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THE p-LAPLACIAN OPERATORS WITH POTENTIAL TERMS

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Abstract. In this paper, we deal with the discrete p-Laplacian operators with a potential term having the smallest nonnegative eigenvalue. Such operators are classified as its smallest eigenvalue is positive or zero. We discuss differences between them such as an existence of solutions of p-Laplacian equations on networks and properties of the energy functional. Also, we give some examples of Poisson equations which suggest a difference between linear types and nonlinear types. Finally, we study characteristics of the set of a potential those involving operator has the smallest positive eigenvalue.

1. Introduction

Networks are good abstracts of various structures such as nervous systems, molecules, economies, webs and so forth. Many phenomena on networks are represented by equations involving an operator, called the discrete Laplacian Δ_ω which is interpreted as a diffusion equation on a network. By the reason, the discrete Laplacian on a network has been studied by a lot of authors over recent years [1, 3, 4, 5, 6, 7].

In [4, 5, 6, 7], Morrow and his collaborators have studied forward problems and inverse conductivity problems involving a discrete Laplacian on a network. Networks they dealt with are electric networks such as lattice and circular forms. Also, they have established useful algorithms for identifying conductivity of edges in the given network.

In [1], Chung and Berenstein introduced another approach, partial differential equations on networks, of studying problems for discrete Laplacian equations on a network. They adapted discrete analogues of some notions on vector calculus such as an integration, a directional derivative, a gradient and a Laplacian. They proved some fundamental properties for a discrete Laplacian, for example, Green's theorem, maximum principle and Dirichlet's principle. They

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also studied the solvability of forward problems, such as Dirichlet and Neumann boundary value problems, and the global uniqueness of inverse problems.

However, most phenomena on a networks are not expressed by mathematical equations by linear operators on networks. For this reason, many researchers have considered various nonlinear operators. In [2], Chung and Kim introduced a nonlinear operator, the discrete p-Laplacian $\Delta_{p,\omega}$ on a network which is a nonlinear generalization of a discrete Laplacian, and the typical eigenvalue problem for the operator. They discussed the existence of a solution of the Poisson equation, Dirichlet and Neumann boundary value problems for the equation

$$\Delta_{p,\omega} u = f$$

on a network.

In [8], Chung, Kim and Park introduced a more general operator, the discrete p-Laplacian $\mathcal{L}_p^V$ with potential term V on a network and the typical eigenvalue problem for the operator. They discussed the existence and uniqueness of solution of the Poisson equation

$$-\Delta_{p,\omega} u + V|u|^{p-2}u = f$$

on a network when the smallest eigenvalue of the operator is positive.

In this paper, we consider *the discrete p-Laplacian $\mathcal{L}_p^V$ with potential term V on a network* by its smallest eigenvalue and study properties of the opator such as the solvability of homogeneous equations and Poisson equations, relations between the potential term and the energy functional and so forth.

This paper is organized as follows:

In Section 1, we study vector calculus on networks and recall useful results in [8]. In Section 2, we classify the p-Laplacian operator with a potential on a network whose smallest eigenvalue is nonnegative into two cases and discuss properties of operators in each cases. In [8], Park and Chung verified an existence of Possion equations

$$-\Delta_{p,\omega} u + V|u|^{p-2}u = f$$

on G under the condition the smallest eigenvalue is just positive. We give some examples, Poisson equations when the smallest eigenvalue is zero. Since p-Laplacian operator is nonlinear if $p \neq 2$, we have a difference between linear and nonlinear elliptic equations. Further, we study a relation between eigenpairs and potentials under a monotone condition on potentials. Finally, in Section 3, we discuss characteristics of the set $\mathcal{A}_0$ of potentials whose involving operator's smallest eigenvalue is positive. The set is open, unbounded and strictly convex. Also, we study a behavior of a potential term V on a straight line where $V \in \mathcal{A}_0$ or $V \in \partial\mathcal{A}_0$. When $V \in \partial\mathcal{A}_0$, the smallest eigenvalue of $\mathcal{L}_p^V$ varies positively or negatively. We get an equivalence condition of the tangent of the line such that the smallest eigenvalue of $\mathcal{L}_p^V$ varies positively.

2. Preliminaries

In this section, we start with graph theoretic notions frequently used throughout this paper.

A *graph* $G = G(V, E)$ consists of a finite set $V(G)$ (or simply V) of *vertices* and a subset $E(G)$ (or simply E) of $V \times V$ whose elements are called *edges*. By $\{x, y\} \in E$ or $x \sim y$, we mean that two vertices x and y are joined by an *edge*. A graph G is said to be *simple* if it has neither multiple edges nor loops. Also, we say that a graph G is *connected* if for every pair of verices x and y, there is a finite sequence $\{x_j\}_{j=0}^{n}$(termed a *path*) of vertices such that $x = x_0 \sim x_1 \sim \cdots \sim x_n = y$.

A *weight* on a graph $G(V, E)$ is a function $\omega : V \times V \to [0, \infty)$ satisfying

(i) $\omega(x, y) = \omega(y, x), \quad x, y \in V$

(ii) $\omega(x, y) = 0 \quad \text{if and only if } \{x, y\} \notin E.$

In particular, a weight ω satisfying

$$\omega(x, y) = 1 \quad \text{if} \quad x \sim y$$

is called the *standard weight*. A graph G associated with a weight ω is said to be a *weighted graph* or a *network*, denoted by (G,ω) (or simply G). The *degree* of a vertex x, denoted by $d_\omega x$, is defined by

$$d_\omega x := \sum_{y \in G} \omega(x, y).$$

From now on, all graphs of given networks in this paper are assumed to be simple and connected.

Throughout this paper, a function on a network is a understood as a function defined just on the set of vertices of the graph. Conventionally used, we denote by $x \in V$ or $x \in G$ the fact that x is a *vertex* in G.

Let G=(G, ω) be a network and $f : G \to \mathbb{R}$ a function. The *integration* of f is defined by

$$\int_G f d_\omega x \quad (\text{or} \int_G f) \quad := \sum_{x \in G} f(x) d_\omega x.$$

For $1 < p < \infty$, the *p-directional derivative* of f to the direction $y \in G$ is defined by

$$D_{p,\omega,y} f(x) := |f(y) - f(x)|^{p-2}(f(y) - f(x))\sqrt{\frac{\omega(x, y)}{d_\omega x}}$$

for $x \in G$. The *p-gradient* of f is defined by

$$\nabla_{p,\omega} f(x) := (D_{p,\omega,y} f(x))_{y \in G}$$

for $x \in G$. Also, the *p-Laplacian* of f is defined by

$$\Delta_{p,\omega} f(x) := \sum_{y \in G} |f(y) - f(x)|^{p-2}(f(y) - f(x))\frac{\omega(x, y)}{d_\omega x}$$

for all $x \in G$. In the case $p = 2$, we write simply $D_{\omega,y}$, ∇_ω and Δ_ω instead of $D_{2,\omega,y}$, $\nabla_{2,\omega}$ and $\Delta_{2,\omega}$, respectively. It is easy to see that these are nonlinear except for $p = 2$.

The next theorem is proved in the paper [2] by Chung and Kim.

Theorem 2.1. *Let G=(G, ω) be a network. Then for any pair of functions $f : G \to \mathbb{R}$ and $h : G \to \mathbb{R}$, we have*

$$\int_G f(-\Delta_{p,\omega} h) = \frac{1}{2}\int_G \nabla_\omega f \cdot \nabla_{p,\omega} h.$$

Let $V : G \to \mathbb{R}$ be a function. The *p-Laplacian operator $\mathcal{L}_p^V$ with potential* V is defined by

$$\mathcal{L}_p^V u := -\Delta_{p,\omega} u + V|u|^{p-2}u$$

on G for all $u : G \to \mathbb{R}$. Also we define the corresponding *energy functional* Q_V by

$$Q_V(u) := \int_G u\mathcal{L}_p^V u$$

for all $u : G \to \mathbb{R}$. By the above theorem, we have

$$\begin{aligned} Q_V(u) &= \frac{1}{2}\int_G \nabla_\omega u \cdot \nabla_{p,\omega} u + \int_G V|u|^p \\ &= \frac{1}{2}\sum_{x,y\in G} |u(y) - u(x)|^p \omega(x, y) + \sum_{x\in G} V(x)|u(x)|^p d_\omega x \end{aligned}$$

for all $u : G \to \mathbb{R}$.

On both continuous media and discrete media [2, 8], there has been a lot of discussions on solvability of *Poisson equations* of the form

$$\mathcal{L}_p^V u = -\Delta_{p,\omega} u + V|u|^{p-2}u = f$$

on G. Among those results, Chung, Kim and Park provided many useful properties of the eigenvalue problem of the form

$$\mathcal{L}_p^V u = -\Delta_{p,\omega} u + V|u|^{p-2}u = \lambda|u|^{p-2}u \tag{1}$$

on G for some $\lambda \in \mathbb{R}$.

Theorem 2.2 ([8])**.** *Let G=(G, ω) be a network. For any function $V : G \to \mathbb{R}$, there is a non-zero solution $\phi : G \to \mathbb{R}$ to the eigenvalue problem*

$$-\Delta_{p,\omega}\phi + V|\phi|^{p-2}\phi = \lambda|\phi|^{p-2}\phi$$

on G for some $\lambda \in \mathbb{R}$.

Theorem 2.3 ([8])**.** *Let G=(G, ω) be a network. For any function $V : G \to \mathbb{R}$, and λ_0 be defined by*

$$\lambda_0 := \inf_{u\not\equiv 0} \frac{Q_V(u)}{\int_G |u|^p}.$$

Then there is a nonzero function $\phi_0 : G \to \mathbb{R}$ *such that* $\frac{Q_V(\phi_0)}{\int_G |\phi_0|^p} = \lambda_0$. *Further, the function* ϕ_0 *is a solution of the eigenvalue problem*

$$-\Delta_{p,\omega} + V|u|^{p-2}u = \lambda_0|u|^{p-2}u.$$

Also they proved the following theorem which guarantees the solvability of the Poisson equation and the uniqueness of the solution.

Theorem 2.4 ([8]). *Let* G=(G, ω) *be a network and* $V : G \to \mathbb{R}$ *a function. The following are equivalent.*

i) $\lambda_0 > 0$.

ii) *If a function* $f : G \to \mathbb{R}$ *is nonnegative, then there is a unique solution* u_0 *satisfying*

$$-\Delta_{p,\omega}u + V|u|^{p-2}u = f$$

on G.

Moreover, u_0 *is nonnegative.*

3. Classification by the smallest eigenvalues

In this section, we study a p-Laplacian equation

$$\mathcal{L}_p^V u = -\Delta p, \omega u + V|u|^{p-2}u = 0$$

on a network G=(G, ω) which operator $\mathcal{L}_p^V$ has the smallest nonnegative eigenvalue. For a function $V : G \to \mathbb{R}$, we define the smallest eigenvalue of the operator $\mathcal{L}_p^V$ by

$$\lambda_0(V) = \lambda_0 := \inf_{u \not\equiv 0} \frac{Q_V(u)}{\int_G |u|^p}.$$

By the definition, $\frac{Q_V(u)}{\int_G |u|^p} \geq \lambda_0$ for all nonzero $u : G \to \mathbb{R}$ and hence $Q_V(u)$ is nonnegative for all nonzero u whenever $\lambda_0 \geq 0$. We say that Q_V is *strictly positive* if $Q_V(u) > 0$ for all $u \not\equiv 0$ and *degenerately positive* if Q_V is nonnegative and $Q_V(u_0) = 0$ for some $u_0 \not\equiv 0$. Q_V is said *nonpositive* otherwise. Clearly, $Q_V(0) = 0$ for any function $V : G \to \mathbb{R}$. By the reason, Q_V is either strictly positive or degenerately positive whenever Q_V is nonnegative. The following two theorems deal with relations between Q_V and the solvability of the equation of the form

$$\mathcal{L}_p^V u = -\Delta_{p,\omega}u + V|u|^{p-2}u = 0 \tag{2}$$

on G.

Theorem 3.1. *Let* G=(G, ω) *be a network. For any function* $V : G \to \mathbb{R}$ *satisfying that* Q_V *is nonnegative, the following are equivalent.*

i) Q_V *is degenerately positive.*

ii) $\lambda_0 = 0$.

iii) *The equation*

$$-\Delta_{p,\omega}u + V|u|^{p-2}u = 0$$

on G *has a positive solution.*

Proof. i)⇒ii) Since Q_V is degenerately positive, there is a nonzero function u_0 such that $Q_V(u_0) = 0$. Then $\lambda_0 = 0$, since Q_V is assumed to be nonnegative.

ii)⇒iii) Let ϕ_0 be a positive eigenfunction corresponding to λ_0. Then we have

$$-\Delta_{p,\omega}\phi_0 + V|\phi_0|^{p-2}\phi_0 = \lambda_0|\phi_0|^{p-2}\phi_0.$$

on G. Since $\lambda_0 = 0$, the equation has a positive solution ϕ_0.

iii)⇒i) If the equation has a positive solution ϕ_0, then $Q_V(\phi_0) = \int_G \phi_0 \mathcal{L}_p^V \phi_0 = 0$, since $\mathcal{L}_p^V \phi_0 = 0$. So, Q_V is degenerately positive. □

Theorem 3.2. *Let G=(G, ω) be a network. For any function $V : G \to \mathbb{R}$ satisfying that Q_V is nonnegative, the following are equivalent.*

i) *Q_V is strictly positive.*

ii) $\lambda_0 > 0$.

iii) *The equation*

$$-\Delta_{p,\omega}u + V|u|^{p-2}u = 0$$

on G has only a trivial solution.

Proof. i)⇒ii) By the definition of λ_0, there is a nonzero function ϕ_0 such that $\frac{Q_V(\phi_0)}{\int_G |\phi_0|^p} = \lambda_0$. Since ϕ_0 is nonzero, $Q_V(\phi_0) > 0$ by the assumption. So, $\lambda_0 > 0$.

ii)⇒iii) Since ϕ_0 is nonnegative, the equation has a unique solution by Theorem 2.4. Clearly, a zero function satisfies the equation. So, there is only a trivial solution.

iii)⇒i) Suppose Q_V is not strictly positive. Since Q_V is assumed to be nonnegative, Q_V is degenerately positive. Then the equation has a positive solution by the above theorem. □

Now, consider the Poisson equation

$$-\Delta_{p,\omega}u + V|u|^{p-2}u = f$$

on a network G=(G, ω). In [8], the existence of solutions of the equation is already verified when $\lambda_0(V) > 0$. In the case $p = 2$ and $\lambda_0(V) = 0$, the equation has infinitely many solutions if f is orthogonal to an eigenfunction ϕ_0 corresponding to $\lambda_0(V)$ by the fact that $\mathcal{L}_2^V$ can be interpreted as a matrix. In case $p = 3$, nonlinear, consider a network G such that $V(G) = \{x, y\}$ with the standard weight. By simple calculations, we can deduce the following. For the function V such that $V(x) = 1$ and $V(y) = -\frac{1}{4}$, $\lambda_0(V) = 0$. Also, the function ϕ_0 such that $\phi_0(x) = 1$ and $\phi_0(y) = 2$. Define $f(x) = -2$ and $f(y) = 1$. Then the function u_0 such that $u_0(x) = -1$ and $u_0(y) = 0$ satisfies the equation. Further, u_0 is the unique solution of the equation.

Consider two functions V and W on a network G such that $V(x) \geq W(x)$ for all $x \in G$. By the definition of the smallest eigenvalue, we deduce that $\lambda_0(V) \geq \lambda_0(W)$.

Theorem 3.3. *Let G=(G, ω) be a network. For two functions V and W on G satisfying $V \geq W$ on G, $\lambda_0(V) > \lambda_0(W)$ whenever $V \not\equiv W$.*

Proof. Let ϕ_0 and ψ_0 be eigenfunctions of $\mathcal{L}_p^V$ and $\mathcal{L}_p^W$, respectively such that $\int_G |\phi_0|^p = \int_G |\psi_0|^p = 1$. Note that

$$Q_V(u) = Q_W(u) + \int_G (V - W)|u|^p$$

and $\int_G (V - W)|u|^p \geq 0$ for all $u : G \to \mathbb{R}$. Then we have

$$\lambda_0(V) = Q_{V_2}(\phi_0) + \int_G (V - W)|\phi_0|^p \geq Q_W(\psi_0).$$

Suppose $V \not\equiv W$. Then $\int_G (V - W)|\phi_0|^p$ is positive, since $|\phi_0|$ is positive and the set $\{x \in G | V(x) \neq W(x)\}$ is nonempty. So we have the following.

$$\lambda_0(V) = Q_W(\phi_0) + \int_G (V - W)|\phi_0|^p > Q_W(\phi_0).$$

Since ψ_0 is an eigenfunction of $\lambda_0(W)$,

$$Q_W(\phi_0) \geq Q_W(\psi_0) = \lambda_0(W).$$

Thus $\lambda_0(V) > \lambda_0(W)$. □

Following corollaries are derived from the above theorem.

Corollary 3.4. *Let V and W be functions on a network G such that $V \geq W$ and $V \not\equiv W$ on G. If Q_W is nonnegative, then Q_V is strictly positive. If Q_V is degenerately positive, then Q_W is nonpositive.*

Corollary 3.5. *Let V and W be functions on a network G where $(\lambda_0(V), \phi_0)$ and $(\lambda_0(W), \psi_0)$ the smallest eigenpair for $\mathcal{L}_p^V$, $\mathcal{L}_p^W$, respectively. Then the followings are equivalent:*

i) *ϕ_0 and ψ_0 are linearly dependent.*

ii) *$V - W \equiv \lambda_0(V) - \lambda_0(W)$ on G.*

Proof. i)⇒ii) Since $\phi_0 = t\psi_0$ for some $t \in \mathbb{R} \setminus \{0\}$, ϕ_0 is also an eigenfunction corresponding to $\lambda_0(W)$. Then

$$-\Delta_p \phi_0 + W|\phi_0|^{p-2}\phi_0 = \lambda_0(W)|\phi_0|^{p-2}\phi_0$$

on G. Since

$$-\Delta_p \phi_0 + V|\phi_0|^{p-2}\phi_0 = \lambda_0(V)|\phi_0|^{p-2}\phi_0$$

on G, we have

$$(V - W)|\phi_0|^{p-2}\phi_0 = (\lambda_0(V) - \lambda_0(W))|\phi_0|^{p-2}\phi_0$$

on G. Then we have $V - W = \lambda_0(V) - \lambda_0(W)$ on G, since ϕ_0 is a positive function.

ii)⇒i) Suppose $V - W \equiv \lambda_0(V) - \lambda_0(W)$ on G. Put $\psi_0 := \frac{\phi_{0,1}}{\int_G |\phi_{0,1}|^p}$. Then $(\lambda_0(V), \psi_0)$ is also an eigenpair and $\int_G |\psi_0|^p = 1$. We can deduce that $Q_W(\psi_0) = \lambda_0(W)$, since

$$\lambda_0(V) = Q_W(\psi_0) + \int_G (V - W)|\psi_0|^p = Q_W(\psi_0) + \lambda_0(V) - \lambda_0(W).$$

That is, $(\lambda_0(W), \psi_0)$ is an eigenpair of $\mathcal{L}_p^W$. Therefore, ϕ_0 and ψ_0 are linearly dependent. $\square$

If Q_V and Q_W are degenerately positive, we have the following result.

Corollary 3.6. *Let Q_V and Q_W be degenerately positive. Then $V \equiv W$ on G if and only if ϕ_0, ψ_0 are linearly dependent.*

4. Geometric characteristics of the set of potential terms

In this section, we study geometric characteristics of the set of potentials having positive smallest eigenvalue.

Lemma 4.1. *Let V and W be a function such that Q_V and Q_W are strictly positive. Then $Q_{tV+(1-t)W}$ is strictly positive for all $t \in (0,1)$.*

Proof. We note that

$$Q_{tV+(1-t)W}(u) = tQ_V(u) + (1-t)Q_W(u)$$

for all $u : G \to \mathbb{R}$. Since Q_V and Q_W are strictly positive, $Q_{tV+(1-t)W}$ is nonnegative for all $t \in (0,1)$. Suppose $Q_{tV+(1-t)W}$ is degenerately positive for some $t \in (0,1)$. Then there is a nonzero function u_0 such that $Q_{tV+(1-t)W}(u_0) = 0$. Then we have

$$Q_V(u_0) = Q_W(u_0) = 0.$$

That is, both Q_V and Q_W are degenerately positive. $\square$

Let $\mathcal{A}_r$ be the set of all functions $V : G \to \mathbb{R}$ such that $\lambda_0(V) > r$. The above lemma means that the set $\mathcal{A}_0$ is convex. Since V is a function defined on a finite set $V(G)$, we regard a function V as a point in an n-dimensional Euclidean space where n=$|V(G)|$. Hence λ_0 is a real-valued function on $\mathbb{R}^n$. Moreover, the function $\lambda_0(\cdot)$ is continuous and hence the set $\mathcal{A}_r$ is open. Also, the boundary of $\mathcal{A}_0$ consists of all functions V such that $\lambda_0(V) = 0$. It is easy to see that a zero function is in the boundary of $\mathcal{A}_0$.

Theorem 4.2. *The set $\mathcal{A}_0$ is strictly convex.*

Proof. By Lemma 4.1, the set $\mathcal{A}_0$ is convex. It suffices to show that the boundary of $\mathcal{A}_0$ doesn't have any line segment. Suppose the boundary has a line segment. Take functions V and W in the boundary. For any $t \in$(0,1), $tV+(1-t)W$ is also in the boundary. That is, $\lambda_0(tV+(1-t)W) = 0$. Then there is a nonzero function u_0 such that $Q_{tV+(1-t)W}(u_0) = 0$. Since

$$Q_{tV+(1-t)W}(u_0) = tQ_V(u_0) + (1-t)Q_W(u_0),$$

and both Q_V and Q_W are degenerately positive, $Q_V(u_0) = Q_W(u_0) = 0$. Then $V = W$ on G. $\square$

If a function $V : G \to \mathbb{R}$ is positive, then Q_V is strictly positive. So, the set of all positive functions is a subset of $\mathcal{A}_0$. Therefore, $\mathcal{A}_0$ is unbounded. Further, there is a function $V \in \mathcal{A}_0$ which is not positive by Theorem 4.2. Let $V : G \to \mathbb{R}$ be a function such that $\lambda_0(V) = 0$ and $V(x_0) < 0$ for some x_0. For any $u : G \to \mathbb{R}$,

$$\begin{aligned}Q_V(u) =& \frac{1}{2}\sum_{x,y\in G}|u(y)-u(x)|^p\omega(x,y)\\ &+\sum_{x\neq x_0}V(x)|u(x)|^p d_\omega x + V(x_0)|u(x_0)|^p d_\omega x_0.\end{aligned}$$

Put $u = \delta_{x_0}$ where $\delta_{x_0}(x) = 1$ if $x = x_0$ and $\delta_{x_0}(x) = 0$ otherwise. Then

$$Q_V(\delta_{x_0}) = d_\omega x_0 + V(x_0)d_\omega x_0.$$

Note that δ_{x_0} is not an eigenfunction corresponding to $\lambda_0(V)$, since δ_{x_0} is not a positive function. Also, Q_V is degenerately positive. Thus $Q_V(\delta_{x_0}) > 0$. That is, $V(x_0) > -1$. In all, we have the following result.

Theorem 4.3. *The set $\mathcal{A}_0$ is strictly convex, open, unbounded. The boundary of $\mathcal{A}_0$ contains the origin. Further,*

$$\mathcal{A}_0 \subseteq \bigcap_{x_0\in G}\{V : G \to \mathbb{R} | V(x_0) > -1\}.$$

The following theorem deals with how Q_V varies when $V \in \mathcal{A}_0$ moves on a straight line. Since $\mathcal{A}_0$ is open, some neighborhood of V is contained in $\mathcal{A}_0$. The direction of the line would be important.

Theorem 4.4. *Let Q_V be strictly positive and W a function such that $W(x_0) < 0$ for some $x_0 \in G$. Then there exist $\tau_+ > 0$ and $\tau_- \in [-\infty, 0)$ satisfying the following*:

i) Q_{V+tW} *is strictly positive for $t \in (\tau_-, \tau_+)$,*
ii) Q_{V+tW} *is degenerately positive for $t = \tau_+$(or τ_- if τ_- is finite),*
iii) Q_{V+tW} *is nonpositive otherwise.*

Proof. Since $\mathcal{A}_0$ is open, there is $\epsilon > 0$ such that $\lambda_0(V + tW) > 0$ for all $t \in (-\epsilon, \epsilon)$. Let I be the largest interval containing $(-\epsilon,\epsilon)$ such that $\lambda_0(V+tW) > 0$ for all $t \in I$. Note that

$$Q_{V+tW}(u) = Q_V(u) + t\int_G W|u|^p$$

for all $u : G \to \mathbb{R}$ and $t \in \mathbb{R}$. Since W is a function such that $W(x_0) < 0$ for some $x_0 \in G$, there is a function u_0 such that $\int_G W|u_0|^p < 0$. So, $Q_{V+tW}(u_0) < 0$ if $t > -\frac{Q_V(u_0)}{\int_G W|u_0|^p}$. Thus, I has the supremum which is finite. Let τ_+ be the supremum of I. Also, if there is a function u_0 such that $\int_G W|u_0|^p > 0$, then $Q_{V+tW}(u_0) < 0$ whenever $t > -\frac{Q_V(u_0)}{\int_G W|u_0|^p}$ (If $W(x) \le 0$ for all $x \in G$, then there is no function such that $\int_G W|u_0|^p > 0$). Thus, I has the infimum which is finite or not. Let τ_- be the infimum of I. By construction, conditions i) and ii) hold. By Theorem 4.2, there is no $t > \tau_+$ such that Q_{V+tW} is degenerately positive. Also there is no $t > \tau_+$ such that Q_{V+tW} is strictly positive by the intermediate value theorem. This guarantees condition iii). $\square$

Let V be a function in the boundary of $\mathcal{A}_0$. When V moves on a straight line, Q_V varies a strictly positive functional or not. We can guess that the reason is also related to the direction of the straight line.

Theorem 4.5. *Let Q_V be degenerately positive and u_0 a function such that $Q_V(u_0) = 0$. Then for any potential $W : G \to \mathbb{R}$, the following are equivalent.*

i) $\int_G W|u_0|^p > 0$.

ii) *there is $\tau_+ \in (0, \infty]$ such that Q_{V+tW} is strictly positive for all $t \in (0, \tau_+)$.*

Proof. ii) $\Rightarrow$ i). Since Q_{V+tW} is strictly positive for all $t \in (0, \tau_+)$, $\lambda_0(V + tW) > 0$. Then

$$0 < \lambda_{0,t} \le \frac{Q_{V+tW}(u_0)}{\int_G |u_0|^p} = \frac{t \int_G W|u_0|^p}{\int_G |u_0|^p}.$$

Since t is positive, $\int_G W|u_0|^p > 0$.

i) $\Rightarrow$ ii) Suppose that there is a positive sequence $\{t_n\}$ converging to 0 such that Q_{V+t_nW} is not strictly positive for all n. Since $\lambda_0(\cdot)$ is continuous,

$$\lim_{n\to\infty} \lambda_0(V + t_nW) = \lambda_0(V).$$

Let $\phi_{0,n}$ be a positive function such that $Q_{V+t_nW}(\phi_{0,n}) = \lambda_0(V + t_nW)$ and $\int_G |\phi_{0,n}|^p = 1$ for all n. Since $\int_G |\phi_{0,n}|^p = 1$ for all n, there is a convergent subsequence $\{\phi_{0,n_j}\}$ of $\{\phi_{0,n}\}$. Then

$$\lim_{j\to\infty} \lambda_0(V+t_{n_j}W) = \lim_{j\to\infty} Q_{V+t_{n_j}W}(\phi_{0,n_j}) = \lim_{j\to\infty} (Q_V(\phi_{0,n_j}) + t_{n_j} \int_G W|\phi_{0,n_j}|^p).$$

Also

$$\lambda_0(V) = \lim_{j\to\infty} (Q_V(\phi_{0,n_j}) + t_{n_j} \int_G W|\phi_{0,n_j}|^p) = Q_V(\phi_0),$$

where $\phi_0 = \lim_{j\to\infty} \phi_{0,n_j}$. That is, ϕ_0 is an eigenfunction corresponding to $\lambda_0(V)$. If u_0 is a positive function such that $\int_G |u_0|^p = 1$, then $\phi_0 = u_0$. Also, for any n_j,

$$0 \ge \lambda_0(V + t_{n_j}) = Q_V(\phi_{0,n_j}) + t_{n_j} \int_G W|\phi_{0,n_j}|^p \ge t_{n_j} \int_G W|\phi_{0,n_j}|^p.$$

So, $\int_G W|u_0|^p \le 0$. $\square$

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