Commun. Korean Math. Soc. **26** (2011), No. 4, pp. 611–622
http://dx.doi.org/10.4134/CKMS.2011.26.4.611

BILINEAR AND BILATERAL GENERATING FUNCTIONS OF HEAT TYPE POLYNOMIALS SUGGESTED BY JACOBI POLYNOMIALS

Mumtaz Ahmad Khan, Abdul Hakim Khan, and Manoj Singh

Abstract. The present paper deals with generalization of several families of bilinear and bilateral generating functions for the heat type polynomials suggested by Jacobi polynomials with different argument.

1. Introduction

This is the continuation to previous papers [1] and [2] in which we have obtained some formulae for the heat type polynomials $P_{n,\lambda,\mu}(x,u)$ suggested by Jacobi polynomials and is defined as

$$P_{n,\lambda,\mu}(x,u) = (4u)^n \left(\lambda+\frac{1}{2}\right)_n {}_2F_1\left[\begin{array}{l} -n, \lambda+\mu+n; \\ \lambda+\frac{1}{2}\ ; \end{array} -\frac{x^2}{4u}\right] \tag{1.1}$$

$$= (4u)^n\ n!\ P_n^{(\lambda-\frac{1}{2},\mu-\frac{1}{2})}(-\frac{x^2}{4u}). \tag{1.2}$$

In view of the relation (see, E. D. Rainville [3], Th. 20, p. 60), a relation (1.1) can be written in the elegant form

$$P_{n,\lambda,\mu}(x,u) = \left(\lambda+\frac{1}{2}\right)_n (x^2+4u)^n {}_2F_1\left[\begin{array}{l} -n, \frac{1}{2}-\mu-n; \\ \lambda+\frac{1}{2}\ ; \end{array} \frac{x^2}{x^2+4u}\right]. \tag{1.3}$$

By means of the relation,

$$P_{n,\lambda,\mu}(\sqrt{x^2+4u}\ i, u) = (-1)^n\ P_{n,\mu,\lambda}(x,u)$$

which we obtain by replacing x by $\sqrt{x^2+4u}\,i$ and t by $(-t)$ in (see [1], eq.(2.2)),

$$\sum_{n=0}^{\infty} \frac{P_{n,\lambda,\mu}(x,u)}{\left(\lambda+\frac{1}{2}\right)_n \left(\mu+\frac{1}{2}\right)_n n!} t^n = {}_0F_1\left[\begin{array}{l} -\ ; \\ \lambda+\frac{1}{2}\ ; \end{array} x^2 t\right] {}_0F_1\left[\begin{array}{l} -\ ; \\ \mu+\frac{1}{2}\ ; \end{array} (x^2+4u)t\right], \tag{1.4}$$

Received July 19, 2010.
2010 *Mathematics Subject Classification.* Primary 42C05; Secondary 33C45.
Key words and phrases. Laguerre polynomials, Rice polynomials, Gauss hypergeometric function, Appell's functions, Kampé de Fériet's double hypergeometric function, Lauricella's functions, Saran's functions.

another form of (1.1) is obtained as

$$P_{n,\lambda,\mu}(x,u) = (-1)^n (4u)^n \left(\mu + \frac{1}{2}\right)_n {}_2F_1\left[\begin{array}{l} -n, \lambda+\mu+n; \\ \mu+\frac{1}{2}\ ; \end{array} \frac{x^2+4u}{4u}\right]. \tag{1.5}$$

Next we rewrite (1.1), (1.3) and (1.5) by reversing the order of summation in the form

$$P_{n,\lambda,\mu}(x,u) = \frac{(\lambda+\mu)_{2n}}{(\lambda+\mu)_n} x^{2n} {}_2F_1\left[\begin{array}{l} -n,\ \frac{1}{2}-\lambda-n; \\ 1-\lambda-\mu-2n; \end{array} -\frac{4u}{x^2}\right], \tag{1.6}$$

$$P_{n,\lambda,\mu}(x,u) = \left(\mu + \frac{1}{2}\right)_n x^{2n} {}_2F_1\left[\begin{array}{l} -n,\ \frac{1}{2}-\lambda-n; \\ \mu+\frac{1}{2}; \end{array} \frac{x^2+4u}{x^2}\right], \tag{1.7}$$

$$P_{n,\lambda,\mu}(x,u) = \frac{(\lambda+\mu)_{2n}}{(\lambda+\mu)_n} (x^2+4u)^n {}_2F_1\left[\begin{array}{l} -n,\ \frac{1}{2}-\mu-n; \\ 1-\lambda-\mu-2n; \end{array} \frac{4u}{x^2+4u}\right]. \tag{1.8}$$

In the earlier paper [2] it was proved that

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{P_{m+n,\lambda-n,\mu-n}(x,u)}{n!} t^n \\ &= [1+(x^2+u)t]^{\lambda-\frac{1}{2}}\ [1+x^2t]^{\mu-\frac{1}{2}}\ P_{m,\lambda,\mu}[x^2(1+(x^2+4u)t), u]. \end{aligned} \tag{1.9}$$

Using Euler's transformation (see, E. D. Rainville [3], Th. 20, p. 60)

$${}_2F_1[a,\ b;\ c;\ z] = (1-z)^{-a}\ {}_2F_1\left[a,\ c-b;\ c;\ \frac{z}{z-1}\right]. \tag{1.10}$$

We rewrite (1.9) as

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{P_{m+n,\lambda-n,\mu-n}(x,u)}{n!} t^n \\ &= (\lambda+\mu+m)_m (x^2+4u)^m \left(\frac{4u}{x^2+4u}\right)^{-(\lambda-\frac{1}{2})} \\ &\quad \times (1+x^2t)^{\lambda+\mu+m-1}\ {}_2F_1\left[\begin{array}{l} -(\lambda-\frac{1}{2}-m), 1-\lambda-\mu-m; \\ 1-\lambda-\mu-2m\ ; \end{array} \frac{4u}{(x^2+4u)(1+x^2t)}\right]. \end{aligned} \tag{1.11}$$

Some of the definition and notations used in this paper are as follows: Saran's functions for three variables are given by (see [4]).

$$\begin{aligned} &F_E\left[\alpha_1,\ \alpha_1,\ \alpha_1,\ \beta_1,\ \beta_2,\ \beta_3;\ \gamma_1,\ \gamma_2,\ \gamma_3;\ x,y,z\right] \\ &= \sum_{m,n,p=0}^{\infty} \frac{(\alpha_1)_{m+n+p}\ (\beta_1)_m\ (\beta_2)_{n+p}}{(\gamma_1)_m\ (\gamma_2)_n\ (\gamma_3)_p\ m!\ n!\ p!} x^m\ y^n\ z^p\ , \end{aligned} \tag{1.12}$$

$$\begin{aligned} &F_G\left[\alpha_1,\ \alpha_1,\ \alpha_1,\ \beta_1,\ \beta_2,\ \beta_3;\ \gamma_1,\ \gamma_2,\ \gamma_2;\ x,y,z\right] \\ &= \sum_{m,n,p=0}^{\infty} \frac{(\alpha_1)_{m+n+p}\ (\beta_1)_m\ (\beta_2)_n\ (\beta_3)_p}{(\gamma_1)_m\ (\gamma_2)_{n+p}\ m!\ n!\ p!} x^m\ y^n\ z^p\ , \end{aligned} \tag{1.13}$$

$$F_N\left[\alpha_1,\ \alpha_2,\ \alpha_3,\ \beta_1,\ \beta_2,\ \beta_1;\ \gamma_1,\ \gamma_2,\ \gamma_2;\ x,y,z\right]$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha_1)_m\ (\alpha_2)_n\ (\alpha_3)_p\ (\beta_1)_{m+p}\ (\beta_2)_n}{(\gamma_1)_m\ (\gamma_2)_{n+p}\ m!\ n!\ p!} x^m\ y^n\ z^p. \tag{1.14}$$

Lauricella's hypergeometric functions for n variables is defined by (see [9])

$$F_A^{(n)}\left[a, b_1, \dots, b_n;\ c_1, \dots, c_n;\ x_1, \dots, x_n\right]$$

$$= \sum_{m_1,\dots,m_n=0}^{\infty} \frac{(a)_{m_1+\dots+m_n}\ (b_1)_{m_1} \dots (b_n)_{m_n}}{(c_1)_{m_1} \dots (c_n)_{m_n}}\ \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!} \tag{1.15}$$

$$\max\{|x_1|, \dots, |x_n|\} < 1;$$

$$F_C^{(n)}\left[a, b;\ c_1, \dots, c_n;\ x_1, \dots, x_n\right]$$

$$= \sum_{m_1,\dots,m_n=0}^{\infty} \frac{(a)_{m_1+\dots\dots+m_n}\ (b)_{m_1+\dots+m_n}}{(c_1)_{m_1} \dots (c_n)_{m_n}}\ \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!} \tag{1.16}$$

$$\max\left\{\sqrt{|x_1|}, \dots, \sqrt{|x_n|}\right\} < 1;$$

$$F_D^{(n)}\left[a, b_1, \dots, b_n;\ c\ ;\ x_1, \dots, x_n\right]$$

$$= \sum_{m_1,\dots,m_n=0}^{\infty} \frac{(a)_{m_1+\dots+m_n}\ (b_1)_{m_1} \dots (b_n)_{m_n}}{(c)_{m_1+\dots+m_n}}\ \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!} \tag{1.17}$$

$$\max\{|x_1|, \dots, |x_n|\} < 1.$$

Confluent form of Lauricella's functions for n variables is defined by (see [9])

$$\psi_2^{(n)}\left[a, c_1, \dots, c_n;\ x_1, \dots, x_n\right]$$

$$= \sum_{m_1,\dots,m_n=0}^{\infty} \frac{(a)_{m_1+\dots+m_n}}{(c_1)_{m_1} \dots (c_n)_{m_n}}\ \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!}. \tag{1.18}$$

Further, Kampé de Fériet's type general double hypergeometric series (cf. Srivastava, H. M. and Manocha, H. L. [9]) is defined as

$$F_{l:m;n}^{p:q;k}\left[\begin{array}{l}(a_p):(b_q);(c_k);\\(\alpha_l):(\beta_m);(\gamma_n)\end{array}\ x,y\right] = \sum_{r,s=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_{r+s}\ \prod_{j=1}^{q}(b_j)_r\ \prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{l}(\alpha_j)_{r+s}\ \prod_{j=1}^{m}(\beta_j)_r\ \prod_{j=1}^{n}(\gamma_j)_s}. \tag{1.19}$$

Similarly, a general triple hypergeometric series $F^{(3)}[x,y,z]$ (cf. Srivastava, H. M. [7], p. 428) is defined as

$$F^{(3)}[x,y,z] = F^{(3)}\left[\begin{array}{l}(a)::(b);(b');(b'')\ :(c);(c');(c'')\ ;\\(e)::(g);(g');(g''):(h);(h');(h'');\end{array}\ x,y,z\right]$$

$$= \sum_{m,n,p=0}^{\infty} \Lambda(m,n,p) \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!}, \tag{1.20}$$

where for convenience

$$\Lambda(m,n,p) = \frac{\prod_{j=1}^{A} (a_j)_{m+n+p} \prod_{j=1}^{B} (b_j)_{m+n} \prod_{j=1}^{B'} (b'_j)_{n+p} \prod_{j=1}^{B''} (b''_j)_{p+m}}{\prod_{j=1}^{E} (e_j)_{m+n+p} \prod_{j=1}^{G} (g_j)_{m+n} \prod_{j=1}^{G'} (g'_j)_{n+p} \prod_{j=1}^{G''} (g''_j)_{p+m}}$$

$$\times \frac{\prod_{j=1}^{C} (c_j)_m \prod_{j=1}^{C'} (c'_j)_n \prod_{j=1}^{C''} (c''_j)_p}{\prod_{j=1}^{H} (h_j)_m \prod_{j=1}^{H'} (h'_j)_n \prod_{j=1}^{H''} (h''_j)_p}. \tag{1.21}$$

2. Bilinear generating function

The poynomials $P_{n,\lambda,\mu}(x,u)$ admit the following generating functions which are recorded similar to Srivastava and Singhal [10] as given below.

In view of the relation (1.8) and ([1], eq.2.16),

$$\begin{aligned} P_{m+n,\lambda,\mu-n}(x,u) = &\left(\lambda + \frac{1}{2}\right)_{m+n} (4u)^{m+n} \left(\frac{x^2+4u}{4u}\right)^{-\lambda-\mu-m} \\ &\times {}_2F_1 \left[\begin{array}{l} \lambda+m+n+\frac{1}{2}, \lambda+\mu+m ; \\ \lambda+\frac{1}{2} ; \end{array} \frac{x^2}{x^2+4u} \right], \end{aligned} \tag{2.1}$$

we get the bilinear generating functions in the subsequent manner as

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{P_{m+n,\lambda,\mu-n}(x,u)\ P_{n,\gamma,\delta-n}(y,u)}{\left(\alpha+\frac{1}{2}\right)_n\ n!} t^n \\ &= \left(\lambda+\frac{1}{2}\right)_m (4u)^m \left(\frac{x^2+4u}{4u}\right)^{-\lambda-\mu-m} \\ &\times F_G \Bigg[\lambda+m+\frac{1}{2},\ \lambda+m+\frac{1}{2},\ \lambda+m+\frac{1}{2},\ \lambda+\mu+m,\ \gamma+\delta,\ -\left(\delta-\frac{1}{2}\right); \\ &\qquad \lambda+\frac{1}{2},\ \alpha+\frac{1}{2},\ \alpha+\frac{1}{2};\ \frac{x^2}{x^2+4u},\ 4u(y^2+4u)t,\ 16u^2t\Bigg]. \end{aligned} \tag{2.2}$$

Proof of (2.2): L.H.S. of (2.2) is equal to

$$\begin{aligned} &= \left(\lambda+\frac{1}{2}\right)_m (4u)^m \left(\frac{x^2+4u}{4u}\right)^{-\lambda-\mu-m} \sum_{n=0}^{\infty} \frac{(\lambda+m+\frac{1}{2})_n(\gamma+\delta-n)_{2n}}{(\alpha+\frac{1}{2})_n(\gamma+\delta-n)_n\ n!} \\ &\times {}_2F_1 \left[\begin{array}{l} \lambda+m+n+\frac{1}{2}, \lambda+\mu+m; \\ \lambda+\frac{1}{2}; \end{array} \frac{x^2}{x^2+4u} \right] {}_2F_1 \left[\begin{array}{l} -n\ , -(\delta-\frac{1}{2})\ ; \\ 1-\gamma-\delta-n; \end{array} \frac{4u}{y^2+4u} \right] \\ &\times (4u(y^2+4u)t)^n \end{aligned}$$

$$= \left(\lambda+\frac{1}{2}\right)_m (4u)^m \left(\frac{x^2+4u}{4u}\right)^{-\lambda-\mu-m} \sum_{n,r=0}^{\infty} \frac{(\lambda+m+\frac{1}{2})_{n+r}(\gamma+\delta)_n(\lambda+\mu+m)_r}{(\alpha+\frac{1}{2})_n\ (\lambda+\frac{1}{2})_r}$$

$$\times \sum_{s=0}^{n} \frac{(-n)_s(-(\delta-\frac{1}{2}))_s}{(1-\gamma-\delta-n)_s\ n!\ r!\ s!} \left(\frac{x^2}{x^2+4u}\right)^r \left(\frac{4u}{y^2+4u}\right)^s (4u(y^2+4u)t)^n$$

$$= \left(\lambda+\frac{1}{2}\right)_m (4u)^m \left(\frac{x^2+4u}{4u}\right)^{-\lambda-\mu-m} \sum_{n,r=0}^{\infty} \frac{(\lambda+m+\frac{1}{2})_{n+r}(\gamma+\delta)_n(\lambda+\mu+m)_r}{(\alpha+\frac{1}{2})_n\ (\lambda+\frac{1}{2})_r}$$

$$\times \sum_{s=0}^{n} \frac{(-(\delta-\frac{1}{2}))_s\Gamma(\gamma+\delta+n-s)}{\Gamma(\gamma+\delta)(n-s)!\ r!\ s!} \left(\frac{x^2}{x^2+4u}\right)^r \left(\frac{4u}{y^2+4u}\right)^s (4u(y^2+4u)t)^n$$

$$= \left(\lambda+\frac{1}{2}\right)_m (4u)^m \left(\frac{x^2+4u}{4u}\right)^{-\lambda-\mu-m} \sum_{n,r,s=0}^{\infty} \frac{(\lambda+m+\frac{1}{2})_{n+r+s}(\lambda+\mu+m)_r}{(\alpha+\frac{1}{2})_{n+s}\ (\lambda+\frac{1}{2})_r}$$

$$\times \frac{((-(\delta-\frac{1}{2}))_s(\gamma+\delta)_n}{n!\ r!\ s!} \left(\frac{x^2}{x^2+4u}\right)^r \left(\frac{4u}{y^2+4u}\right)^s (4u(y^2+4u)t)^{n+s}$$

yields the R.H.S. of (2.2), or, equivalently

(2.3)

$$\sum_{n=0}^{\infty} \frac{P_{m+n,\lambda-n,\mu-n}(x,u)\ P_{n,\gamma-n,\delta-n}(y,u)}{\left(\alpha+\frac{1}{2}\right)_n\ n!} t^n$$

$$= (\lambda+\mu+m)_m\ (x^2+4u)^m \left(\frac{x^2+4u}{4u}\right)^{\lambda-\frac{1}{2}}$$

$$\times F_G\left[1-\lambda-\mu-m, 1-\lambda-\mu-m, 1-\lambda-\mu-m, -\left(\lambda-\frac{1}{2}\right)-m, -\left(\gamma-\frac{1}{2}\right),\right.$$

$$\left. -\left(\delta-\frac{1}{2}\right); 1-\lambda-\mu-2m,\ \alpha+\frac{1}{2},\ \alpha+\frac{1}{2};\ \frac{4u}{x^2+4u},\ x^2(y^2+4u)t,\ x^2y^2t\right],$$

where in (2.2) and (2.3) F_G is the Saran's function defined by eq.(1.13).

By using the relation (1.3), which in conjunction with (1.10) we obtain the bilinear generating functions similar to Sharma and Manocha [5] in the elegant form,

(2.4)

$$\sum_{n=0}^{\infty} \frac{\left(\alpha+\frac{1}{2}\right)_n\ P_{m+n,\lambda,\mu-n}(x,u)\ P_{n,\gamma,\delta}(y,u)}{\left(\gamma+\frac{1}{2}\right)_n \left(\delta+\frac{1}{2}\right)_n\ n!} t^n$$

$$= \left(\lambda+\frac{1}{2}\right)_m (4u)^m \left(\frac{x^2+4u}{4u}\right)^{-\lambda-\mu-m}$$

$$\times F_E\left[\lambda+m+\frac{1}{2},\ \lambda+m+\frac{1}{2},\ \lambda+m+\frac{1}{2},\ \lambda+\mu+m,\ \alpha+\frac{1}{2},\ \alpha+\frac{1}{2};\right.$$

$$\left.\lambda+\frac{1}{2},\ \gamma+\frac{1}{2},\ \delta+\frac{1}{2};\ \frac{x^2}{x^2+4u}, 4uy^2t,\ 4u(y^2+4u)t\right]$$

or, equivalently

(2.5)

$$\sum_{n=0}^{\infty} \frac{\left(\alpha+\frac{1}{2}\right)_n \; P_{m+n,\lambda-n,\mu-n}(x,u) \; P_{n,\gamma,\delta}(y,u)}{\left(\gamma+\frac{1}{2}\right)_n \left(\delta+\frac{1}{2}\right)_n \; n!} t^n$$

$$= (\lambda+\mu+m)_m (x^2+4u)^m \left(\frac{x^2+4u}{4u}\right)^{\lambda-\frac{1}{2}}$$

$$\times F_E \Bigg[1-\lambda-\mu-m, 1-\lambda-\mu-m, 1-\lambda-\mu-m, -\left(\lambda-\frac{1}{2}\right)-m,$$

$$\alpha+\frac{1}{2}, \alpha+\frac{1}{2}; 1-\lambda-\mu-2m, \; \gamma+\frac{1}{2}, \; \delta+\frac{1}{2}; \frac{4u}{x^2+4u}, -x^2y^2t, -x^2(y^2+4u)t\Bigg].$$

In view of the definition (1.1) and (1.3), we further obtain the bilinear generating function for $P_{n,\lambda,\mu}(x,u)$ as

(2.6)

$$\sum_{n=0}^{\infty} \frac{(\lambda+\mu+2m)_n \left(\alpha+\frac{1}{2}\right)_n \; P_{m,\lambda+m,\mu+n}(x,u) \; P_{n,\gamma,\delta}(y,u)}{\left(\gamma+\frac{1}{2}\right)_n \left(\delta+\frac{1}{2}\right)_n \; n!} t^n$$

$$= \left(\lambda+m+\frac{1}{2}\right)_m (4u)^m F_E \Bigg[\lambda+\mu+2m, \; \lambda+\mu+2m, \; \lambda+\mu+2m, -m,$$

$$\alpha+\frac{1}{2}, \; \alpha+\frac{1}{2}; \; \lambda+m+\frac{1}{2}, \; \gamma+\frac{1}{2}, \; \delta+\frac{1}{2}; \; \frac{-x^2}{4u}, y^2t, (y^2+4u)t\Bigg],$$

where in (2.4), (2.5) and (2.6) F_E is the Saran's function defined by eq.(1.12).

3. Bilateral generating functions

In term of the Lauricella's triple hypergeometric series F_4, F_8 and F_7 (which, in the notations used by Saran [4], are F_E, F_G and F_S, respectively), we obtain three bilateral generating functions for the polynomials $P_{n,\lambda,\mu}(x,u)$ by using the relation (1.3), each of which involved the Gaussian hypergeometric ${}_2F_1$ function are as follows:

(3.1)

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n \; P_{n,\lambda,\mu}(x,u) \; {}_2F_1\left[\alpha+n, \; a; \; b; \; y\right]}{\left(\lambda+\frac{1}{2}\right)_n \left(\mu+\frac{1}{2}\right)_n \; n!} t^n$$

$$= F_E \left[\alpha \; , \; \alpha \; , \; \alpha \; , \; a \; , \; \beta \; , \; \beta \; ; \; b \; , \; \lambda+\frac{1}{2} \; , \; \mu+\frac{1}{2} \; ; \; y \; , \; x^2t \; , \; (x^2+4u)t\right],$$

(3.2)

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n \; P_{n,\lambda-n,\mu-n}(x,u) \; {}_2F_1\left[\alpha+n, \; a; \; b; \; y\right]}{(\beta)_n \; n!} t^n$$

$$= F_G \left[\alpha, \; \alpha, \; \alpha, \; a, -\left(\lambda-\frac{1}{2}\right), -\left(\mu-\frac{1}{2}\right); \; b, \; \beta, \; \beta; \; y, -(x^2+4u)t, -x^2t\right],$$

(3.3)

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n\ P_{n,\lambda-n,\mu-n}(x,u)\ {}_2F_1\left[a,\ b;\ \beta+n;\ y\right]}{(\beta)_n\ n!} t^n$$
$$= F_S\left[a,\ \alpha,\ \alpha,\ b, -\left(\lambda-\frac{1}{2}\right), -\left(\mu-\frac{1}{2}\right);\ \beta,\ \beta,\ \beta;\ y, -(x^2+4u)t, -x^2t\right].$$

Now, with the help of linear generating function (1.11), we obtain bilateral generating functions, each involving a polynomials $P_{n,\lambda,\mu}(x,u)$ and one of the Appell functions F_1, F_2 and F_4 in the form,

(3.4)

$$\sum_{n=0}^{\infty} \frac{P_{m+n,\lambda-n,\mu-n}(x,u)\ F_1\left[-n,\ \alpha,\ \beta;\ \gamma;\ y,z\right]}{n!} t^n$$
$$= (\lambda+\mu+m)_m(x^2+4u)^m \left(\frac{x^2}{x^2+4u}\right)^{-(\lambda-\frac{1}{2})} (1+x^2t)^{\lambda+\mu+m-1}$$
$$\times F_G\left[1-\lambda-\mu-m, 1-\lambda-\mu-m, 1-\lambda-\mu-m, -\left(\lambda-\frac{1}{2}\right)-m, \alpha, \beta;\right.$$
$$\left. 1-\lambda-\mu-2m,\ \gamma,\ \gamma;\ \frac{4u}{(x^2+4u)(1+x^2t)}, \frac{x^2yt}{1+x^2t}, \frac{x^2zt}{1+x^2t}\right],$$

(3.5)

$$\sum_{n=0}^{\infty} \frac{P_{m+n,\lambda-n,\mu-n}(x,u)\ F_2\left[-n,\ \alpha,\ \beta;\ \gamma,\ \delta;\ y,z\right]}{n!} t^n$$
$$= (\lambda+\mu+m)_m(x^2+4u)^m \left(\frac{x^2}{x^2+4u}\right)^{-(\lambda-\frac{1}{2})} (1+x^2t)^{\lambda+\mu+m-1}$$
$$\times F_A^{(3)}\left[1-\lambda-\mu-m, -\left(\lambda-\frac{1}{2}\right)-m,\ \alpha,\ \beta; 1-\lambda-\mu-2m,\ \gamma,\ \delta;\right.$$
$$\left.\frac{4u}{(x^2+4u)(1+x^2t)},\ \frac{x^2yt}{1+x^2t},\ \frac{x^2zt}{1+x^2t}\right]$$

and

(3.6)

$$\sum_{n=0}^{\infty} \frac{P_{m+n,\lambda-n,\mu-n}(x,u)\ F_4\left[-n,\ \alpha;\ \beta,\ \gamma;\ y,z\right]}{n!} t^n$$
$$= (\lambda+\mu+m)_m(x^2+4u)^m \left(\frac{x^2}{x^2+4u}\right)^{-(\lambda-\frac{1}{2})} (1+x^2t)^{\lambda+\mu+m-1}$$
$$\times F_E\left[1-\lambda-\mu-m, 1-\lambda-\mu-m, 1-\lambda-\mu-m, -\left(\lambda-\frac{1}{2}\right)-m, \alpha, \alpha;\right.$$
$$\left. 1-\lambda-\mu-2m,\ \beta,\ \gamma;\ \frac{4u}{(x^2+4u)(1+x^2t)}, \frac{x^2yt}{1+x^2t}, \frac{x^2zt}{1+x^2t}\right],$$

where, in (3.4) and (3.6) F_G and F_E are Saran's functions defined by (1.13) and (1.12).

By involving the Laguerre polynomials (see [3], p. 200, eq.1) with the polynomials $P_{n\lambda,\mu}(x,u)$, would readily yield the bilateral generating functions in the form

(3.7)
$$\sum_{n=0}^{\infty}\binom{m+n}{n}\frac{L_{m+n}^{(\alpha)}(x)\ P_{n,\gamma,\delta}(y,u)}{\left(\gamma+\frac{1}{2}\right)_n\left(\delta+\frac{1}{2}\right)_n}t^n$$
$$=\binom{\alpha+m}{m}\ e^x\ \psi_2^{(3)}\left[\alpha+m+1;\alpha+1,\gamma+\frac{1}{2},\delta+\frac{1}{2};-x,\ y^2t,\ (y^2+4u)t\right],$$

(3.8)
$$\sum_{n=0}^{\infty}\binom{m+n}{n}\frac{L_{m+n}^{(\alpha)}(x)\ P_{n,\gamma,\delta-n}(y,u)}{\left(\gamma+\frac{1}{2}\right)_n}t^n$$
$$=\binom{\alpha+m}{m}\ e^x\ (1-4ut)^{-\alpha-m-1}\psi_1\left[\alpha+m+1\ ,\ \gamma+\delta\ ;\ \gamma+\frac{1}{2}\ ,\ \alpha+1\ ;\ \frac{y^2t}{1-4ut},\ \frac{x}{1-4ut}\right]$$

or, equivalently

$$\sum_{n=0}^{\infty}\binom{m+n}{n}\frac{L_{m+n}^{(\alpha)}(x)\ P_{n,\gamma-n,\delta-n}(y,u)}{(1-\gamma-\delta)_n}t^n$$
$$=\binom{\alpha+m}{m}\ e^x\ (1+y^2t)^{-\alpha-m-1} \tag{3.9}$$
$$\times\psi_1\left[\alpha+m+1\ ,-\left(\gamma-\frac{1}{2}\right);\alpha+1\ ,\ 1-\gamma-\delta;\frac{-4ut}{1+y^2t}\ ,\frac{-x}{1+y^2t}\right],$$

where, in (3.7) $\psi_2^{(3)}$ is the confluent form of Lauricella function defined by eq.(1.18) with $n=3$ and in (3.8) and (3.9) ψ_1 is the confluent hypergeometric function of two variables (see [9], eq.41).

Further, involving the Rice polynomials (see [3], p. 287, eq.1) with the relation (2.1), the polynomials $P_{n,\lambda,\mu}(x,u)$ leads the bilateral generating functions as:

(3.10)
$$\sum_{n=0}^{\infty}\frac{P_{m+n,\lambda,\mu-n}(x,u)\ H_n^{(\gamma-n,\delta-n)}(\zeta,\rho,y)}{(-\gamma-\delta)_n}t^n$$
$$=\left(\lambda+\frac{1}{2}\right)_m(4u)^m\left(\frac{x^2+4u}{4u}\right)^{-\lambda-\mu-m}$$
$$\times F_A^{(3)}\left[\lambda+m+\frac{1}{2},\lambda+\mu+m,-\gamma,\ \zeta;\lambda+\frac{1}{2},-\gamma-\delta,\rho;\frac{x^2}{x^2+4u},-4ut,4uyt\right].$$

Proof of (3.10): L.H.S. of (3.10) is equal to

$$= \left(\lambda+\frac{1}{2}\right)_m (4u)^m \left(\frac{x^2+4u}{4u}\right)^{-\lambda-\mu-m} \sum_{n,r=0}^{\infty} \binom{\gamma}{n} \frac{(\lambda+m+\frac{1}{2})_{n+r}(\lambda+\mu+m)_r}{(-\gamma-\delta)_n(\lambda+\frac{1}{2})_r}$$

$$\times \frac{(-n)_s(1+\gamma+\delta-n)_s(\zeta)_s}{(1+\gamma-n)_s(\rho)_s\ r!\ s!} \left(\frac{x^2}{x^2+4u}\right)^r (y)^s(4ut)^n$$

$$= \left(\lambda+\frac{1}{2}\right)_m (4u)^m \left(\frac{x^2+4u}{4u}\right)^{-\lambda-\mu-m} \sum_{n,r,s=0}^{\infty} \frac{(\lambda+m+\frac{1}{2})_{n+r+s}(-\gamma)_{n+r}}{(-\gamma-\delta)_{n+s}(\lambda+\frac{1}{2})_r}$$

$$\times \frac{(\lambda+\mu+m)_r(\zeta)_s(1+\gamma+\delta-n-s)_s}{(\rho)_s(1+\gamma-n-s)_s\ n!\ r!\ s!} \left(\frac{x^2}{x^2+4u}\right)^r (-y)^s(-4ut)^{n+s}$$

$$= \left(\lambda+\frac{1}{2}\right)_m (4u)^m \left(\frac{x^2+4u}{4u}\right)^{-\lambda-\mu-m} \sum_{n,r,s=0}^{\infty} \frac{(\lambda+m+\frac{1}{2})_{n+r+s}(\lambda+\mu+m)_r(-\gamma)_n}{(\lambda+\frac{1}{2})_r(-\gamma-\delta)_n}$$

$$\times \frac{(\zeta)_s(-\gamma+n)_s(1+\gamma+\delta-n-s)_s}{(\rho)_s(-\gamma-\delta+n)_s(1+\gamma-n-s)_s\ n!\ r!\ s!} \left(\frac{x^2}{x^2+4u}\right)^r (4uyt)^s(-4ut)^n.$$

Since,

$$\frac{(-\gamma+n)_s(1+\gamma+\delta-n-s)_s}{(-\gamma-\delta+n)_s(1+\gamma-n-s)_s} = 1$$

yields the R.H.S. of (3.10), or, equivalently

$$\sum_{n=0}^{\infty} \frac{P_{m+n,\lambda-n,\mu-n}(x,u)\ H_n^{(\gamma-n,\delta-n)}(\zeta,\rho,y)}{(-\gamma-\delta)_n} t^n$$

$$= (\lambda+\mu+m)_m(x^2+4u)^m \left(\frac{x^2+4u}{x^2}\right)^{\lambda-\frac{1}{2}} \tag{3.11}$$

$$\times F_A^{(3)}\left[1-\lambda-\mu-m, -\left(\lambda-\frac{1}{2}\right)-m\ , -\gamma,\ \zeta;\ 1-\lambda-\mu-2m,\right.$$

$$\left. -\gamma-\delta\ ,\ \rho\ ;\ \frac{4u}{x^2+4u}\ ,\ x^2t\ , -x^2yt\right].$$

Similarly, using the definition (1.8) another form of (3.11) is obtained in the form,

(3.12)

$$\sum_{n=0}^{\infty} \frac{P_{m+n,\lambda-n,\mu-n}(x,u)\ H_n^{(\gamma-n,\delta-n)}(\zeta,\rho,y)}{(-\gamma-\delta)_n} t^n$$

$$= (\lambda+\mu+m)_m\ x^{2m} \left(\frac{x^2}{x^2+4u}\right)^{\mu-\frac{1}{2}}$$

$$\times F_A^{(3)}\left[1-\lambda-\mu-m, -\left(\mu-\frac{1}{2}\right)-m\ , -\gamma, \zeta; 1-\lambda-\mu-2m, -\gamma-\delta, \rho;\right.$$

$$\left. -\frac{4u}{x^2}\ ,\ (x^2+4u)t\ , -(x^2+4u)yt\right],$$

where in relation (3.5), (3.10), (3.11) and (3.12), $F_A^{(3)}$ denote the Lauricella's triple hypergeometric function defined by eq.(1.15) with $n = 3$.

By appealing the results of [8], we obtain the bilateral generating functions in the form,

(3.13)
$$\sum_{n=0}^{\infty} \frac{(\gamma)_n(\delta)_n\ P_{n,\lambda,\mu}(x,u)}{(\lambda+\frac{1}{2})_n(\mu+\frac{1}{2})_n\ n!} F_C^{(S)}\left[\gamma+n\ ,\ \delta+n\ ;\rho_1,\ldots,\rho_s\ ;z_1,\ldots,z_s\right]t^n$$
$$= F_C^{(S+2)}\left[\gamma\ ,\ \delta\ ;\rho_1,\ldots,\rho_s\ ,\lambda+\frac{1}{2}\ ,\mu+\frac{1}{2}\ ;z_1,\ldots,z_s\ ,x^2t\ ,(x^2+4u)t\right]$$

and

(3.14)
$$\sum_{n=0}^{\infty} \frac{(\gamma)_n\ P_{n,\lambda-n,\mu-n}(x,u)}{(\delta)_n\ n!} F_D^{(S)}\left[\gamma+n\ ,\nu_1,\ldots,\nu_s\ ;\delta+n\ ;z_1,\ldots,z_s\right]t^n$$
$$= F_D^{(S+2)}\left[\gamma,\nu_1,\ldots,\nu_s,-\left(\lambda-\frac{1}{2}\right),-\left(\mu-\frac{1}{2}\right);\delta;z_1,\ldots,z_s,-(x^2+4u)t,-x^2t\right],$$

where in (3.13) and (3.14) $F_C^{(S+2)}$ and $F_D^{(S+2)}$ denote the Lauricella's triple hypergeometric function defined by eq.(1.16) and (1.17) with $n = s + 2$.

Making the use of [6, p. 27, eq.(15)], which in conjunction with the generating relation (1.11), we get bilateral relation in the pretty form as follows:

(3.15)
$$\sum_{n=0}^{\infty} \frac{P_{m+n,\lambda-n,\mu-n}(x,u)}{n!}\ F^{u+1:\ p;\ r}_{\ v\ :\ q;\ s}\left[\begin{matrix} -n\ ,\ (e_u)\ :\ (a_p)\ ;\ (c_r)\ ; \\ (f_v)\ :\ (b_q)\ ;\ (d_s)\ ; \end{matrix}\ y,z\right]t^n$$
$$= (\lambda+\mu+m)_m(x^2+4u)^m\left(\frac{x^2+4u}{x^2}\right)^{\lambda-\frac{1}{2}}(1+x^2t)^{\lambda+\mu+m-1}$$
$$\times F^{(3)}\left[\begin{matrix} 1-\lambda-\mu-m\ ::\ldots;\ (e_u)\ ;\ldots:\ -\left(\lambda-\frac{1}{2}\right)-m\ ;\ (a_p)\ ;\ (c_r)\ ; \\ \ldots\ldots\ldots\ldots\ldots::\ldots;\ (f_v)\ ;\ldots:\ 1-\lambda-\mu-2m\ ;\ (b_q)\ ;\ (d_s)\ ; \end{matrix}\right.$$
$$\left.\frac{4u}{(x^2+4u)(1+x^2t)}\ ,\ \frac{x^2yt}{1+x^2t}\ ,\ \frac{x^2zt}{1+x^2t}\right].$$

The bilateral generating function (3.15) unifies the results (3.4), (3.5) and (3.6) involving the polynomials $P_{n,\lambda,\mu}(x,u)$. Also,

(3.16)
$$\sum_{n=0}^{\infty} \frac{P_{m+n,\lambda-n,\mu-n}(x,u)}{n!}\ F^{u\ :\ p+1;\ r}_{v\ :\ q\ ;\ s}\left[\begin{matrix} (e_u)\ :-n\ ,\ (a_p)\ ;\ (c_r)\ ; \\ (f_v)\ :\ (b_q)\ ;\ (d_s)\ ; \end{matrix}\ y,z\right]t^n$$
$$= (\lambda+\mu+m)_m(x^2+4u)^m\left(\frac{x^2+4u}{x^2}\right)^{\lambda-\frac{1}{2}}(1+x^2t)^{\lambda+\mu+m-1}$$

$$\times F^{(3)}\left[\begin{array}{l} \ldots :: 1-\lambda-\mu-m\ ;\ (e_u)\ ;\ldots :\ -\left(\lambda-\frac{1}{2}\right)-m\ ;\ (a_p)\ ;\ (c_r)\ ; \\ \ldots :: \ldots\ldots\ldots\ldots\ldots ;\ (f_v)\ ;\ldots :\ 1-\lambda-\mu-2m;\ (b_q)\ ;\ (d_s)\ ; \end{array}\right.$$
$$\left.\frac{4u}{(x^2+4u)(1+x^2t)}\ ,\ \frac{x^2yt}{1+x^2t}\ ,\ z\right],$$

where in (3.15) and (3.16), $F^{(3)}[x,y,z]$ is a general triple hypergeometric series defined by eq.(1.20).

The bilateral generating function (3.16) is complementary to the above relation (3.15).

References

[1] M. A. Khan, A. H. Khan, and M. Singh, *A note on heat type polynomials suggested by Jacobi polynomials*, Communicated for publication.

[2] ———, *Linear, bilinear and trilinear generating functions for heat type polynomials suggested by Jacobi polynomils*, Communicated for publication.

[3] E. D. Rainville, *Special Functions*, MacMillan, New York, 1960, reprinted by Chelsea Publishing Company, Bronx, New York, 1971.

[4] S. Saran, *Hypergeometric functions of three variables*, Ganita **5** (1954), 77–91.

[5] B. L. Sharma and H. L. Manocha, *Some generating functions of Jacobi polynomials*, Mat. Vesnik **6(21)** (1969), 403–407.

[6] H. M. Srivastava, *A class of bilateral generating functions for the Jacobi polynomials*, J. Korean Math. Soc. **8** (1971), 25–30.

[7] ———, *Generalized Neumann expansion involving hypergeometric functions*, Proc. Cambridge Philos. Soc. **63** (1967), 425–429.

[8] H. M. Srivastava and M. C. Daoust, *Some generating functions for the Jacobi polynomials*, Comment. Math. Univ. St. Paul. **20** (1971), 15–21.

[9] H. M. Srivastava and H. L. Manocha, *A Treatise on Generating Functions*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and sons, New York, Chichester Brisbane, Toronto, 1984.

[10] H. M. Srivastava and J. P. Singhal, *Certain generating functions for Jacobi, Laguerre, and Rice polynomials*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **20** (1972), 355–363.

Mumtaz Ahmad Khan
Department of Applied Mathematics
Faculty of Engineering and Technology
Aligarh Muslim University
Aligarh-202002, U.P., India
E-mail address: mumtaz_ahmad_khan_2008@yahoo.com

Abdul Hakim Khan
Department of Applied Mathematics
Faculty of Engineering and Technology
Aligarh Muslim University
Aligarh-202002, U.P., India
E-mail address: abdulhakim2002@yahoomail.com

Manoj Singh
Department of Applied Mathematics
Faculty of Engineering and Technology
Aligarh Muslim University
Aligarh-202002, U.P., India
E-mail address: manojsingh1181@gmail.com