

ON THE CONVERGENCE OF A NEWTON-LIKE METHOD UNDER WEAK CONDITIONS

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ABSTRACT. We provide a semilocal convergence analysis for a Newton-like method under weak conditions in a Banach space setting. In particular, we only assume that the Gateaux derivative of the operator involved is hemicontinuous. An application is also provided.

1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution x^* of equation

$$(1.1) \quad F(x) + G(x) = 0,$$

where, F is a Gateaux-differentiable operator on an open convex subset D of a Banach space X with values in a Banach space Y , and $G : D \rightarrow Y$ is a continuous operator.

A large number of problems in applied mathematics and also in engineering are solved by finding the solutions of certain equations. For example, dynamic systems are mathematically modeled by difference of differential equations, and their solutions usually represent the states of the systems. For the sake of simplicity, assume that a time-invariant system is driven by the equation $\dot{x} = J(x)$, for some suitable operator J , where x is the state. Then the equilibrium states are determined by solving equation (1.1). Similar equations are used in the case of discrete systems. The unknowns of engineering equations can be functions (difference, differential and integral equations), vectors (systems of linear of nonlinear algebraic equations), or real or complex numbers (single algebraic equations with single unknowns). Except in special cases, the most commonly used solution methods are iterative - when starting from one or several initial approximations a sequence is constructed that converges to a solution of the equation. Iteration methods are also applied for solving optimization problems. In such cases, the iteration sequences converge to an optimal solution of the

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problem at hand. Since all of these methods have the same recursive structure, they can be introduced and discussed in a general framework.

We introduce the Newton-like method (NLM):

$$(1.2) \quad x_{n+1} = x_n - A_n^{-1}(F(x_n) + G(x_n)), \quad (n \geq 0), \quad (x_0 \in D)$$

to generate a sequence $\{x_n\}$ approximating x^* . Here, $A_n \in L(X, Y)$, the space of bounded linear operators from X into Y .

We shall assume

$$(1.3) \quad A_n = A(z_n) = F'(z_n) + [z_n, z_{n-1}; G], \quad (n \geq 0),$$

where $\{z_n\} (n \geq -1)$ is a sequence in D .

Special cases:

- (1) $G = 0$ on D . Then, we obtain Zincenko's iteration [12], [11], [6], [3], [4].
- (2) $z_n = x_n$ ($n \geq -1$). Then, we obtain Catinas' iteration [5], [3], [4].
- (3) $G = 0$ on D , and $z_n = x_n$. Then, we obtain Newton's method [4], [3], [8].
- (4) $F = 0$ on D , and $z_n = x_n$. Then, we obtain Secant method [3], [4], [8].
- (5) Other choices have been given in [3], [4], [9], where semilocal as well as local convergence results have been given, under more general Lipschitz-type conditions on the Fréchet-derivative F' of operator F than in (1)-(4).

There are many examples in the literature where operator F is nowhere Fréchet differentiable but it is differentiable in some other sense (Gateaux or B-differentiable or slantly differentiable) [1], [2], [4], [7], [9], [10]. In [1], [2], [4] we provided convergence results for Newton's method, when F' is only continuous on D . This way we extended further the application of Newton's method. In this study, we extend even further the application of Newton-like method (1.2), by assuming that F' is Gateaux differentiable, and piecewise hemicontinuous on a subset of D , whereas G is only continuous on D . Finally, we provide a class of equations where earlier results cannot apply to solve equations, but ours can.

2. Semilocal convergence analysis of (NLM)

We shall use $U(x_0, r)$ to denote a ball centered at $x_0 \in D$ and of radius $r > 0$. Moreover, $\bar{U}(x_0, r)$ denotes the closure of $U(x_0, r)$.

We need the following definition of hemicontinuity for an operator.

Definition 2.1. An operator $H : D \rightarrow Y$ is said to be hemicontinuous at each $x \in D$, if for all $\varepsilon > 0$, there exists $\delta > 0$ (δ depending on ε) such that if $|t| \leq \delta$, and $z \in X$, then

$$\|H(x + tz) - H(x)\| < \varepsilon.$$

We can now show the main semilocal convergence result for (NLM):

Theorem 2.2. *Let $F : D \subseteq X \rightarrow Y$ be a Gateaux-differentiable operator at each point in some neighborhood of $x_0 \in D$, and $G : D \subseteq X \rightarrow Y$ a continuous operator with divided difference $[x, y; G]$ satisfying*

$$(2.1) \quad [x, y; G](x - y) = G(x) - G(y) \quad \text{for all } x, y \in D.$$

Moreover, assume:

There exist $z_{-1}, z_0 \in D$, and $r > 0$ such that $A_0^{-1} \in L(Y, X)$, $\|A_0^{-1}\| \leq \alpha$,

$$(2.2) \quad \|F'(x) - F'(z_0)\| \leq \varepsilon_0 \quad \text{for all } x \in U(x_0, r),$$

$$(2.3) \quad \|F'(x) - F'(y)\| \leq \varepsilon \quad \text{for all } x, y \in U(x_0, r),$$

$$(2.4) \quad \|[x, y; G] - [u, v; G]\| \leq \varepsilon_1 \quad \text{for all } x, y, u, v \in U(x_0, r);$$

Operator $F'(x)$ is piecewise hemicontinuous for each $x \in U(x_0, r)$;

For

$$(2.5) \quad \beta = \frac{\alpha}{1 - \alpha(\varepsilon_0 + \varepsilon_1)}, \quad \eta = \alpha\|F(x_0) + G(x_0)\|,$$

$$(2.6) \quad 0 < \gamma = \beta(\varepsilon + \varepsilon_1) < 1,$$

$$(2.7) \quad \frac{\eta}{1 - \gamma} < r;$$

and

$$(2.8) \quad \overline{U}(x_0, r) \subseteq D.$$

Then, iteration $\{x_n\}$ generated by the (NLM) is well defined, remains in $U(x_0, r)$ for any sequence $\{z_n\} \subseteq U(x_0, r)$, and converges to a solution x^ of equation $F(x) + G(x) = 0$, which is unique in $\overline{U}(x_0, r)$. Moreover, the following estimates hold for all $n \geq 0$:*

$$(2.9) \quad \|x_n - x^*\| \leq \frac{\gamma^n}{1 - \gamma} \eta.$$

Proof. **Existence of x^* .** We shall show using induction that

$$(2.10) \quad x_n \in U(x_0, r),$$

and

$$(2.11) \quad \|x_{n+1} - x_n\| \leq \gamma \|x_n - x_{n-1}\|.$$

Using (1.2) for $n = 0$, we have by (2.5)

$$\|x_1 - x_0\| \leq \|A_0^{-1}(F(x_0) + G(x_0))\| \leq \eta < r,$$

which implies $x_1 \in U(x_0, r)$. That is estimate (2.10) holds for $n = 1$.

We also have in turn

$$\begin{aligned} (2.12) \quad & F(x_1) + G(x_1) \\ &= F(x_1) + G(x_1) - F(x_0) - G(x_0) - A_0(x_1 - x_0) \\ &= F(x_1) - F(x_0) - F'(z_0)(x_1 - x_0) \end{aligned}$$

$$\begin{aligned}
& -[z_0, z_{-1}; G](x_1 - x_0) + G(x_1) - G(x_0) \\
& = \int_0^1 [F'(x_0 + t(x_1 - x_0)) - F'(z_0)](x_1 - x_0) dt \\
& \quad + ([x_1, x_0; G] - [z_0, z_{-1}; G])(x_1 - x_0),
\end{aligned}$$

so, by (2.3), and (2.4)

$$\begin{aligned}
(2.13) \quad \|F(x_1) + G(x_1)\| & \leq \left\| \int_0^1 [F'(x_0 + t(x_1 - x_0)) - F'(z_0)](x_1 - x_0) dt \right\| \\
& \quad + \|[x_1, x_0; G] - [z_0, z_{-1}; G]\|(x_1 - x_0)\| \\
& \leq (\varepsilon + \varepsilon_1)\|x_1 - x_0\|.
\end{aligned}$$

Moreover, using (2.2), (2.4), and (2.6), we get

$$\begin{aligned}
(2.14) \quad & \|A_0^{-1}\| \|A_0 - A_1\| \\
& \leq \|A_0^{-1}\| \| (F'(z_1) - F'(z_0)) + ([z_1, z_0; G] - [z_0, z_{-1}; G]) \| \\
& \leq \alpha(\varepsilon_0 + \varepsilon_1) < 1.
\end{aligned}$$

It follows from (2.14), and the Banach lemma on invertible operators [4], [8] that A_1^{-1} exists and

$$(2.15) \quad \|A_1^{-1}\| \leq \beta.$$

In view of (1.2) for $n = 1$, we get

$$\begin{aligned}
(2.16) \quad \|x_2 - x_1\| & \leq \|A_1^{-1}(F(x_1) + G(x_1))\| \\
& \leq \|A_1^{-1}\| \|F(x_1) + G(x_1)\| \leq \gamma \|x_1 - x_0\|,
\end{aligned}$$

which shows (2.11) for $n = 1$.

Let us assume (2.10), and (2.11) hold for all $k \leq n$. We have as in (2.12):

$$\begin{aligned}
& F(x_n) + G(x_n) \\
& = F(x_n) + G(x_n) - F(x_{n-1}) - G(x_{n-1}) - A_{n-1}(x_n - x_{n-1}) \\
& = F(x_n) - F(x_{n-1}) - F'(z_{n-1})(x_n - x_{n-1}) - [z_{n-1}, z_{n-2}; G](x_n - x_{n-1}) \\
& \quad + G(x_n) - G(x_{n-1}) \\
& = \int_0^1 [F'(x_{n-1} + t(x_n - x_{n-1})) - F'(z_{n-1})](x_n - x_{n-1}) dt \\
& \quad + ([x_n, x_{n-1}; G] - [z_{n-1}, z_{n-2}; G])(x_n - x_{n-1}),
\end{aligned}$$

so,

$$(2.17) \quad \|F(x_n) + G(x_n)\| \leq (\varepsilon + \varepsilon_1)\|x_n - x_{n-1}\|.$$

We also have

$$\begin{aligned}
(2.18) \quad & \|A_0^{-1}\| \|A_0 - A_n\| \\
& \leq \|A_0^{-1}\| \| (F'(z_n) - F'(z_0)) + ([z_n, z_{n-1}; G] - [z_0, z_{-1}; G]) \|
\end{aligned}$$

$$\leq \alpha(\varepsilon_0 + \varepsilon_1) < 1.$$

In view of (2.18), and the Banach lemma A_n^{-1} exists, and

$$(2.19) \quad \|A_n^{-1}\| \leq \beta.$$

Using (1.2), (2.17), and (2.19), we get

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq \|A_n^{-1}\| \|F(x_n) + G(x_n)\| \\ &\leq \beta(\varepsilon + \varepsilon_1) \|x_n - x_{n-1}\| = \gamma \|x_n - x_{n-1}\|, \end{aligned}$$

which shows (2.11) for all $n \geq 0$.

We also have:

$$\begin{aligned} (2.20) \quad \|x_{n+1} - x_0\| &\leq \|x_{n+1} - x_n\| + \|x_n - x_{n-1}\| + \cdots + \|x_1 - x_0\| \\ &\leq (\gamma^n + \gamma^{n-1} + \cdots + 1)\eta = \frac{1 - \gamma^{n+1}}{1 - \gamma} \eta \leq \frac{\eta}{1 - \gamma} < r, \end{aligned}$$

so, $x_{n+1} \in U(x_0, r)$, which implies (2.10) holds for all $n \geq 0$.

The induction for (2.10) and (2.11) is now completed.

Let $n \geq 2$, and $m \geq 0$. Then we have in turn by (2.11):

$$\begin{aligned} (2.21) \quad &\|x_{n+m} - x_n\| \\ &\leq \|x_{n+m} - x_{n+m-1}\| + \|x_{n+m-1} - x_{n+m-2}\| + \cdots + \|x_{n+1} - x_n\| \\ &\leq (\gamma^{n+m-1} + \gamma^{n+m-2} + \cdots + \gamma^n)\eta = \frac{1 - \gamma^m}{1 - \gamma} \gamma^n \eta \leq \frac{\gamma^n}{1 - \gamma} \eta. \end{aligned}$$

In view of estimate (2.21), sequence $\{x_n\}$ is Cauchy in a Banach space X , and as such it converges to some $x^* \in \overline{U}(x_0, r)$ (since $x^* \in \overline{U}(x_0, r)$ is a closed set).

Moreover, we have

$$\begin{aligned} (2.22) \quad &\|F(x_n) + G(x_n)\| \\ &\leq \|A_n\| \|x_{n+1} - x_n\| \\ &\leq (\|A_n - A_0\| + \|A_0\|) \|x_{n+1} - x_n\| \leq (\|A_0\| + \varepsilon_0 + \varepsilon_1) \|x_{n+1} - x_n\|. \end{aligned}$$

By letting $n \rightarrow \infty$ in (2.22), we obtain $F(x^*) + G(x^*) = 0$.

Uniqueness of x^* . Let y^* be another solution of equation $F(x) + G(x) = 0$ in $U(x_0, r)$. We have in turn

$$\begin{aligned} (2.23) \quad &x^* - y^* \\ &= x^* - y^* - A_0^{-1}[F(x^*) + G(x^*) - F(y^*) - G(y^*)] \\ &= A_0^{-1}[F(y^*) - F(x^*) - F'(z_0)(y^* - x^*) + G(y^*) - G(x^*) - [z_0, z_{-1}; G](y^* - x^*)] \\ &= A_0^{-1} \left(\int_0^1 [F'(x^* + t(y^* - x^*)) - F'(z_0)](y^* - x^*) dt \right. \\ &\quad \left. + ([y^*, x^*; G] - [z_0, z_{-1}; G])(y^* - x^*) \right), \end{aligned}$$

so,

$$(2.24) \quad \|x^* - y^*\| \leq \gamma \|x^* - y^*\| < \|x^* - y^*\|,$$

which is a contradiction.

Hence, we deduce:

$$(2.25) \quad x^* = y^*.$$

That completes the proof of the theorem. \square

3. Applications

We shall provide some classes of operators where the results of Section 2 can apply.

Lemma 3.1. *Let $q : \mathbb{R} \rightarrow \mathbb{R}$ be a continuously differentiable function, and q' be a bounded function on \mathbb{R} . Define function $Q : I \rightarrow I, 1 < p < \infty$ for $I = L^p[a, b]$ by:*

$$Q(y)x = \int_a^x q(y(s))ds.$$

Then, function Q is Gateaux differentiable everywhere, and $Q'(y)$ is a hemi-continuous function.

Proof. We shall make use of the dominated convergence theorem and the mean-value theorem.

Let $y \in I$, and define operator $P(y) : I \rightarrow I$ by

$$P(y)(v)x = \int_a^x v(s)q'(y(s))ds$$

for all $v \in I$. Moreover, for $x \in [a, b]$, $v, y \in I$, and $t \in \mathbb{R} - \{0\}$, define

$$E(y, v, t)x = \frac{1}{t}[q(y(x) + tv(x)) - q(y(x)) - tv(x)q'(y(x))].$$

By hypothesis, there exists $\beta > 0$ such that

$$\beta = \sup_{x \in [a, b]} |q'(x)|.$$

Then, we have:

$$|E(y, v, t)x| \leq 2\beta|v(x)|,$$

and

$$E(y, v, t)x \rightarrow 0 \text{ as } t \rightarrow 0.$$

It follows by the mean-value theorem that

$$\lim_{t \rightarrow 0} \left(\int_a^b \left(\int_a^x |E(y, v, t)s| ds \right)^p dx \right)^{\frac{1}{p}} = 0.$$

Moreover, we obtain in turn:

$$\left| \frac{1}{t}[Q(y + tv) - Q(y) - tP(y)(v)]x \right| \leq \int_a^x |E(y, v, t)s| ds$$

or

$$\begin{aligned} & \frac{1}{|t|} \left(\int_a^b |[Q(y+tv) - Q(y) - tP(y)(v)]x|^p dx \right)^{\frac{1}{p}} \\ & \leq \left(\int_a^b \left(\int_a^x |E(y, v, t)s| ds \right)^p dx \right)^{\frac{1}{p}} \rightarrow 0 \text{ as } t \rightarrow 0, \end{aligned}$$

which implies $P(y)$ is the Gateaux derivative of Q at y .

Furthermore, we have:

$$\begin{aligned} & \lim_{t \rightarrow 0} \|Q'(y+tw) - Q'(y)\| \\ &= \lim_{t \rightarrow 0} \sup_{\|v\| \leq 1} \left(\int_a^b \left| \int_a^x v(s)[q'(y(s)+tw(s)) - q'(y(s))] ds \right|^p dx \right)^{\frac{1}{p}} \\ &\leq \lim_{t \rightarrow 0} \sup_{\|v\| \leq 1} \left(\int_a^b \left(\int_a^x |v(s)[q'(y(s)+tw(s)) - q'(y(s))]| ds \right)^p dx \right)^{\frac{1}{p}} \\ &\leq \lim_{t \rightarrow 0} \sup_{\|v\| \leq 1} \left(\int_a^b \left(\int_a^x |v(s)|^p ds \right) \left(\int_a^x |[q'(y(s)+tw(s)) - q'(y(s))]|^\mu ds \right)^{\frac{p}{\mu}} dx \right)^{\frac{1}{p}} \\ &\leq \lim_{t \rightarrow 0} (b-a)^{\frac{2}{p}} \left(\int_a^b |q'(y(s)+tw(s)) - q'(y(s))|^\mu ds \right)^{\frac{1}{\mu}} \\ &= 0, \end{aligned}$$

where, $\mu > 0$ is a parameter satisfying $\frac{1}{p} + \frac{1}{\mu} = 1$.

That completes the proof of the lemma. \square

We provide an example of a function which is Gateaux but nowhere Fréchet-differentiable. The result is provided in a Banach space setting.

Lemma 3.2. *Assume $q'(x)$ is not a constant function. Then, function Q for $p \in (1, 2)$ is everywhere Gateaux but nowhere Fréchet-differentiable.*

Proof. It follows from Lemma 3.1 that Q is Gateaux differentiable, and the derivative is given by P .

Let us define function

$$T(y, w)x = \int_a^x [q(y(s) + w(s)) - q(y(s)) - w(s)q'(y(s))] ds.$$

We shall show that function Q is nowhere Fréchet-differentiable.

Let $y \in I$. We shall show that there exists $w \in I$ such that the Lebesgue measure of the set $S = \{x \in [a, b] : T(y, w)x \neq 0\}$ is positive. Otherwise, the set

$$S_\gamma = \{x \in [a, b] : T(y, w_\gamma) \neq 0\}$$

has measure zero for all $w_\gamma \in I$ with $w_\gamma(x) = \gamma$ on $[a, b]$ and $\gamma \in \mathbb{R}$. That is

$$\int_a^x [q(y(s) + \gamma) - q(y(s))] ds = \int_a^x \gamma q'(y(s)) ds$$

almost everywhere on $(a, b]$, and so differentiable with respect to x . We then get

$$q(y(x) + \gamma) - q(y(x)) = \gamma q'(y(x)).$$

Moreover, differentiating with respect to γ , we obtain $q'(y(x) + \gamma) = q'(y(x))$, contradicting the hypothesis that $q'(x)$ is not a constant function. Consequently, there exist $w \in I$, $\varepsilon > 0$ such that $m(S) > 0$, and $m(C) > 0$, where $C = \{x \in [a, b] : |T(y, w)x| > \varepsilon\}$.

Let $\{C_n\}$ be a sequence in C such that $C_{n+1} \subseteq C_n$, $m(C_n) > 0$ ($n \geq 1$) with $\bigcap_{n \geq 1} C_n = \phi$. Define a sequence $\{\theta_n\}$ of functions in $L^p[a, b]$ by

$$\theta_n(x) = \begin{cases} w(x), & x \in C_n; \\ 0, & x \notin C_n. \end{cases}$$

There exists $\lambda > 0$ such that $|w(x)| \leq \lambda$ for almost all $x \in [a, b]$ (since, $w \in I$). Moreover, we have:

$$\frac{\|Q(y+\theta_n)-Q(y)-P(y)(\theta_n)\|}{\|\theta_n\|} = \frac{\left(\int_a^b |T(y, \theta_n)x| dx\right)^{\frac{1}{p}}}{\|\theta_n\|} \geq \frac{\varepsilon m(C_n)^{\frac{1}{p}}}{\lambda(m(C_n))^{\frac{1}{p}}} = \frac{\varepsilon}{\lambda} > 0,$$

whereas, $\|\theta_n\| = \left(\int_a^b |\theta_n|^p dx\right)^{\frac{1}{p}} \rightarrow 0$ as $n \rightarrow \infty$. Hence, Q is nowhere Fréchet differentiable.

That completes the proof the lemma. \square

By simply replacing (2.4) by (3.2) (see below) and following verbatim the steps of the proof of Theorem 2.2 we arrive at:

Theorem 3.3. *Let $F : D \subseteq X \rightarrow Y$ be a Gateaux-differentiable operator at each point in some neighborhood of $x_0 \in D$, $G : D \subseteq X \rightarrow Y$ an operator and*

$$(3.1) \quad A_n = F'(z_n) \quad (n \geq 0).$$

Moreover, assume:

There exist $z_0 \in D$, and $r > 0$ such that $A_0^{-1} \in L(Y, X)$, $\|A_0^{-1}\| \leq \alpha$,

$$\begin{aligned} \|F'(x) - F'(z_0)\| &\leq \varepsilon_0, \\ \|F'(x) - F'(y)\| &\leq \varepsilon \end{aligned}$$

and

$$(3.2) \quad \|G(x) - G(y)\| \leq \varepsilon_2 \|x - y\|$$

for all $x, y \in U(x_0, r)$;

Operator $F'(x)$ is piecewise hemicontinuous for each $x \in U(x_0, r)$;

For

$$\begin{aligned} \beta_0 &= \frac{\alpha}{1-\alpha\varepsilon_0}, \quad \eta = \alpha\|F(x_0) + G(x_0)\|, \\ 0 < \gamma_0 &= \beta_0(\varepsilon + \varepsilon_2) < 1, \\ \frac{\eta}{1-\gamma_0} &< r; \end{aligned}$$

and

$$\overline{U}(x_0, r) \subseteq D.$$

Then, iteration $\{x_n\}$ generated by the (NLM) is well defined, remains in $U(x_0, r)$ for any sequence $\{z_n\} \subseteq U(x_0, r)$ and converges to a solution x^* of equation $F(x) + G(x) = 0$, which is unique in $\overline{U}(x_0, r)$.

Moreover, the following estimates hold for all $n \geq 0$:

$$\|x_n - x^*\| \leq \frac{\gamma_0^n}{1 - \gamma_0} \eta.$$

Below, we provide an example for Theorem 3.3:

Example 3.4. Let $X = Y = L^p[a, b]$. Define operator F on X by

$$F(y)x = y(x) + \int_a^x \sin(y(s))ds$$

almost everywhere, $x \in [a, b]$, $y \in L^p[a, b]$. Set $A(x) = F'(x)$, and further assume that $G : X \rightarrow X$ is any continuous operator satisfying (3.2) for some $\varepsilon_2 > 0$. Let also $v, w \in L^p[a, b]$. Then, we have in turn:

$$|(I - F'(v))w(x)| = \left| \int_a^x \cos(v(s))w(s)ds \right| \leq \int_a^x |w(s)|ds,$$

so,

$$\|(I - F'(v))w\| = \left(\int_a^b |(I - F'(v))w(x)|^p dx \right)^{\frac{1}{p}} \leq (b - a)^{\frac{1}{p}} \|w\|.$$

That is we have

$$\|I - F'(v)\| \leq (b - a)^{\frac{1}{p}} = \delta.$$

Assume $\delta \in [0, 1)$. Then $F'(v)^{-1}$ exists, and

$$\|F'(v)^{-1}\| \leq \frac{1}{1 - \delta} = \alpha.$$

Let $v_1, v_2 \in U(x_0, r)$ for some r to be specified later. Then we have in turn:

$$\begin{aligned} |(F'(v_1) - F'(v_2))w(x)| &\leq \left| \int_a^x [\cos(v_1(s)) - \cos(v_2(s))]w(s)ds \right| \\ &\leq 2 \int_a^x |w(s)|ds, \end{aligned}$$

so,

$$\|F'(v_1) - F'(v_2)\| \leq 2\delta \quad \text{for all } v_1, v_2 \in X.$$

Then, we can set $\varepsilon_0 = \varepsilon = 2\delta$. Choose: $r > \frac{\eta}{1 - \gamma_0}$, where $\eta = \alpha\|F(x_0) + G(x_0)\|$, provided that

$$0 < \gamma_0 = (2\delta + \varepsilon_2)\alpha < 1.$$

Then, all hypotheses of Theorem 3.3 are satisfied. Therefore, there exists a unique solution y^* in $\overline{U}(x_0, r)$ of equation (1.1), which can be obtained as the limit of (NLM) (1.2).

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