, \xi) = 0$, we have

$$\nabla^*_X(A^*_\xi Y) - \nabla^*_Y(A^*_\xi X) - A^*_\xi[X, Y] + \tau(X)A^*_\xi Y - \tau(Y)A^*_\xi X = 0 \tag{2.7}$$

for all $X, Y \in \Gamma(TM)$. Thus the equation (2.6) reduces to

$$R(X, Y)\xi = \{C(Y, A^*_\xi X) - C(X, A^*_\xi Y) - 2d\tau(X, Y)\}\xi. \tag{2.8}$$

□

Proposition 2.5. *Let M be a half lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then the local second fundamental form D and C of M and $S(TM)$ respectively satisfy the following equation, for all $X, Y, Z \in \Gamma(TM)$:*

$$\epsilon\{\phi(X)D(Y, Z) - \phi(Y)D(X, Z)\} = \phi(X)C(Y, PZ) - \phi(Y)C(X, PZ). \tag{2.9}$$

Proof. Taking the scalar product with any L to (1.18), we have

$$\bar{g}(\bar{R}(X,Y)Z,\, L) = \epsilon\{(\nabla_X D)(Y,Z) - (\nabla_Y D)(X,Z) + \rho(X)B(Y,Z) - \rho(Y)B(X,Z)\}.$$

Also, taking the scalar product with any $Z \in \Gamma(TM)$ to (1.20), we have

$$\begin{aligned}\bar{g}(\bar{R}(X,Y)Z,\, L) &= \epsilon\{(\nabla_X D)(Y,Z) - (\nabla_Y D)(X,Z)\\ &\quad + \rho(X)B(Y,Z) - \rho(Y)B(X,Z) + \phi(Y)D(X,Z) - \phi(X)D(Y,Z)\}\\ &\quad + \phi(X)g(A_N Y,Z) - \phi(Y)g(A_N X,Z).\end{aligned}$$

Comparing the last two equations, for all $X,\, Y,\, Z \in \Gamma(TM)$, we have

$$\epsilon\{\phi(X)D(Y,Z) - \phi(Y)D(X,Z)\} = \phi(X)g(A_N Y,Z) - \phi(Y)g(A_N X,Z). \tag{2.10}$$

Replace Z by PZ to (2.10) and use (1.9), we have the equation (2.9). □

3. Real half lightlike submanifolds

Let $\bar{M} = (\bar{M}, J, \bar{g})$ be a real $2m$-dimensional indefinite Kaehler manifold, where $\bar{g}$ is a semi-Riemannian metric of index $q = 2v,\ 0 < v < m$ and J is an almost complex structure on $\bar{M}$ satisfying, for all $X,\, Y \in \Gamma(T\bar{M})$,

$$J^2 = -I, \quad \bar{g}(JX, JY) = \bar{g}(X,Y), \quad (\bar{\nabla}_X J)Y = 0. \tag{3.1}$$

An indefinite complex space form, denoted by $\bar{M}(c)$, is a connected indefinite Kaehler manifold of constant holomorphic sectional curvature c such that

$$\begin{aligned}\bar{R}(X,Y)Z = \frac{c}{4}\{&\bar{g}(Y,Z)X - \bar{g}(X,Z)Y + \bar{g}(JY,Z)JX\\ &- \bar{g}(JX,Z)JY + 2\bar{g}(X,JY)JZ\},\ \forall\, X,\, Y,\, Z \in \Gamma(TM).\end{aligned} \tag{3.2}$$

Theorem 3.1 ([6, 7])**.** *Let M be a real half lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then there exist a screen $S(TM)$ such that*

$$J(S(TM)^{\perp}) \subset S(TM).$$

Note 2. Although $S(TM)$ is not unique, it is canonically isomorphic to the factor vector bundle $TM^* = TM/Rad(TM)$ considered by Kupeli [8]. Thus all screens $S(TM)$ are mutually isomorphic. For this reason, we consider only real half lightlike submanifolds equipped with a screen $S(TM)$ such that $J(S(TM)^{\perp}) \subset S(TM)$. We call such a screen $S(TM)$ the *generic screen* of M.

By Theorem 3.1, the generic screen $S(TM)$ is expressed as follow:

$$S(TM) = \{J(Rad(TM)) \oplus J(ltr(TM))\} \oplus_{orth} J(S(TM^{\perp}) \oplus_{orth} H_o,$$

where H_o is a non-degenerate almost complex distribution on M with respect to J, i.e., $J(H_o) = H_o$. Denote $H' = J(ltr(TM)) \oplus_{orth} J(S(TM^{\perp}))$. In this case, the general decompositions (1.1) and (1.3) reduce to

$$TM = H \oplus H', \quad T\bar{M} = H \oplus H' \oplus tr(TM), \tag{3.3}$$

where H is a 2-lightlike almost complex distribution on M such that

$$H = Rad(TM) \oplus_{orth} J(Rad(TM)) \oplus_{orth} H_o.$$

Consider null vector fields U and V, and a non-null vector field W such that

$$U = -JN, \quad V = -J\xi, \quad W = -JL. \tag{3.4}$$

Denote by S the projection morphism of TM on H. Then, by the first equation of (3.3)[denote (3.3)-1], any vector field X on M is expressed as follows

$$X = SX + u(X)U + w(X)W, \quad JX = FX + u(X)N + w(X)L, \tag{3.5}$$

where u, v and w are 1-forms locally defined on M by

$$u(X) = g(X, V), \quad v(X) = g(X, U), \quad w(X) = \epsilon\, g(X, W) \tag{3.6}$$

and F is a tensor field of type $(1, 1)$ globally defined on M by

$$FX = JSX, \quad \forall\, X \in \Gamma(TM).$$

Apply J to (1.4) $\sim$ (1.6) and (1.17) and use (3.1) and (3.4) $\sim$ (3.6), we have

$$B(X, U) = C(X, V),\ C(X, W) = \epsilon D(X, U),\ B(X, W) = \epsilon D(X, V), \tag{3.7}$$

$$\nabla_X U = F(A_N X) + \tau(X)U + \rho(X)W, \tag{3.8}$$

$$\nabla_X V = F(A^*_\xi X) - \tau(X)V - \epsilon\phi(X)W, \tag{3.9}$$

$$\nabla_X W = F(A_L X) + \phi(X)U,\ \forall\, X \in \Gamma(TM), \tag{3.10}$$

$$(\nabla_X F)(Y) = u(Y)A_N X + w(Y)A_L X - B(X, Y)U - D(X, Y)W. \tag{3.11}$$

Definition 4. We say that M is *totally umbilical* [3] if, on any coordinate neighborhood $\mathcal{U}$, there is a smooth vector field $\mathcal{H} \in \Gamma(tr(TM))$ such that

$$h(X, Y) = \mathcal{H}\, g(X, Y), \quad \forall\, X,\, Y \in \Gamma(TM).$$

It is easy to see that M is totally umbilical if and only if, on each coordinate neighborhood $\mathcal{U}$, there exist smooth functions β and δ such that

$$B(X, Y) = \beta g(X, Y),\ D(X, Y) = \delta g(X, Y),\ \forall\, X,\, Y \in \Gamma(TM). \tag{3.12}$$

Theorem 3.2. *Let M be a real half lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then H is an integrable distribution on M if and only if*

$$h(X, FY) = h(FX, Y), \quad \forall\, X,\, Y \in \Gamma(H).$$

Moreover, if M is totally umbilical, then H is a parallel distribution on M.

Proof. Take $Y \in \Gamma(H)$. Then we have $FY = JY \in \Gamma(H)$. Apply J to (1.4) with $Y \in \Gamma(H)$ and use (1.4), (3.1), (3.4) and (3.5), we have

$$B(X, FY) = g(\nabla_X Y, V), \quad D(X, FY) = \epsilon g(\nabla_X Y, W), \tag{3.13}$$

$$(\nabla_X F)(Y) = -B(X, Y)U - D(X, Y)W. \tag{3.14}$$

By straightforward calculations from two equations of (3.13), we have

$$h(X, FY) - h(FX, Y) = g([X, Y], V)N + \epsilon g([X, Y], W)L.$$

If H is an integrable distribution on M, then $[X, Y] \in \Gamma(H)$ for any $X, Y \in \Gamma(H)$. This implies $g([X, Y], V) = g([X, Y], W) = 0$. Thus we have $h(X, FY) = h(FX, Y)$ for all $X, Y \in \Gamma(H)$. Conversely if $h(X, FY) = h(FX, Y)$ for all $X, Y \in \Gamma(H)$, then we have $g([X, Y], V) = g([X, Y], W) = 0$. This imply $[X, Y] \in \Gamma(H)$. Thus H is an integrable distribution on M.

Moreover, if M is totally umbilical, from (3.7)-3 and (3.12), we show that

$$\beta g(X, W) = \epsilon\delta g(X, V), \quad \forall\, X \in \Gamma(TM).$$

Replacing X by W and U in this equation by turns, we have $\beta = 0$ and $\delta = 0$ respectively, i.e., we get $B = D = 0$. Therefore, from (3.13), we have

$$g(\nabla_X Y, V) = g(\nabla_X Y, W) = 0, \ \forall\, X \in \Gamma(TM), \ Y \in \Gamma(H).$$

This imply $\nabla_X Y \in \Gamma(H)$ for all $X, Y \in \Gamma(H)$. Thus H is parallel on M. □

Theorem 3.3. *Let M be a real half lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then F is parallel on H with respect to ∇ if and only if H is a parallel distribution on M.*

Proof. Assume that F is parallel on H with respect to ∇. For any $X, Y \in \Gamma(H)$, we have $(\nabla_X F)Y = 0$. Taking the scalar product with V and W to (3.14) with $(\nabla_X F)Y = 0$, we have $B(X, Y) = 0$ and $D(X, Y) = 0$ for all $X, Y \in \Gamma(H)$ respectively. From (3.13), we have $g(\nabla_X Y, V) = 0$ and $g(\nabla_X Y, W) = 0$. This imply $\nabla_X Y \in \Gamma(H)$ for all $X, Y \in \Gamma(H)$. Thus H is a parallel distribution on M. Conversely if H is a parallel distribution on M, from (3.13), we have

$$B(X, FY) = 0, \ D(X, FY) = 0, \ \forall\, X, Y \in \Gamma(H).$$

For any $Y \in \Gamma(H)$, we show that $F^2 Y = J^2 Y = -Y$. Replace Y by FY to the last equations, we have $B(X, Y) = 0$ and $D(X, Y) = 0$ for any $X, Y \in \Gamma(H)$. By this results and (3.14), we see that F is parallel on D with respect to ∇. □

Proposition 3.4. *Let M be a half lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. For all $X, Y \in \Gamma(TM)$ and $Z \in \Gamma(H)$, the second fundamental forms B and C of M and $S(TM)$ respectively are related by*

$$\begin{aligned} &B(Y, Z)C(X, V) - B(X, Z)C(Y, V) \\ &= B(X, V)C(Y, PZ) - B(Y, V)C(X, PZ). \end{aligned} \tag{3.15}$$

Proof. Apply $\bar{\nabla}_Z$ to (3.13)-1 and use (3.5)-2, (3.7), (3.9) and (3.14), we have

$$\begin{aligned} (\nabla_X B)(Y, FZ) = {} & g(\nabla_X \nabla_Y Z, V) - B(\nabla_X Y, FZ) + B(X, Z)C(Y, V) \\ & + \epsilon\, D(X, Z)D(Y, V) - \tau(X)B(Y, FZ) - \phi(X)D(Y, FZ) \\ & - B(X, F(\nabla_Y Z)) - B(Y, F(\nabla_X Z)) \end{aligned}$$

for all $X, Y \in \Gamma(TM)$ and $Z \in \Gamma(H)$. From this equation, we have

$$\begin{aligned} & (\nabla_X B)(Y, FZ) - (\nabla_Y B)(X, FZ) = g(R(X, Y)Z, V) \\ & + B(X, Z)C(Y, V) - B(Y, Z)C(X, V) \\ & + \epsilon\{D(X, Z)D(Y, V) - D(Y, Z)D(X, V)\} \end{aligned}$$

$$+ \ \tau(Y)B(X,FZ) - \tau(X)B(Y,FZ)$$
$$+ \ \phi(Y)D(X,FZ) - \phi(X)D(Y,FZ).$$

Comparing (2.1) with $Z = FZ$ and the last equation, we obtain

$$\begin{aligned} g(R(X,Y)Z,V) = & \ B(Y,Z)C(X,V) - B(X,Z)C(Y,V) \\ & + \ \epsilon\{D(Y,Z)D(X,V) - D(X,Z)D(Y,V)\} \\ & + \ \phi(X)D(Y,FZ) - \phi(Y)D(X,FZ). \end{aligned} \tag{3.16}$$

Apply the operator ∇_Y to (3.9) and use (3.9), we have

$$\begin{aligned} \nabla_X\nabla_Y V = & \ (\nabla_X F)(A^*_\xi Y) + F(\nabla^*_X(A^*_\xi Y)) - \tau(Y)F(A^*_\xi X) \\ & - \ \epsilon\phi(Y)F(A_L X) - \{C(X, A^*_\xi Y) + X(\tau(Y)) - \tau(X)\tau(Y)\}V \\ & - \ \epsilon\{X(\phi(Y)) - \phi(X)\tau(Y)\}W - \epsilon\phi(X)\phi(Y)U, \ \forall\, X,\, Y \in \Gamma(TM). \end{aligned}$$

Using this equation, (1.12), (1.14), (2.7), (2.8), (3.7) and (3.11), we have

$$\begin{aligned} R(X,Y)V = & \ B(Y,V)A_N X - B(X,V)A_N Y + D(Y,V)A_L X \\ & - D(X,V)A_L Y - F\{R(X,Y)\xi) + \epsilon\phi(Y)A_L X - \epsilon\phi(X)A_L Y\} \\ & - \epsilon\{B(Y,A_L X) - B(X,A_L Y) + 2d\phi(X,Y) + \tau(X)\phi(Y) - \tau(Y)\phi(X)\}W. \end{aligned} \tag{3.17}$$

Taking the scalar product with $Z \in \Gamma(H)$ to (3.17), we have

$$\begin{aligned} g(R(X,Y)Z,V) = & \ B(X,V)g(A_N Y,Z) - B(Y,V)g(A_N X,Z) \\ & + \ \epsilon\{D(X,V)D(Y,Z) - D(Y,V)D(X,Z)\} \\ & + \ \phi(X)D(Y,FZ) - \phi(Y)D(X,FZ) \end{aligned} \tag{3.18}$$

for all $X,\, Y \in \Gamma(TM)$ and $Z \in \Gamma(H)$. Comparing (3.16) and (3.18), we have

$$\begin{aligned} & B(Y,Z)C(X,V) - B(X,Z)C(Y,V) \\ & = B(X,V)g(A_N Y,Z) - B(Y,V)g(A_N X,Z). \end{aligned}$$

From the last equation and (1.9), we obtain (3.15). □

Theorem 3.5. *Let M be a half lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. If M is lightlike transversal umbilical, then M is lightlike transversal geodesic and the induced connection ∇ on M is a metric connection.*

Proof. From (2.4), (3.7) and (3.15), we have

$$C(X,V) = \beta g(X,U), \ D(X,V) = \epsilon\beta g(X,W), \tag{3.19}$$

$$\begin{aligned} & \beta^2 g(Y,Z)g(X,U) - \beta^2 g(X,Z)g(Y,U) \\ = & \ \ \beta g(X,V)C(Y,PZ) - \beta g(Y,V)C(X,PZ) \end{aligned} \tag{3.20}$$

for all $X,\, Y \in \Gamma(TM)$ and $Z \in \Gamma(H)$. Replace X by V to (3.20), we have

$$\beta^2\{g(Y,Z) + u(Y)v(PZ)\} = 0, \ \forall\, Y \in \Gamma(TM), \ \forall\, Z \in \Gamma(H).$$

Replace Y by U and Z by V in this equation, we have $\beta = 0$. Thus M is lightlike transversal geodesic. From (1.10), ∇ is a metric connection of M. □

From Theorem 2.3 and 3.5, we have the following theorem.

Theorem 3.6. *Let M be a half lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. If $Rad(TM)$ is a conformal Killing, then $Rad(TM)$ is a Killing.*

Theorem 3.7. *Let M be an irrotational real half lightlike submanifold of an indefinite complex space form $\bar{M}(c)$. Then we have $c = 0$.*

Proof. Taking the scalar product with ξ to (3.2) and use (2.3) with $\phi = 0$, (3.1) and (3.6), for all X, Y, $Z \in \Gamma(TM)$, we get

$$\frac{c}{4}\{u(X)g(JY,Z) - u(Y)g(JX,Z) + 2u(Z)\bar{g}(X,JY)\} = 0.$$

Taking $X = Z = U$ and $Y = \xi$ in this and use (3.4) and (3.6), we have $c = 0$. □

Corollary 1. *There exist no irrotational real half lightlike submanifolds M of an indefinite complex space form $\bar{M}(c)$ with $c \neq 0$.*

4. Totally umbilical screen distributions

Proposition 4.1. *Let M be a half lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. For all X, $Y \in \Gamma(TM)$, we have the following equation:*

$$\begin{aligned} &B(Y,W)C(X,V) - B(X,W)C(Y,V) \\ &= C(Y,W)B(X,V) - C(X,W)B(Y,V). \end{aligned} \tag{4.1}$$

Proof. Apply the operator ∇_Y to (3.10) and use (3.8), we have

$$\begin{aligned} \nabla_X\nabla_Y W = &(\nabla_X F)(A_L Y) + F(\nabla_X(A_L Y)) + \phi(Y)F(A_N X) \\ &+ \{X(\phi(Y)) + \tau(X)\phi(Y)\}U + \rho(X)\phi(Y)W \end{aligned}$$

for all X, $Y \in \Gamma(TM)$. Using this equation, (3.7) and (3.11), we have

$$\begin{aligned} R(X,Y)W = &B(Y,W)A_N X - B(X,W)A_N Y \\ &+ D(Y,W)A_L X - D(X,W)A_L Y \\ &+ F\{\nabla_X(A_L Y) - \nabla_Y(A_L X) - A_L[X,Y] + \phi(Y)A_N X - \phi(X)A_N Y\} \\ &+ \{B(Y,A_L X) - B(X,A_L Y) + 2d\phi(X,Y) + \tau(X)\phi(Y) - \tau(Y)\phi(X)\}U \\ &+ \{D(Y,A_L X) - D(X,A_L Y) + \rho(X)\phi(Y) - \rho(Y)\phi(X)\}W. \end{aligned} \tag{4.2}$$

Taking the scalar product with V to (4.2) and using (3.6), we have

$$\begin{aligned} g(R(X,Y)W,V) = &B(Y,W)C(X,V) - B(X,W)C(Y,V) \\ &+ \epsilon\{D(Y,W)D(X,V) - D(X,W)D(Y,V)\} \\ &+ B(Y,A_L X) - B(X,A_L Y) + 2d\phi(X,Y) + \tau(X)\phi(Y) - \tau(Y)\phi(X), \end{aligned} \tag{4.3}$$

due to $g(FX,V) = 0$ for all $X \in \Gamma(TM)$.

On the other hand, taking the scalar product with W to (3.17) and using the fact $g(FX,W) = 0$ for all $X \in \Gamma(TM)$, we have

$$g(R(X,Y)W,V) = C(Y,W)B(X,V) - C(X,W)B(Y,V) \tag{4.4}$$

$$+\,\epsilon\{D(Y,W)D(X,V)-D(X,W)D(Y,V)\}$$
$$+\,B(Y,A_LX)-B(X,A_LY)+2d\phi(X,Y)+\tau(X)\phi(Y)-\tau(Y)\phi(X)$$

for all $X,\,Y\in\Gamma(TM)$. Comparing (4.3) and (4.4), we have (4.1). □

Definition 5. We say that $S(TM)$ is *totally umbilical* [3] in M if, on any coordinate neighborhood $\mathcal{U}\subset M$, there is a smooth function γ such that

$$C(X,PY)=\gamma\,g(X,Y),\ \forall\,X,\,Y\in\Gamma(TM). \tag{4.5}$$

In case $\gamma=0\,(\gamma\neq 0)$ on $\mathcal{U}$, we say that $S(TM)$ is *totally geodesic* (*proper totally umbilical*) in M.

The type number $t^*(x)$ of M at any point x is the rank of the operator A^*_ξ.

Theorem 4.2. *Let M be a real half lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. If $S(TM)$ is proper totally umbilical, then $t^*(x)=1$ and M is irrotational.*

Proof. Assume that $\gamma\neq 0$. From (3.7), (3.15) and (4.5), we have

$$B(X,U)=\gamma g(X,V),\quad D(X,U)=\epsilon\gamma g(X,W), \tag{4.6}$$

$$B(Y,Z)g(X,V)-B(X,Z)g(Y,V) \tag{4.7}$$
$$=\ B(X,V)g(Y,PZ)-B(Y,V)g(X,PZ)$$

for all $X,\,Y\in\Gamma(TM)$ and $Z\in\Gamma(H)$. Taking $Y=U$ and $Z=V$ to (4.7) and using (4.6), we show that $B(Y,V)=0$ for all $Y\in\Gamma(TM)$. Replace Y by U to (4.7) and use the facts that $B(Z,U)=0$ and $B(Y,V)=0$, we have

$$B(X,Z)=0,\ \forall\,X\in\Gamma(TM),\ \forall\,Z\in\Gamma(H). \tag{4.8}$$

On the other hand, from (4.1), (4.5) and (4.8), we get

$$B(X,W)g(Y,V)=B(Y,W)g(X,V),\ \forall\,X\in\Gamma(TM).$$

Replace Y by U in this and use the fact $B(U,W)=0$ due to (4.6), we have

$$B(X,W)=0,\ \forall\,X\in\Gamma(TM). \tag{4.9}$$

From (4.6)-1, (4.8) and (4.9) we have $t^*(x)=1$.

Take $X=\xi$ in (4.6)-2 we have $\phi(U)=0$. Replace Y by U to (2.9) and use (4.6)-2 and the facts that $\phi(U)=0$ and $\gamma\neq 0$, we have

$$\phi(X)g(Z,W)=\phi(X)g(U,PZ),\quad \forall\,X,\,Z\in\Gamma(TM).$$

Take $Z=W$ in this equation we get $\phi(X)=0$ for all $X\in\Gamma(TM)$. Thus, by Note 1, we show that M is irrotational. Therefore we have our assertions. □

Corollary 2. *Let M be a real half lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ such that $S(TM)$ is totally umbilical. If $t^*(x)>1$, then $S(TM)$ is totally geodesic.*

Theorem 4.3. *Let M be a real half lightlike submanifold of an indefinite complex space form $\bar{M}(c)$. If $S(TM)$ is totally umbilical in M, then $c=0$.*

Proof. Taking the scalar product with ξ to (3.2) and use (2.3), we have

$$\begin{aligned}&\frac{c}{4}\{u(X)g(JY,Z)-u(Y)g(JX,Z)+2u(Z)\bar{g}(X,JY)\}\\&=\phi(X)D(Y,Z)-\phi(Y)D(X,Z),\quad \forall\, X,\,Y,\,Z\in\Gamma(TM).\end{aligned}$$

Taking $X=Z=U$ and $Y=\xi$ in this and using (3.4), (3.6) and (4.2), we have

$$-\frac{3}{4}\,c=\epsilon\,\gamma\,\{\phi(U)g(\xi,W)-\phi(\xi)g(U,W)\}=0,\quad \text{i.e.,}\quad c=0.$$ □

Corollary 3. *There exist no real half lightlike submanifolds M of an indefinite complex space form $\bar{M}(c)$, $c\neq 0$ such that $S(TM)$ is totally umbilical.*

Theorem 4.4. *Let M be a real half lightlike submanifold of an indefinite complex space form $\bar{M}(c)$ such that $S(TM)$ is totally umbilical in M. Then $S(TM)$ is totally geodesic in M.*

Proof. Taking the scalar product with N to (1.18) and using (1.13), (1.14) and the fact that $c=0$ due to Theorem 4.3, we have

$$\bar{g}(R(X,Y)Z,\,N)=\epsilon\{\rho(X)D(Y,Z)-\rho(Y)D(X,Z)\},\tag{4.10}$$

for all $X,\,Y,\,Z\in\Gamma(TM)$. Using (1.7) and (1.8), we have

$$\begin{aligned}\bar{g}(R(X,Y)PZ,\,N)&=(\nabla_X C)(Y,PZ)-(\nabla_Y C)(X,PZ)\\&\quad+C(X,PZ)\tau(Y)-C(Y,PZ)\tau(X).\end{aligned}\tag{4.11}$$

Using (1.10), (4.5), (4.10) and (4.11), we get

$$\begin{aligned}&\gamma\,\{B(Y,PZ)\eta(X)-B(X,PZ)\eta(Y)\}\\&+\epsilon\,\{D(Y,PZ)\rho(X)-D(X,PZ)\rho(Y)\}\\=\;&\{X[\gamma]-\gamma\tau(X)\}g(Y,PZ)-\{Y[\gamma]-\gamma\tau(Y)\}g(X,PZ)\end{aligned}$$

for any $X,\,Y,\,Z\in\Gamma(TM)$. Replacing X by ξ in this equation, we have

$$\gamma B(Y,PZ)+\epsilon D(Y,PZ)\rho(\xi)+\phi(PZ)\rho(Y)=\{\xi[\gamma]-\gamma\tau(\xi)\}g(Y,PZ)\tag{4.12}$$

for all $Y,\,Z\in\Gamma(TM)$. From the second equations of (1.9) and (4.6) we get $\phi(U)=0$ and $D(U,U)=0$. Taking $Y=PZ=U$ in (4.12) and using the fact $B(U,U)=\gamma$ due to (4.6), we have $\gamma=0$. Thus we have our theorem. □

The induced Ricci type tensor $R^{(0,\,2)}$ of M is defined by

$$R^{(0,\,2)}(X,Y)=trace\{Z\to R(Z,X)Y\},\ \forall\, X,\,Y\in\Gamma(TM).\tag{4.13}$$

Consider the induced quasi-orthonormal frame field $\{\xi;W_a\}$ on M such that $Rad(TM)=Span\{\xi\}$ and $S(TM)=Span\{W_a\}_{a=1}^m$. Using this frame field and (4.13), we obtain

$$R^{(0,\,2)}(X,Y)=\sum_{a=1}^{m}\epsilon_a\,g(R(W_a,X)Y,\,W_a)+\bar{g}(R(\xi,X)Y,\,N),\tag{4.14}$$

where $\epsilon_a=g(W_a,W_a)$ is the sign of W_β. In general, the induced Ricci type tensor $R^{(0,\,2)}$ is not symmetric [2, 3, 4]. A tensor field $R^{(0,\,2)}$ of M is called its *induced Ricci tensor* of M if it is symmetric. A symmetric $R^{(0,2)}$ tensor will be denoted by *Ric*. M is called *Ricci flat* if its Ricci tensor vanishes on $\bar{M}$. If $\dim(M) > 2$ and $Ric = \kappa g$ where κ is a constant, then M is called to be an *Einstein manifold*. For $\dim(M) = 2$, any M is Einstein but κ is not necessarily constant.

Theorem 4.5. *Let M be an irrotational real half lightlike submanifold of an indefinite complex space form $\bar{M}(c)$ such that $S(TM)$ is totally umbilical. Then $R^{(0,2)}$ is an induced Ricci tensor of M. Moreover, if M is an Einstein manifold, then M is Ricci flat.*

Proof. Using (1.18), (1.19) and Theorem 4.3 and 4.4, (4.14) reduces to

$$R^{(0,2)}(X,Y) = D(X,Y)trA_L - \epsilon g(A_L X, A_L Y) + \rho(X)\phi(Y), \tag{4.15}$$

where trA_L is the trace of A_L. Thus $R^{(0,2)}$ is a symmetric Ricci tensor *Ric*. Let M be an Einstein manifold. Replacing Y by U in (4.15) and using the facts $\phi(U) = 0$ and $g(A_L U, X) = g(A_L U, X) - \phi(U)\eta(X) = D(X,U) = 0$ for any $X \in \Gamma(TM)$, we obtain $\kappa g(X,U) = 0$ for all $X \in \Gamma(TM)$. Replacing X by V in this equation, we have $\kappa = 0$. Thus M is Ricci flat. □

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