Commun. Korean Math. Soc. **26** (2011), No. 4, pp. 551–555
http://dx.doi.org/10.4134/CKMS.2011.26.4.551

A SHORT PROOF OF AN IDENTITY FOR CUBIC PARTITION FUNCTION

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Abstract. In this note, we will give a short proof of an identity for cubic partition function.

1. Introduction

Let $p(n)$ denote the number of the unrestricted partitions of n, defined by $\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=0}^{\infty} \frac{1}{1-q^n}$. One of the celebrated results about $p(n)$ is the theorem which was proved by Watson [8]: if $k \geq 1$, then for every nonnegative integer n

$$p(5^k n + r_k) \equiv 0 \pmod{5^k}, \tag{1}$$

where r_k is the reciprocal modulo 5^k of 24. Recently, the notion of cubic partitions of a natural number n, named by Kim [5], was introduced by Chan [1] in connection with Ramanujan's cubic continued fraction. By defintion, the generating function of the number of cubic partitions of n is

$$\sum_{n=0}^{\infty} a(n)q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)(1-q^{2n})}. \tag{2}$$

Chan [1] from the Ramanujan's cubic continued fraction

$$v(q) := \frac{q^{\frac{1}{3}}}{1} {}_{+} \frac{q+q^3}{1} {}_{+} \frac{q^2+q^4}{1} {}_{+} \cdots \qquad |q| \leq 1$$

derived an elegant identity: let $x(q) = q^{-\frac{1}{3}} v(q)$, then

$$\frac{1}{x(q)} - q^{\frac{1}{3}} - 2q^{\frac{2}{3}} x(q) = \frac{(q^{\frac{1}{3}}; q^{\frac{1}{3}})_\infty (q^{\frac{2}{3}}; q^{\frac{2}{3}})_\infty}{(q^3; q^3)_\infty (q^6; q^6)_\infty},$$

where we set for $|q| \leq 1$, $(c;q)_\infty := \prod_{n=0}^{\infty}(1-cq^k)$. From this he obtained the generating function for $a(3n+2)$ [1, Theorem 1]:

Received June 9, 2010.
2010 *Mathematics Subject Classification.* Primary 11F11, 11P83.
Key words and phrases. q-series identities, modular functions, cubic partition.

Theorem 1.1.

$$\sum_{n=0}^{\infty} a(3n+2)q^n = 3\frac{(q^3;q^3)_\infty^3 (q^6;q^6)_\infty^3}{(q;q)_\infty^4 (q^2;q^2)_\infty^4}. \tag{3}$$

In this note, we will give a short proof of Theorem 1.1. The proof of Theorem 1.1 by Chan used identities involved Ramanujan's cubic continued fraction. Our proof depends on meromorphic modular functions on $\Gamma_0(6)$ and $\Gamma_0(18)$.

2. Preliminaries

Let $\mathbb{H} := \{z \in \mathbb{C} | \mathrm{Im}(z) > 0\}$ denote the complex upper half plane, for a positive integer N, define the congruence subgroup $\Gamma_0(N)$ of $SL_2(\mathbb{Z})$ by $\Gamma_0(N) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \middle| c \equiv 0 \pmod{N} \right\}$. Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ act on the complex upper half plane by the linear fractional transformation $\gamma z := \frac{az+b}{cz+d}$. Let $f(z)$ be a function on $\mathbb{H}$ which satisfies $f(\gamma z) = f(z)$, if $f(z)$ is meromorphic on $\mathbb{H}$ and at all the cusps of $\Gamma_0(N)$, then we call $f(z)$ a meromorphic modular function with respect to $\Gamma_0(N)$. The set of all such functions is denoted by $\mathcal{M}_0(\Gamma_0(N))$.

Dedekind's eta function is defined by $\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n)$, where $q = e^{2\pi i z}$ and $\mathrm{Im}(z) > 0$. A function $f(z)$ is called an eta-product if it can be written in the form of $f(z) = \prod_{\delta|N} \eta^{r_\delta}(\delta z)$, where N is a natural number and r_δ is an integer. The following proposition due to Gordon, Hughes [4] and Newman [7] which is useful to verify whether an eta-product is a modular function.

Proposition 2.1. *If $f(z) = \prod_{\delta|N} \eta^{r_\delta}(\delta z)$ is an eta-product with $\frac{1}{2}\sum_{\delta|N} r_\delta = 0$ satisfies the conditions*:

$$\sum_{\delta|N} \delta r_\delta \equiv 0 \pmod{24}, \quad \sum_{\delta|N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24}, \quad \prod_{\delta|N} \delta^{r_\delta} \in \mathbb{Q}^2,$$

then $f(z)$ is in $\mathcal{M}_0(\Gamma_0(N))$.

The following proposition due to Ligozat [6] which gives the analytic orders of an eta-product at the cusps of $\Gamma_0(N)$.

Proposition 2.2. *Let c, d and N be positive integers with $d|N$ and $(c,d) = 1$. If $f(z)$ is an eta-product satisfying the conditions in Proposition 2.1 for N, then the order of vanishing of $f(z)$ at the cusp $\frac{c}{d}$ is*

$$\frac{N}{24} \sum_{\delta|N} \frac{(d,\delta)^2 r_\delta}{(d, \frac{N}{d}) d\delta}.$$

Let p be a prime, and $f(q) = \sum_{n \geq n_0}^{\infty} a(n)q^n$ be a formal power series, we define $U_p f(q) = \sum_{pn \geq n_0} a(pn)q^n$. If $f(z) \in \mathcal{M}_0(\Gamma_0(N))$, then $f(z)$ has an expansion at the point $i\infty$ of the form $f(z) = \sum_{n=n_0}^{\infty} a(n)q^n$ where $q = e^{2\pi i z}$

and $\mathrm{Im}(z) > 0$. We call this expansion the Fourier series of $f(z)$. Moreover we define $U_p f(z)$ to be the result of applying U_p to the Fourier series of $f(z)$.

We use the results on the U_3-operator (we write U for U_3 in the following) acting on modular functions on $\mathcal{M}_0(\Gamma_0(6))$ and $\mathcal{M}_0(\Gamma_0(18))$ stated by Gordon and Hughes [4]. We know that $\Gamma_0(6)$ has 4 cusps, represented by $0, \frac{1}{2}, \frac{1}{3}, \frac{1}{6}(= i\infty)$, $\Gamma_0(18)$ has 8 cusps, represented by $0, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{6}, \frac{5}{6}, \frac{1}{9}, \frac{1}{18}(= i\infty)$. By Ligozat's formula on the analytic orders of an eta-product, if $f(z)$ is in $\mathcal{M}_0(\Gamma_0(N))$, then $f(z)$ has the same order at the cusps which have the same denominators. The order of $U_3(f(z))$ at a cusp r of $\Gamma_0(6)$ is denoted by $ord_r U(f)$, and the order of $f(z)$ at a cusp of s of $\Gamma_0(18)$ is denoted by $ord_s f$.

Proposition 2.3. *Let $f(z)$ be an eta-product in $\mathcal{M}_0(\Gamma_0(18))$. Then $U_3(f(z))$ is in $\mathcal{M}_0(\Gamma_0(6))$, and*

$$(4)\quad ord_0 U(f) \geq \min(ord_0 f, ord_{\frac{1}{3}} f),\quad ord_{\frac{1}{2}} U(f) \geq \min(ord_{\frac{1}{2}} f, ord_{\frac{1}{6}} f),$$

$$(5)\quad ord_{\frac{1}{3}} U(f) \geq \frac{1}{3} ord_{\frac{1}{9}} f,\qquad ord_{\frac{1}{6}} U(f) \geq \frac{1}{3} ord_{\frac{1}{18}} f.$$

Moreover, $U(f)$ has no poles on $\mathbb{H}$ except the cusps.

3. Proof of Theorem 1.1

Let the eta-product

$$F := F(z) = \frac{\eta(9z)}{\eta(z)} \frac{\eta(18z)}{\eta(2z)},$$

put $N = 18$, we find that $F(z)$ satisfies the conditions of Newman-Gordon-Hughes's theorem, i.e., Proposition 2.1, so $F(z)$ is in $\mathcal{M}_0(\Gamma_0(18))$. We use Ligozat's formula to calculate the orders of $F(z)$ at the cusps $\frac{c}{d}$, for $d = 1, 2, 3, 6, 9, 18$. We give the calculation of the case of $d = 1$ as an example:

$$\begin{aligned} ord_0 F &= \frac{18}{24 \times (1, \frac{18}{1})} \sum_{\delta|18} \frac{(1,\delta)^2}{\delta} r_\delta \\ &= \frac{18}{24} \times \left(\frac{(1,9)^2}{9} \times 1 + \frac{(1,18)^2}{18} \times 1 + \frac{(1,1)^2}{1} \times (-1) + \frac{(1,2)^2}{2} \times (-1) \right) \\ &= -1. \end{aligned}$$

Similar calculations give

$$ord_{\frac{1}{2}} F = -1,\quad ord_{\frac{1}{3}} F = 0,\quad ord_{\frac{1}{6}} F = 0,\quad ord_{\frac{1}{9}} F = 1,\quad ord_{\frac{1}{18}} F = 1.$$

By Proposition 2.3, the orders of $U(F)$ at the cusps of $\Gamma_0(6)$ satisfy

$$ord_0 U(F) \geq -1,\quad ord_{\frac{1}{2}} U(F) \geq -1,\quad ord_{\frac{1}{3}} U(F) \geq 1,\quad ord_{\frac{1}{6}} U(F) \geq 1$$

and $U(F)$ is holomorphic on $\mathbb{H}$. We note that the poles of $U(F)$ only appear at the cusps 0 and $\frac{1}{2}$. We define another eta-product

$$A := A(z) = \frac{\eta^4(3z)}{\eta^4(z)} \frac{\eta^4(6z)}{\eta^4(2z)}.$$

By Proposition 2.1, we find that A is in $\mathcal{M}_0(\Gamma_0(6))$. Ligozat's formula on the orders of an eta-product gives

$$ord_0 A = -1, \quad ord_{\frac{1}{2}} A = -1, ord_{\frac{1}{3}} A = 1, \quad ord_{\frac{1}{6}} A = 1$$

and A is holomorphic and non-zero elsewhere. Since the Riemann surface $(\mathbb{H} \cup \mathbb{Q} \cup i\infty)/\Gamma_0(6)$ has genus 0, $\mathcal{M}_0(\Gamma_0(6))$ has one generator as a field. The orders of A show that $U(F) = cA$. Since

$$\begin{aligned} F &= q \prod_{n=1}^{\infty} \frac{(1-q^{9n})(1-q^{18n})}{(1-q^n)(1-q^{2n})} = q + q^2 + 3q^3 + 4q^4 + 9q^5 + 12q^6 + \cdots , \\ A &= q \prod_{n=1}^{\infty} \frac{(1-q^{3n})^4(1-q^{6n})^4}{(1-q^n)^4(1-q^{2n})^4} = q + 4q^2 + 18q^3 + 52q^4 + \cdots . \end{aligned}$$

So $U(F) = 3q + 12q^2 + 54q^3 + \cdots$. The comparison of the coefficients of $U(f)$ and A shows that $c = 3$, so $U(F) = 3A$. On the other hand,

$$F = q \prod_{n=1}^{\infty} \frac{(1-q^{9n})(1-q^{18n})}{(1-q^n)(1-q^{2n})} = \left(\sum_{n=1}^{\infty} a(n-1)q^n \right) \prod_{n=1}^{\infty} (1-q^{9n})(1-q^{18n}).$$

Apply U-operator again on both sides of above, we have

$$U(F) = 3A = \left(\sum_{n=0}^{\infty} a(3n-1)q^n \right) \prod_{n=1}^{\infty} (1-q^{3n})(1-q^{6n}). \tag{6}$$

Put

$$A = q \prod_{n=1}^{\infty} \frac{(1-q^{3n})^4(1-q^{6n})^4}{(1-q^n)^4(1-q^{2n})^4}$$

into above, we obtain the identity:

$$\sum_{n=0}^{\infty} a(3n+2)q^n = 3 \frac{(q^3;q^3)_\infty^3 (q^6;q^6)_\infty^3}{(q;q)_\infty^4 (q^2;q^2)_\infty^4}. \tag{7}$$

Which is Theorem 1.1.

4. Closing remarks

We outline a proof of Theorem 1 in [2]. The ideal is similar to the paper [9]. Firstly we apply the Proposition 2.3 to A^i and FA^i for $i \geq 1$, we can express A^i (resp. FA^i) as a polynomial in A of degree at most $3i$ (resp. $3i+1$). This is the part corresponding to Proposition 1 and Proposition 2 in [2]. Next we use the initial values of A^i to calculate the elementary symmetric functions σ_i $(i = 1, 2, 3)$ of $U(A(\frac{z+t}{3}))$ $(t = 0,\ 1,\ 2)$ which are polynomials in A with integers as coefficients. Then by the Newton recurrence for power sums, we get for all $i \geq 3$

$$U(A^i) = \sigma_1 U(A^{i-1}) - \sigma_2 U(A^{i-2}) + \sigma_3 U(A^{i-3}).$$

Hence for $i \geq 1$, $U(A^i) \in \mathbb{Z}[A]$. Moreover $U(FA^i)$ satisfies the same recurrence as $U(A^i)$ also $U(FA^i)$ is in $\mathbb{Z}[A]$ and By induction we obtain the lower bounds of 3-adic orders of these coefficients. The last step is almost the same as Proposition 3 and Theorem 4.

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