Commun. Korean Math. Soc. **26** (2011), No. 4, pp. 623–634
http://dx.doi.org/10.4134/CKMS.2011.26.4.623

ON A CLASS OF $N(k)$-QUASI EINSTEIN MANIFOLDS

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Abstract. The object of the present paper is to study $N(k)$-quasi Einstein manifolds. Existence of $N(k)$-quasi Einstein manifolds are proved. Physical example of $N(k)$-quasi Einstein manifold is also given. Finally, Weyl-semisymmetric $N(k)$-quasi Einstein manifolds have been considered.

1. Introduction

A Riemannian or a semi-Riemannian manifold (M^n, g), $n = \dim M \geq 2$, is said to be an Einstein manifold if the following condition

$$S = \frac{r}{n} g \tag{1.1}$$

holds on M, where S and r denote the Ricci tensor and the scalar curvature of (M^n, g) respectively. According to ([4], p. 432), (1.1) is called the Einstein metric condition. Einstein manifolds play an important role in Riemannian geometry as well as in general theory of relativity. Also Einstein manifolds form a natural subclass of various classes of Riemannian or semi-Riemannian manifolds by a curvature condition imposed on their Ricci tensor ([4], pp. 432–433). For instance, every Einstein manifold belongs to the class of Riemannian manifolds (M^n, g) realizing the following relation:

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{1.2}$$

where a, b are smooth functions and η is a non-zero 1-form such that

$$g(X,\xi) = \eta(X), \qquad g(\xi,\xi) = \eta(\xi) = 1 \tag{1.3}$$

for all vector fields X.

A non-flat Riemannian manifold (M^n, g) $(n > 2)$ is defined to be a quasi Einstein manifold [8] if its Ricci tensor S of type $(0,2)$ is not identically zero and satisfies the condition (1.2). We shall call η the associated 1-form and the unit vector field ξ is called the generator of the manifold.

Received April 12, 2010.
2010 *Mathematics Subject Classification.* 53C25, 53Z05.
Key words and phrases. quasi Einstein manifolds, $N(k)$-quasi Einstein manifolds, pseudo Ricci symmetric spacetimes, conformal curvature tensor.

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Quasi Einstein manifolds arose during the study of exact solutions of the Einstein field equations as well as during considerations of quasi-umbilical hypersurfaces of semi-Euclidean spaces. So many studies about Einstein field equations are done. For example, in [16], El Naschie turned the tables on the theory of elementary particles and showed that we could derive the expectation number of elementary particles of the standard model using Einstein's unified field equation or more precisely his somewhat forgotten strength criteria directly and without resorting to quantum field theory [15]. He also discussed possible connections between Gödel's classical solution of Einstein's field equations and E-infinity in [17]. Also quasi Einstein manifolds have some importance in the general theory of relativity. For instance, the Robertson-Walker spacetime are quasi Einstein manifolds [14]. Further, quasi Einstein manifold can be taken as a model of the perfect fluid spacetime in general relativity [11].

The study of quasi Einstein manifolds was continued by M. C. Chaki [5], S. Guha [18], U. C. De and G. C. Ghosh [12], [13] and many others.

Let R denotes the Riemannian curvature tensor of a Riemannian manifold M. The k-nullity distribution $N(k)$ of a Riemannian manifold M [30] is defined by

$$N(k) : p \longrightarrow N_p(k) = \{Z \in T_pM : R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y]\},$$

k being some smooth function. In a quasi Einstein manifold M, if the generator ξ belongs to some k-nullity distribution $N(k)$, then M is said to be a $N(k)$-quasi Einstein manifold [31]. In fact k is not arbitrary as the following:

Lemma 1 ([20]). *In an n-dimensional $N(k)$-quasi Einstein manifold it follows that*

$$k = \frac{a+b}{n-1}. \tag{1.4}$$

In [31] it was shown that an n-dimensional conformally flat quasi Einstein manifold is an $N(\frac{a+b}{n-1})$-quasi Einstein manifold and in particular a 3-dimensional quasi Einstein manifold is an $N(\frac{a+b}{2})$-quasi Einstein manifold. Also in [19] Özgür, cited some physical examples of $N(k)$-quasi Einstein manifolds. All these motivated us to study such a manifold.

In 1967, R. N. Sen and M. C. Chaki [23] studied certain curvature restrictions on a certain kind of conformally flat space of class one and they obtained the following expressions of the covariant derivative of Ricci tensor:

$$R_{ij,l} = 2\lambda_l R_{ij} + \lambda_i R_{lj} + \lambda_j R_{il}, \tag{1.5}$$

where λ_i is a non-zero covariant vector and ',' denotes covariant differentiation with respect to the metric tensor g_{ij}.

Later in 1988 M. C. Chaki [6] called a non-flat Riemannian manifold a pseudo Ricci symmetric manifold if its Ricci tensor satisfies (1.5). In index free notation this can be stated as follows:

A non-flat Riemannian manifold is called pseudo Ricci symmetric and denoted by $(PRS)_n$ if the Ricci tensor S of type $(0,2)$ of the manifold is non-zero and satisfies the condition

$$(\nabla_X S)(Y,Z) = 2B(X)S(Y,Z) + B(Y)S(X,Z) + B(Z)S(X,Y), \tag{1.6}$$

where ∇ denotes the Levi-Civita connection and B is a non-zero 1-form such that

$$g(X,U) = B(X) \tag{1.7}$$

for all vector fields X; U being the vector field corresponding to the associated 1-form B. Here U is called the basic vector field of the manifold. Pseudo Ricci symmetric manifold is a particular case of a weakly Ricci symmetric manifold introduced by Tamassy and Binh [29]. It is known [6] that in a $(PRS)_n$ if the scalar curvature r is constant, then $r = 0$. A concrete example of a $(PRS)_n$ is given by F. Ozen and S. Altay [21]. Also R. N. Sen and M. C. Chaki [23] studied hypersurfaces of a conformally flat space of class-one and obtained in a natural way the notion of the pseudo Ricci symmetric manifold. On the other hand Chaki [6] proved the existence of a pseudo Ricci symmetric manifold by considering a linear connection on a Riemannian manifold.

Also in [22] S. Ray-Guha proved that a conformally flat perfect fluid pseudo Ricci symmetric spacetime obeying Einstein equation without cosmological constant and having the basic vector field of pseudo Ricci symmetric spacetime as the velocity vector field of the fluid is infinitesimally spatially isotropic relative to the velocity vector field. In [10] De and Gazi proved that a pseudo Ricci symmetric quasi Einstein perfect fluid spacetime represents the equation of state in the radiation era in the evolution of our universe. So pseudo Ricci symmetric manifolds also have some importance in the general theory of relativity.

On the other hand conformal curvature tensor play an important role in differential geometry and also in general theory of relativity. The Weyl conformal curvature tensor C of a Riemannian manifold (M^n, g) $n > 3$ is defined by

$$\begin{aligned} &C(X,Y)Z \\ =&R(X,Y)Z - \frac{1}{n-2}[g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y] \\ &+ \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{1.8}$$

where r is the scalar curvature and Q is the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor S, that is, $g(QX,Y) = S(X,Y)$. If the dimension $n = 3$, then the conformal curvature tensor becomes identically zero.

As is well known, a Riemannian manifold (M,g) is said to be (locally) conformally flat if for any point $p \in M$ there exists a neighborhood U of p and a positive smooth function $f : U \longrightarrow \mathbb{R}$ such that fg is a flat metric.

The study of conformally flat Riemannian manifolds is a classical field of research in Riemannian geometry. Now if $S(X,Y) = 0$, then (1.8) leads to $C(X,Y)Z = R(X,Y)Z$. Since $R(X,Y)Z$ characterize the gravitational field, therefore, it is the Weyl tensor which describes the true gravitational fields in a vacuum region. Also it is known [25] that all conformally flat perfect fluid solutions are of embedding class one and are therefore all contained either in generalized interior Schwarzschild type metrics or in the generalized Friedman type metrics.

In 1963 M. C. Chaki and B. Gupta [7] introduced the notion of conformally symmetric manifold.

A Riemannian manifold of dimension $n > 3$ is said to be conformally symmetric if the conformal curvature tensor C, defined by (1.8), satisfies the condition

$$\nabla C = 0,$$

where ∇ denotes the operator of covariant derivatives with respect to the metric tensor g.

It is known [24] that if a conformally symmetric spacetime of general relativity admits an infinitesimal conformal symmetry X, then it is either locally of type O or of type N. If of type N, and the Einstein tensor is invariant under X, then the spacetime represents a plane fronted gravitational waves with parallel rays.

The paper is organized as follows:

In Section 2 we prove the existence of an $N(k)$-quasi Einstein manifold by two examples. Section 3 contains a physical example of $N(k)$-quasi Einstein manifold. Finally, we consider $N(k)$-quasi Einstein manifold satisfying the condition $R(X,Y).C = 0$, where C is the Weyl conformal curvature tensor.

2. Examples of $N(k)$-quasi Einstein manifolds

Example 2.1. A special para-Sasakian manifold with vanishing D-concircular curvature tensor V is an $N(k)$-quasi Einstein manifold.

Let M^n be a Riemannian manifold admitting a unit concircular vector field ξ such that $\nabla_X\xi = \varepsilon(-X + \eta(X)\xi)$, $\eta(X) = g(X,\xi)$, $\varepsilon = \pm 1$. Then M^n is called a special para-Sasakian manifold ([1], [2], [3]). Recently G. Chuman [9] introduced the notion of a D-concircular curvature tensor V. V is given by the following equation

$$\begin{aligned}
&V(X,Y,Z,W)\\
&= {}'R(X,Y,Z,W) + \frac{r+2(n-1)}{(n-1)(n-2)}[g(X,Z)g(Y,W) - g(Y,Z)g(X,W)]\\
&\quad - \frac{r+n(n-1)}{(n-1)(n-2)}[g(X,Z)\eta(Y)\eta(W) - g(Y,Z)\eta(X)\eta(W)\\
&\quad + g(Y,W)\eta(Z)\eta(X) - g(X,W)\eta(Y)\eta(Z)],
\end{aligned} \tag{2.1}$$

where $'R(X,Y,Z,W) = g(R(X,Y)Z,W))$ for $R(X,Y)Z$ the curvature tensor of type $(1,3)$. If $V(X,Y,Z,W) = 0$, then from (2.1) we get that the curvature tensor $'R(X,Y,Z,W)$ is of the form

$$\begin{aligned}
&'R(X,Y,Z,W)\\
&=\frac{r+2(n-1)}{(n-1)(n-2)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]\\
&+\frac{r+n(n-1)}{(n-1)(n-2)}[g(X,Z)\eta(Y)\eta(W) - g(Y,Z)\eta(X)\eta(W)\\
&+g(Y,W)\eta(Z)\eta(X) - g(X,W)\eta(Y)\eta(Z)].
\end{aligned} \tag{2.2}$$

Putting $X = W = e_i$ in (2.2) where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over i, $1 \leq i \leq n$, we get

$$S(Y,Z) = ag(Y,Z) + b\eta(Y)\eta(Z), \tag{2.3}$$

where $a = \frac{r+n-1}{n-1}$ and $b = -\frac{r+n(n-1)}{n-1}$. Therefore, $\frac{a+b}{n-1} = -1$. Hence a special para-Sasakian manifold with vanishing D-concircular curvature tensor is an $N(-1)$-quasi Einstein manifold.

Example 2.2. Let $(\mathbb{R}^4, g)$ be a 4–dimensional Lorentzian space endowed with the Lorentzian metric g given by

$$ds^2 = g_{ij}dx^i dx^j = (x^4)^{\frac{4}{3}}\left[(dx^1)^2 + (dx^2)^2 + (dx^3)^2\right] - (dx^4)^2$$

$(i,j = 1,2,3,4)$. Then $(\mathbb{R}^4, g)$ is an $N\left(\frac{2\left[1-18(x^4)^4\right]}{9(x^4)^2}\right)$-quasi Einstein spacetime.

Let us consider a Lorentzian metric g on $\mathbb{R}^4$ by

$$ds^2 = g_{ij}dx^i dx^j = (x^4)^{\frac{4}{3}}\left[(dx^1)^2 + (dx^2)^2 + (dx^3)^2\right] - (dx^4)^2 \tag{2.4}$$

$(i,j = 1,2,3,4)$. Here the signature of g is $(+,+,+,-)$ which is Lorentzian. Then the only non-vanishing components of the Christoffel symbols and the curvature tensors are

$$\begin{aligned}
&\Gamma^1_{14} = \Gamma^2_{24} = \Gamma^3_{34} = \frac{2}{3x^4}, \qquad \Gamma^4_{11} = \Gamma^4_{22} = \Gamma^4_{33} = \frac{2}{3}(x^4)^{\frac{1}{3}},\\
&R_{1441} = R_{2442} = R_{3443} = -\frac{2}{9(x^4)^{2/3}}
\end{aligned}$$

and the components obtained by the symmetry properties. The non-vanishing components of the Ricci tensors and their covariant derivatives are:

$$\begin{aligned}
&R_{11} = R_{22} = R_{33} = \frac{2}{9(x^4)^{2/3}}, \qquad R_{44} = -\frac{2}{3(x^4)^2},\\
&R_{11,4} = R_{22,4} = R_{33,4} = -\frac{4}{9(x^4)^{5/3}}; \;\; R_{44,4} = \frac{4}{3(x^4)^3}.
\end{aligned} \tag{2.5}$$

It can be easily shown that the scalar curvature r of the resulting space $(\mathbb{R}^4, g)$ is $r = \frac{4}{3(x^4)^2}$, which is non-vanishing and non-constant. Now we shall show that this $(\mathbb{R}^4, g)$ is a $N(k)$-quasi Einstein spacetime.

To show that the spacetime under consideration is an $N(k)$-quasi Einstein spacetime, let us choose the scalar functions a, b and the 1–form η as follows:

$$a = \frac{2}{9(x^4)^2}, \qquad b = -4(x^4)^2, \tag{2.6}$$

$$\eta_i(x) = \begin{cases} \frac{1}{3(x^4)^2} & \text{for } i = 4 \\ 0 & \text{otherwise,} \end{cases} \tag{2.7}$$

at any point $x \in \mathbb{R}^4$. Now the equation (1.2) reduces to the equations

$$R_{11} = ag_{11} + b\eta_1\eta_1, \tag{2.8}$$

$$R_{22} = ag_{22} + b\eta_2\eta_2 \tag{2.9}$$

$$R_{33} = ag_{33} + b\eta_3\eta_3, \tag{2.10}$$

and

$$R_{44} = ag_{44} + b\eta_4\eta_4, \tag{2.11}$$

since, for the other cases (1.2) holds trivially. By (2.6) and (2.7) we get

$$\begin{aligned} \text{R.H.S. of (2.8)} &= ag_{11} + b\eta_1\eta_1 \\ &= \frac{2}{9(x^4)^2} \times (x^4)^{\frac{4}{3}} \\ &= \frac{2}{9(x^4)^{2/3}} = R_{11} \\ &= \text{L.H.S. of (2.8).} \end{aligned}$$

Again for (2.11) we have

$$\begin{aligned} \text{R.H.S. of (2.11)} &= ag_{44} + b\eta_4 E_4 \\ &= -\frac{2}{9(x^4)^2} - 4(x^4)^2 \times \left\{\frac{1}{3(x^4)^2}\right\}^2 \\ &= -\frac{2}{3(x^4)^2} = R_{44} \\ &= \text{L.H.S. of (2.11).} \end{aligned}$$

By similar argument it can be shown that (2.9) and (2.10) are also true. So, $(\mathbb{R}^4, g)$ is an $N\left(\frac{[1-18(x^4)^4]}{9(x^4)^2}\right)$-quasi Einstein spacetime.

3. Physical example of an $N(k)$-quasi Einstein manifold

Example 3.1. This example is concerned with example of an $N(k)$-quasi Einstein manifold in general relativity by the coordinate free method of differential geometry. In this method of study the spacetime of general relativity is regarded as a connected four-dimensional semi-Riemannian manifold (M^4, g) with Lorentz metric g with signature $(-,+,+,+)$. The geometry of the Lorentz manifold begins with the study of causal character of vectors of the manifold. It is due to this causality that the Lorentz manifold becomes a convenient choice for the study of general relativity.

Here we consider a perfect fluid $(PRS)_4$ spacetime of non-zero scalar curvature and having the basic vector field U as the timelike vector field of the fluid, that is, $g(U,U) = -1$.

For the perfect fluid spacetime, we have the Einstein equation without cosmological constant as

$$S(X,Y) - \frac{1}{2}rg(X,Y) = \kappa T(X,Y), \tag{3.1}$$

where κ is the gravitational constant, T is the energy-momentum tensor of type (0,2) given by

$$T(X,Y) = (\sigma + p)B(X)B(Y) + pg(X,Y), \tag{3.2}$$

with σ and p as the energy density and isotropic pressure of the fluid respectively. Using (3.2) in (3.1) we get

$$S(X,Y) - \frac{1}{2}rg(X,Y) = \kappa[(\sigma + p)B(X)B(Y) + pg(X,Y)]. \tag{3.3}$$

Taking a frame field and contracting (3.3) over X and Y we have

$$r = \kappa(\sigma - 3p). \tag{3.4}$$

Using (3.4) in (3.3) we see that

$$S(X,Y) = \kappa\left[(\sigma + p)B(X)B(Y) + \frac{(\sigma - p)}{2}g(X,Y)\right]. \tag{3.5}$$

Putting $Y = U$ in (3.5) and since $g(U,U) = -1$, we get

$$S(X,U) = -\frac{\kappa}{2}[\sigma + 3p]B(X). \tag{3.6}$$

Again for $(PRS)_4$ spacetime [6], $S(X,U) = 0$. This condition will be satisfied by the equation (3.6) if

$$\sigma + 3p = 0 \quad \text{as} \quad \kappa \neq 0 \quad \text{and} \quad A(X) \neq 0. \tag{3.7}$$

Using (3.4) and (3.7) in (3.5) we see that

$$S(X,Y) = \frac{r}{3}[B(X)B(Y) + g(X,Y)]. \tag{3.8}$$

Thus we can state the following:

A perfect fluid pseudo Ricci symmetric spacetime is an $N(\frac{2r}{9})$-quasi Einstein manifold.

4. $N(k)$-quasi Einstein manifold satisfying $R(X,Y).C = 0$

In an $N(k)$-quasi Einstein manifold [20] we have the following relations:

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y], \tag{4.1}$$

which is equivalent to

$$R(X,\xi)Y = k[\eta(Y)X - g(X,Y)\xi] = -R(\xi,X)Y. \tag{4.2}$$

$$S(X,\xi) = (n-1)k\eta(X). \tag{4.3}$$

$$k = \frac{a+b}{n-1}. \tag{4.4}$$

Now for Weyl conformal curvature tensor C we have

$$\begin{aligned} &C(X,Y)Z \\ =&R(X,Y)Z - \frac{1}{n-2}[g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y] \\ &+ \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{4.5}$$

where r is the scalar curvature and Q is the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor S, that is,

$$g(QX,Y) = S(X,Y).$$

Hence

$$\begin{aligned} &\eta(C(X,Y)Z) \\ =&\frac{1}{n-2}\Big[\left(\frac{r}{n-1} - k\right)\{\eta(X)g(Y,Z) - \eta(Y)g(X,Z)\} \\ &- \{\eta(X)S(Y,Z) - \eta(Y)S(X,Z)\}\Big]. \end{aligned} \tag{4.6}$$

Putting $Z = \xi$ in (4.6) we get

$$\eta(C(X,Y)\xi) = 0. \tag{4.7}$$

Again putting $X = \xi$ in (4.6) we have

$$\begin{aligned} &\eta(C(\xi,Y)Z) \\ =&\frac{1}{n-2}\Big[\left(\frac{r}{n-1} - k\right)\{g(Y,Z) - \eta(Y)\eta(Z)\} \\ &- \{S(Y,Z) - (n-1)k\eta(Y)\eta(Z)\}\Big]. \end{aligned} \tag{4.8}$$

A Riemannian manifold (M^n, g), $n \geq 3$, is called semisymmetric ([26], [27], [28]) if

$$R.R = 0 \tag{4.9}$$

holds on M.

A Riemannian manifold (M^n, g), $n > 3$, is said to be Weyl-semisymmetric if

$$R.C = 0 \tag{4.10}$$

holds on M.

Now for Weyl-semisymmetric $N(k)$-quasi Einstein manifold we have

$$R(\xi, X).C(Y, Z)W = 0$$

or,

$$\begin{aligned}&R(\xi, X)C(Y, Z)W - C(R(\xi, X)Y, Z)W\\ &-C(Y, R(\xi, X)Z)W - C(Y, Z)R(\xi, X)W = 0.\end{aligned}$$

Using (4.2) we get from here

$$\begin{aligned}&k[{}'C(Y, Z, W, X)\xi - \eta(C(Y, Z)W)X + \eta(Y)C(X, Z)W\\ &+ \eta(Z)C(Y, X)W + \eta(W)C(Y, Z)X - g(X, Y)C(\xi, Z)W\\ &- g(X, Z)C(Y, \xi)W - g(X, W)C(Y, Z)\xi] = 0,\end{aligned} \tag{4.11}$$

where ${}'C(X, Y, Z, W) = g(C(X, Y)Z, W)$. Equation (4.11) implies that either $k = 0$, that is,

$$a + b = 0, \quad \text{follows from (4.4)} \tag{4.12}$$

or,

$$\begin{aligned}&{}'C(Y, Z, W, X)\xi - \eta(C(Y, Z)W)X + \eta(Y)C(X, Z)W\\ &+\eta(Z)C(Y, X)W + \eta(W)C(Y, Z)X - g(X, Y)C(\xi, Z)W\\ &-g(X, Z)C(Y, \xi)W - g(X, W)C(Y, Z)\xi = 0.\end{aligned} \tag{4.13}$$

Now in (4.13) taking inner product with ξ we get

$$\begin{aligned}&{}'C(Y, Z, W, X) - \eta(C(Y, Z)W)\eta(X) + \eta(Y)\eta(C(X, Z)W)\\ &+ \eta(Z)\eta(C(Y, X)W) + \eta(W)\eta(C(Y, Z)X) - g(X, Y)\eta(C(\xi, Z)W)\\ &- g(X, Z)\eta(C(Y, \xi)W) - g(X, W)\eta(C(Y, Z)\xi) = 0.\end{aligned} \tag{4.14}$$

Again putting $X = Y = \{e_i\}$, where $\{e_i\}$, $i = 1, 2, \ldots, n$ be an orthonormal basis of the tangent space at any point and then taking the sum for $1 \leq i \leq n$ we have from (4.14)

$$\eta(C(\xi, Z)W) = 0. \tag{4.15}$$

In virtue of (4.7) and (4.15), (4.14) gives us

$$\begin{aligned}&{}'C(Y, Z, W, X) - \eta(C(Y, Z)W)\eta(X) + \eta(Y)\eta(C(X, Z)W)\\ &+ \eta(Z)\eta(C(Y, X)W) + \eta(W)\eta(C(Y, Z)X) = 0.\end{aligned} \tag{4.16}$$

Again (4.8) together with (4.15) gives us

$$S(Y,Z) = \left(\frac{r}{n-1} - k\right) g(Y,Z) - \left(\frac{r}{n-1} - nk\right) \eta(Y)\eta(Z). \tag{4.17}$$

Using (4.6) and (4.17) we get from (4.16) that

$$'C(Y,Z,W,X) = 0$$

which implies that the manifold is conformally flat. Thus we can state the following theorem:

Theorem 4.1. *In a Weyl-semisymmetric $N(k)$-quasi Einstein manifold either the sum of the associated scalars is zero or, the manifold is conformally flat.*

Conclusions

Quasi Einstein manifolds arose during the study of exact solutions of the Einstein field equations as well as during considerations of quasi-umbilical hypersurfaces of semi-Euclidean spaces. In this paper we have proved that a special para-Sasakian manifold with vanishing D-concircular curvature tensor V is an $N(k)$-quasi Einstein manifold. Then we find a metric of a four-dimensional $N(k)$-quasi Einstein manifold. Also we give a physical example of $N(k)$-quasi Einstein manifold. Moreover, we have considered $N(k)$-quasi Einstein manifold satisfying the condition $R(X,Y).C = 0$, where C is the Weyl conformal curvature tensor.

Acknowledgement. The authors are thankful to the referee for his valuable suggestions towards the improvement of this paper.

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