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JORDAN (φ, ψ)-DERIVATIONS IN JB^*-TRIPLES

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Abstract. Using algebraic methods, we prove that every Jordan (φ, ψ)-derivation is a (φ, ψ)-derivation under certain conditions. In particular, we conclude that every Jordan θ-derivation is a θ-derivation.

1. Introduction

A (complex) JB^*-triple is a complex Banach space $(\mathcal{J}, \|\cdot\|)$ with a continuous triple product $\mathcal{J} \times \mathcal{J} \times \mathcal{J} \ni (x, y, z) \mapsto \{x, y, z\} \in \mathcal{J}$, which is symmetric (i.e., $\{x, y, z\} = \{z, y, x\}$ for all $x, y, z \in \mathcal{J}$) and bi-linear in the outer variables and conjugate linear in the middle variable, and satisfies:

(i) (Jordan Identity)
$L(a, b)\{x, y, z\} = \{L(a, b)x, y, z\} - \{x, L(b, a)y, z\} + \{x, y, L(a, b)z\}$ for all $a, b, x, y, z \in \mathcal{J}$, where $L(a, b)x := \{a, b, x\}$;

(ii) The operator $L(a, a) : \mathcal{J} \to \mathcal{J}$ is Hermitian, i.e., $\|e^{itL(a,a)}\| = 1$, and has positive spectrum in the Banach algebra $B(\mathcal{J})$ of all bounded linear operators;

(iii) $\|L(a, a)a\| = \|a\|^3$ for all $a \in \mathcal{J}$.

Every C^*-algebra is a (complex) JB^*-triple with respect to $\{a, b, c\} = \frac{1}{2}(ab^*c + cb^*a)$. Also, every JB^*-algebra is a JB^*-triple with respect to $\{a, b, c\} = (a \circ b^*) \circ c + (c \circ b^*) \circ a - (a \circ c) \circ b^*$. Conversely, every JB^*-triple with a unitary element e (which satisfies, by definition, $\{e, e, a\} = \{a, e, e\} = a$ for all a) is a unital JB^*-algebra with product $a \circ b = \{a, e, b\}$, involution $a^* = \{e, a, e\}$, and unit e. We refer to [1, 4, 8, 12, 13] for recent surveys on the theory of JB^*-triples.

By a bounded symmetric domain we mean a bounded domain D in a Banach space such that for each $z \in D$ there exists a smooth automorphism $s_z \in \mathrm{Aut}(D)$ of period two on D having z as the only fixed point. It is known [9] that the category of bounded symmetric domains is equivalent to the category of JB^*-triples. These categories can be applied in modeling transmission line

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theory and in special relativity [5, 6]. In a general setting, Jordan algebras, as non-commutative associative algebras, act as a model for quantum mechanics.

Let $\mathcal{J}$ be a complex JB^*-triple and $\varphi, \psi : \mathcal{J} \to \mathcal{J}$ be $\mathbb{C}$-linear mappings. A $\mathbb{C}$-linear mapping $D : \mathcal{J} \to \mathcal{J}$ is

(i) a (φ, ψ)-derivation if

$$D(\{x, y, z\}) = \{D(x), \varphi(y), \psi(z)\} + \{\varphi(x), D(y), \psi(z)\} + \{\varphi(x), \psi(y), D(z)\}$$

for all $x, y, z \in \mathcal{J}$. If $\varphi = \psi = \theta$, then D is called a θ-derivation;

(ii) a Jordan (φ, ψ)-derivation on $\mathcal{J}$ if

$$D(\{x, x, x\}) = \{D(x), \varphi(x), \psi(x)\} + \{\varphi(x), D(x), \psi(x)\} + \{\varphi(x), \psi(x), D(x)\}$$

for all $x \in \mathcal{J}$. If $\varphi = \psi = \theta$, then D is called a Jordan θ-derivation.

It is obvious that every Jordan (φ, ψ)-derivation is a Jordan (ψ,φ)-derivation.

Throughout this paper, $\mathcal{J}$ is a complex JB^*-triple, $D, \varphi, \psi : \mathcal{J} \to \mathcal{J}$ are $\mathbb{C}$-linear mappings and $A^D_{\varphi,\psi} : \mathcal{J} \times \mathcal{J} \times \mathcal{J} \to \mathcal{J}$ is a mapping defined by

$$A^D_{\varphi,\psi}(x, y, z) := \{D(x), \varphi(y), \psi(z)\} + \{\varphi(x), D(y), \psi(z)\} + \{\varphi(x), \psi(y), D(z)\}$$

for all $x, y, z \in \mathcal{J}$. It is clear that the mapping $A^D_{\varphi,\psi}$ is bi-linear in the outer variables and conjugate linear in the middle variable. Also, $A^D_{\varphi,\psi}(x, y, z) = A^D_{\psi,\varphi}(z, y, x)$ for all $x, y, z \in \mathcal{J}$.

I. N. Herstein [7] has proved that any Jordan derivation on a 2-torsion free prime ring is a derivation. M. Brešar [3] showed that Herstein's result is true in 2-torsion free semiprinie rings. In [1], the authors have studied the stability of θ-derivations and posed the following problem (see also [10]):

Problem 1.1. Is every Jordan θ-derivation a θ-derivation?

The second author gave an affirmative answer to Problem 1.1, cf. [11]. In the present paper, we use algebraic methods to prove that every Jordan (φ, ψ)-derivation is a (φ, ψ)-derivation if and only if $A^D_{\varphi,\psi} = A^D_{\psi,\varphi}$, so we give an affirmative answer to Problem 1.1 in a more general setting.

2. Main result

We start our work with the following lemma.

Lemma 2.1. *Let $D : \mathcal{J} \to \mathcal{J}$ be a Jordan (φ, ψ)-derivation. Then*

$$\begin{aligned} 2D(\{x, x, y\}) &= (A^D_{\varphi,\psi} + A^D_{\psi,\varphi})(x, x, y), \\ D(\{x, y, x\}) &= A^D_{\varphi,\psi}(x, y, x) \end{aligned}$$

for all $x, y \in \mathcal{J}$.

Proof. Since $D : \mathcal{J} \to \mathcal{J}$ is a Jordan (φ, ψ)-derivation,

$$A^D_{\varphi,\psi}(x, x, x) = D(\{x, x, x\})$$

for all $x \in \mathcal{J}$. We therefore have

$$\begin{aligned} &D(\{x+y,x+y,x+y\}) \\ &= \{D(x)+D(y),\varphi(x)+\varphi(y),\psi(x)+\psi(y)\} \\ &\quad + \{\varphi(x)+\varphi(y),D(x)+D(y),\psi(x)+\psi(y)\} \\ &\quad + \{\varphi(x)+\varphi(y),\psi(x)+\psi(y),D(x)+D(y)\} \\ &= D(\{x,x,x\})+D(\{y,y,y\})+A^D_{\varphi,\psi}(x,x,y) \\ &\quad + A^D_{\varphi,\psi}(y,x,x)+A^D_{\varphi,\psi}(x,y,y)+A^D_{\varphi,\psi}(y,y,x) \\ &\quad + A^D_{\varphi,\psi}(x,y,x)+A^D_{\varphi,\psi}(y,x,y) \end{aligned} \tag{2.1}$$

for all $x, y \in \mathcal{J}$. On the other hand,

$$\begin{aligned} \{x+y,x+y,x+y\} = \{x,x,x\}+\{y,y,y\}+2\{x,x,y\} \\ + 2\{x,y,y\}+\{x,y,x\}+\{y,x,y\} \end{aligned}$$

for all $x, y \in \mathcal{J}$. Hence

$$\begin{aligned} D(\{x+y,x+y,x+y\}) = D(\{x,x,x\})+D(\{y,y,y\}) \\ + 2D(\{x,x,y\})+2D(\{x,y,y\}) \\ + D(\{x,y,x\})+D(\{y,x,y\}) \end{aligned} \tag{2.2}$$

for all $x, y \in \mathcal{J}$. It follows from (2.1) and (2.2) that

$$\begin{aligned} &2D(\{x,x,y\})+2D(\{x,y,y\})+D(\{x,y,x\})+D(\{y,x,y\}) \\ &= A^D_{\varphi,\psi}(x,x,y)+A^D_{\varphi,\psi}(y,x,x)+A^D_{\varphi,\psi}(x,y,y) \\ &\quad + A^D_{\varphi,\psi}(y,y,x)+A^D_{\varphi,\psi}(x,y,x)+A^D_{\varphi,\psi}(y,x,y) \end{aligned} \tag{2.3}$$

for all $x, y \in \mathcal{J}$. Replacing y by $-y$ in (2.3), we obtain

$$\begin{aligned} &-2D(\{x,x,y\})+2D(\{x,y,y\})-D(\{x,y,x\})+D(\{y,x,y\}) \\ &= -A^D_{\varphi,\psi}(x,x,y)-A^D_{\varphi,\psi}(y,x,x)+A^D_{\varphi,\psi}(x,y,y) \\ &\quad + A^D_{\varphi,\psi}(y,y,x)-A^D_{\varphi,\psi}(x,y,x)+A^D_{\varphi,\psi}(y,x,y) \end{aligned} \tag{2.4}$$

for all $x, y \in \mathcal{J}$. If we subtract (2.4) from (2.3), we infer that

$$\begin{aligned} &2D(\{x,x,y\})+D(\{x,y,x\}) \\ &= A^D_{\varphi,\psi}(x,x,y)+A^D_{\varphi,\psi}(y,x,x)+A^D_{\varphi,\psi}(x,y,x) \end{aligned} \tag{2.5}$$

for all $x, y \in \mathcal{J}$. Replacing y by iy in (2.5), we get

$$\begin{aligned} &2D(\{x,x,y\})-D(\{x,y,x\}) \\ &= A^D_{\varphi,\psi}(x,x,y)+A^D_{\varphi,\psi}(y,x,x)-A^D_{\varphi,\psi}(x,y,x) \end{aligned} \tag{2.6}$$

for all $x, y \in \mathcal{J}$, so we get from (2.5) and (2.6) that

$$\begin{aligned} 2D(\{x,x,y\}) &= A^D_{\varphi,\psi}(x,x,y)+A^D_{\varphi,\psi}(y,x,x) \\ D(\{x,y,x\}) &= A^D_{\varphi,\psi}(x,y,x) \end{aligned}$$

for all $x, y \in \mathcal{J}$. Since $A^D_{\varphi,\psi}(y, x, x) = A^D_{\psi,\varphi}(x, x, y)$ for all $x, y \in \mathcal{J}$, the proof is completed. □

Theorem 2.2. *Let $D : \mathcal{J} \to \mathcal{J}$ be a Jordan (φ, ψ)-derivation. Then D is a (φ, ψ)-derivation if and only if $A^D_{\varphi,\psi} = A^D_{\psi,\varphi}$.*

Proof. Clearly

$$\{x + z, y, x + z\} = \{x, y, x\} + \{x, y, z\} + \{z, y, x\} + \{z, y, z\} \tag{2.7}$$

for all $x, y, z \in \mathcal{J}$. Hence

$$\begin{aligned} &D(\{x + z, y, x + z\}) \\ &= D(\{x, y, x\}) + D(\{x, y, z\}) + D(\{z, y, x\}) + D(\{z, y, z\}) \end{aligned} \tag{2.8}$$

for all $x, y, z \in \mathcal{J}$. Since D is a Jordan (φ, ψ)-derivation, Lemma 2.1 yields that

$$\begin{aligned} D(\{x + z, y, x + z\}) &= A^D_{\varphi,\psi}(x + z, y, x + z) \\ &= A^D_{\varphi,\psi}(x, y, x) + A^D_{\varphi,\psi}(x, y, z) \\ &\quad + A^D_{\varphi,\psi}(z, y, x) + A^D_{\varphi,\psi}(z, y, z) \\ &= D(\{x, y, x\}) + A^D_{\varphi,\psi}(x, y, z) \\ &\quad + A^D_{\varphi,\psi}(z, y, x) + D(\{z, y, z\}) \end{aligned} \tag{2.9}$$

for all $x, y, z \in \mathcal{J}$. We get from (2.8) and (2.9) that

$$D(\{x, y, z\}) + D(\{z, y, x\}) = A^D_{\varphi,\psi}(x, y, z) + A^D_{\varphi,\psi}(z, y, x)$$

for all $x, y, z \in \mathcal{J}$. Hence

$$2D(\{x, y, z\}) = (A^D_{\varphi,\psi} + A^D_{\psi,\varphi})(x, y, z) \tag{2.10}$$

for all $x, y, z \in \mathcal{J}$. If D is a (φ, ψ)-derivation, then $D(\{x, y, z\}) = A^D_{\varphi,\psi}(x, y, z)$ for all $x, y, z \in \mathcal{J}$. Therefore (2.10) implies that $A^D_{\varphi,\psi} = A^D_{\psi,\varphi}$.

Conversely, if $A^D_{\varphi,\psi} = A^D_{\psi,\varphi}$, then it follows from (2.10) that $D(\{x, y, z\}) = A^D_{\varphi,\psi}(x, y, z)$ for all $x, y, z \in \mathcal{J}$. Thus D is a (φ, ψ)-derivation. □

Corollary 2.3. *Every Jordan θ-derivation is a θ-derivation.*

We end the paper with the following interesting problem (see also [3, 2]):

Problem 2.4. Under which conditions is any Jordan (φ, ψ)-derivation on a JB^*-triple is continuous?

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