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BIPOLAR FUZZY a-IDEALS OF BCI-ALGEBRAS

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Abstract. The notion of bipolar fuzzy a-ideals of BCI-algebras is introduced, and their properties are investigated. Relations between bipolar fuzzy subalgebras, bipolar fuzzy ideals and bipolar fuzzy a-ideals are discussed. Conditions for a bipolar fuzzy ideal to be a bipolar fuzzy a-ideal are provided. Characterizations of bipolar fuzzy a-ideals are given. Using a finite collection of a-ideals, a bipolar fuzzy a-ideal is established.

1. Introduction

Fuzzy set theory is established in the paper [8]. In the traditional fuzzy sets, the membership degrees of elements range over the interval $[0, 1]$. The membership degree expresses the degree of belongingness of elements to a fuzzy set. The membership degree 1 indicates that an element completely belongs to its corresponding fuzzy set, and the membership degree 0 indicates that an element does not belong to the fuzzy set. The membership degrees on the interval $(0, 1)$ indicate the partial membership to the fuzzy set. Sometimes, the membership degree means the satisfaction degree of elements to some property or constraint corresponding to a fuzzy set (see [1, 9]). In the viewpoint of satisfaction degree, the membership degree 0 is assigned to elements which do not satisfy some property. The elements with membership degree 0 are usually regarded as having the same characteristics in the fuzzy set representation. By the way, among such elements, some have irrelevant characteristics to the property corresponding to a fuzzy set and the others have contrary characteristics to the property. The traditional fuzzy set representation cannot tell apart contrary elements from irrelevant elements. Only with the membership degrees ranged on the interval [0, 1], it is difficult to express the difference of the irrelevant elements from the contrary elements in fuzzy sets. If a set representation could express this kind of difference, it would be more informative than the traditional fuzzy set representation. Based on these observations, Lee [5] introduced an extension of fuzzy sets named bipolar-valued fuzzy sets.

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Using the the notion of bipolar-valued fuzzy set, Jun and Song [3] and Lee [4] discussed subalgebras and ideals of BCK/BCI-algebras based on bipolar-valued fuzzy sets.

This paper is a continuation of the paper [3] and [4]. We introduce the notion of bipolar fuzzy a-ideals of BCI-algebras, and investigate their properties. We discuss relations between bipolar fuzzy subalgebras, bipolar fuzzy ideals and bipolar fuzzy a-ideals. We give conditions for a bipolar fuzzy ideal to be a bipolar fuzzy a-ideal, and provide characterizations of bipolar fuzzy a-ideals. Using a finite collection of a-ideals, we establish a bipolar fuzzy a-ideal.

2. Preliminaries

2.1. Basic results on BCK/BCI-algebras

Let $K(\tau)$ be the class of all algebras of type $\tau = (2, 0)$. By a *BCI-algebra* we mean a system $(X; *, 0) \in K(\tau)$ in which the following axioms hold:

(I) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,
(II) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,
(III) $(\forall x \in X)\ (x * x = 0)$,
(IV) $(\forall x, y \in X)\ (x * y = 0, y * x = 0 \Rightarrow x = y)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)\ (0 * x = 0)$,

then X is called a *BCK-algebra*. Any BCK/BCI-algebra X satisfies the following axioms:

(a1) $(\forall x \in X)\ (x * 0 = x)$,
(a2) $(\forall x, y, z \in X)\ (x \leq y \Rightarrow x * z \leq y * z,\ z * y \leq z * x)$,
(a3) $(\forall x, y, z \in X)\ ((x * y) * z = (x * z) * y)$,
(a4) $(\forall x, y, z \in X)\ ((x * z) * (y * z) \leq x * y)$

where $x \leq y$ if and only if $x * y = 0$. A subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$. A nonempty subset I of a BCK/BCI-algebra X is called an *ideal* of X if it satisfies:

(I1) $0 \in I$,
(I2) $(\forall x \in X)\ (\forall y \in I)\ (x * y \in I \Rightarrow x \in I)$.

Note that every ideal I of a BCK/BCI-algebra X has the following property.

$$(\forall x \in X)\,(\forall y \in I)\,(x \leq y \Rightarrow x \in I). \tag{2.1}$$

A nonempty subset I of a BCI-algebra X is called an *a-ideal* of X if it satisfies (I1) and

(I3) $(\forall x, y \in X)\ (\forall z \in I)\ ((x * z) * (0 * y) \in I \Rightarrow y * x \in I)$.

A fuzzy set μ in a BCK/BCI-algebra X is said to be a *fuzzy subalgebra* of X if it satisfies:

$$(\forall x, y \in X)\,(\mu(x * y) \geq \min\{\mu(x), \mu(y)\}). \tag{2.2}$$

A fuzzy set μ in a BCK/BCI-algebra X is said to be a *fuzzy ideal* of X if it satisfies:

(2.3) $$(\forall x \in X)(\mu(0) \geq \mu(x));$$

(2.4) $$(\forall x, y \in X)(\mu(x) \geq \min\{\mu(x * y),\, \mu(y)\}).$$

Note that every fuzzy ideal μ of a BCK/BCI-algebra X is order reversing, i.e., if $x \leq y$ then $\mu(x) \geq \mu(y)$.

2.2. Basic results on bipolar-valued fuzzy set

As an extension of fuzzy sets, Lee [5] introduced the notion of bipolar-valued fuzzy sets. So, this subsection is based on his paper (see [5, 6]). Fuzzy sets are a kind of useful mathematical structure to represent a collection of objects whose boundary is vague. There are several kinds of fuzzy set extensions in the fuzzy set theory, for example, intuitionistic fuzzy sets, interval-valued fuzzy sets, vague sets etc. Bipolar-valued fuzzy sets are an extension of fuzzy sets whose membership degree range is enlarged from the interval $[0, 1]$ to $[-1, 1]$. Bipolar-valued fuzzy sets have membership degrees that represent the degree of satisfaction to the property corresponding to a fuzzy set and its counter-property. In a bipolar-valued fuzzy set, the membership degree 0 means that elements are irrelevant to the corresponding property, the membership degrees on $(0, 1]$ indicate that elements somewhat satisfy the property, and the membership degrees on $[-1, 0)$ indicate that elements somewhat satisfy the implicit counter-property (see [5]). Let X be the universe of discourse. A *bipolar-valued fuzzy set* φ in X is an object having the form

$$\varphi = \{(x, \varphi^-(x), \varphi^+(x)) \mid x \in X\},$$

where $\varphi^- : X \to [-1, 0]$ and $\varphi^+ : X \to [0, 1]$ are mappings. The positive membership degree $\varphi^+(x)$ denoted the satisfaction degree of an element x to the property corresponding to a bipolar-valued fuzzy set $\varphi = \{(x, \varphi^-(x), \varphi^+(x)) \mid x \in X\}$, and the negative membership degree $\varphi^-(x)$ denotes the satisfaction degree of x to some implicit counter-property of $\varphi = \{(x, \varphi^-(x), \varphi^+(x)) \mid x \in X\}$. If $\varphi^+(x) \neq 0$ and $\varphi^-(x) = 0$, it is the situation that x is regarded as having only positive satisfaction for $\varphi = \{(x, \varphi^-(x), \varphi^+(x)) \mid x \in X\}$. If $\varphi^+(x) = 0$ and $\varphi^-(x) \neq 0$, it is the situation that x does not satisfy the property of $\varphi = \{(x, \varphi^-(x), \varphi^+(x)) \mid x \in X\}$ but somewhat satisfies the counter-property of $\varphi = \{(x, \varphi^-(x), \varphi^+(x)) \mid x \in X\}$. It is possible for an element x to be $\varphi^+(x) \neq 0$ and $\varphi^-(x) \neq 0$ when the membership function of the property overlaps that of its counter-property over some portion of the domain (see [6]). For the sake of simplicity, we shall use the symbol $\varphi = (X; \varphi^-, \varphi^+)$ for the bipolar-valued fuzzy set $\varphi = \{(x, \varphi^-(x), \varphi^+(x)) \mid x \in X\}$, and use the notion of bipolar fuzzy sets instead of the notion of bipolar-valued fuzzy sets.

3. Bipolar fuzzy a-ideals

In what follows, let X denote a BCI-algebra unless otherwise specified.

Definition 3.1 ([4])**.** A bipolar fuzzy set $\varphi = (X; \varphi^-, \varphi^+)$ in X is called a *bipolar fuzzy subalgebra* of X if it satisfies:

$$\begin{aligned} \varphi^-(x * y) &\leq \max\{\varphi^-(x), \varphi^-(y)\}, \\ \varphi^+(x * y) &\geq \min\{\varphi^+(x), \varphi^+(y)\} \end{aligned} \tag{3.1}$$

for all $x, y \in X$.

Proposition 3.2 ([4])**.** *If $\varphi = (X; \varphi^-, \varphi^+)$ is a bipolar fuzzy subalgebra of X, then $\varphi^-(0) \leq \varphi^-(x)$ and $\varphi^+(0) \geq \varphi^+(x)$ for all $x \in X$.*

For a bipolar fuzzy set $\varphi = (X; \varphi^-, \varphi^+)$ and $(s, t) \in [-1, 0] \times [0, 1]$, we define

$$\begin{aligned} N(\varphi; s) &:= \{x \in X \mid \varphi^-(x) \leq s\}, \\ P(\varphi; t) &:= \{x \in X \mid \varphi^+(x) \geq t\} \end{aligned} \tag{3.2}$$

which are called the *negative s-cut* of $\varphi = (X; \varphi^-, \varphi^+)$ and the *positive t-cut* of $\varphi = (X; \varphi^-, \varphi^+)$, respectively. The set

$$C(\varphi; (s, t)) := N(\varphi; s) \cap P(\varphi; t)$$

is called the *(s, t)-cut* of $\varphi = (X; \varphi^-, \varphi^+)$. For every $k \in [0, 1]$, if $(s, t) = (-k, k)$ then the set

$$C(\varphi; k) := N(\varphi; -k) \cap P(\varphi; k)$$

is called the *k-cut* of $\varphi = (X; \varphi^-, \varphi^+)$.

Theorem 3.3 ([4])**.** *Let $\varphi = (X; \varphi^-, \varphi^+)$ be a bipolar fuzzy subalgebra of X. Then the following assertions are valid.*

(i) $(\forall s \in [-1, 0])$ $(N(\varphi; s) \neq \emptyset \Rightarrow N(\varphi; s)$ *is a subalgebra of* $X)$.
(ii) $(\forall t \in [0, 1])$ $(P(\varphi; t) \neq \emptyset \Rightarrow P(\varphi; t)$ *is a subalgebra of* $X)$.

Definition 3.4 ([4])**.** A bipolar fuzzy set $\varphi = (X; \varphi^-, \varphi^+)$ in X is called a *bipolar fuzzy ideal* of X if it satisfies:

$$\varphi^-(0) \leq \varphi^-(x) \ \& \ \varphi^+(0) \geq \varphi^+(x); \tag{3.3}$$

$$\begin{aligned} \varphi^-(x) &\leq \max\{\varphi^-(x * y), \varphi^-(y)\}, \\ \varphi^+(x) &\geq \min\{\varphi^+(x * y), \varphi^+(y)\} \end{aligned} \tag{3.4}$$

for all $x, y \in X$.

Definition 3.5. A bipolar fuzzy set $\varphi = (X; \varphi^-, \varphi^+)$ in X is called a *bipolar fuzzy a-ideal* of X if it satisfies (3.3) and

$$\begin{aligned} \varphi^-(y * x) &\leq \max\{\varphi^-((x * z) * (0 * y)), \varphi^-(z)\}, \\ \varphi^+(y * x) &\geq \min\{\varphi^+((x * z) * (0 * y)), \varphi^+(z)\} \end{aligned} \tag{3.5}$$

for all $x, y, z \in X$.

Example 3.6. Consider a BCI-algebra $X = \{0, a, b, c\}$ with the following Cayley table:

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Define a bipolar fuzzy set $\varphi = (X; \varphi^-, \varphi^+)$ in X by

X	0	a	b	c
φ^-	−0.8	−0.8	−0.5	−0.5
φ^+	0.9	0.9	0.3	0.3

Then $\varphi = (X; \varphi^-, \varphi^+)$ is a bipolar fuzzy a-ideal of X.

Proposition 3.7. *If $\varphi = (X; \varphi^-, \varphi^+)$ is a bipolar fuzzy a-ideal of X, then*

$$\varphi^-(x) = \varphi^-(0 * x) \text{ and } \varphi^+(x) = \varphi^+(0 * x)$$

for all $x \in X$.

Proof. Let $\varphi = (X; \varphi^-, \varphi^+)$ be a bipolar fuzzy a-ideal of X. Taking $y = z = 0$ in (3.5) and using (III) and (a1), we get

$$\varphi^-(0 * x) \le \varphi^-(x) \text{ and } \varphi^+(0 * x) \ge \varphi^+(x). \tag{3.6}$$

Setting $x = z = 0$ in (3.5) and using (a1), (III), (3.3) and (3.6), we have

$$\varphi^-(y) = \varphi^-(y * 0) \le \varphi^-(0 * (0 * y)) \le \varphi^-(0 * y)$$

and

$$\varphi^+(y) = \varphi^+(y * 0) \ge \varphi^+(0 * (0 * y)) \ge \varphi^+(0 * y)$$

for all $y \in X$. Hence $\varphi^-(x) = \varphi^-(0*x)$ and $\varphi^+(x) = \varphi^+(0*x)$ for all $x \in X$. □

Theorem 3.8. *Every bipolar fuzzy a-ideal of X is both a bipolar fuzzy subalgebra of X and a bipolar fuzzy ideal of X.*

Proof. Let $\varphi = (X; \varphi^-, \varphi^+)$ be a bipolar fuzzy a-ideal of X. Using (3.5), (a1) and Proposition 3.7, we have

$$\begin{aligned}\varphi^-(x) = \varphi^-(0 * x) &\le \max\{\varphi^-((x * z) * (0 * 0)), \varphi^-(z)\}\\ &= \max\{\varphi^-(x * z), \varphi^-(z)\}\end{aligned}$$

and

$$\begin{aligned}\varphi^+(x) = \varphi^+(0 * x) &\ge \min\{\varphi^+((x * z) * (0 * 0)), \varphi^+(z)\}\\ &= \min\{\varphi^+(x * z), \varphi^+(z)\}\end{aligned}$$

for all $x, z \in X$. Hence $\varphi = (X; \varphi^-, \varphi^+)$ is a bipolar fuzzy ideal of X. Now for any $x, y \in X$, we obtain

$$\begin{aligned}\varphi^-(x * y) &\leq \max\{\varphi^-((x * y) * x), \varphi^-(x)\}\\ &= \max\{\varphi^-(0 * y), \varphi^-(x)\} = \max\{\varphi^-(x), \varphi^-(y)\}\end{aligned}$$

and

$$\begin{aligned}\varphi^+(x * y) &\geq \min\{\varphi^+((x * y) * x), \varphi^+(x)\}\\ &= \min\{\varphi^+(0 * y), \varphi^+(x)\} = \min\{\varphi^+(x), \varphi^+(y)\}.\end{aligned}$$

Therefore $\varphi = (X; \varphi^-, \varphi^+)$ is a bipolar fuzzy subalgebra of X. □

The following example shows that the converse of Theorem 3.8 is not valid.

Example 3.9. (1) Let $X = \{0, a, b\}$ be a BCI-algebra with the following Cayley table:

$*$	0	a	b
0	0	b	a
a	a	0	b
b	b	a	0

Define a bipolar fuzzy set $\varphi = (X; \varphi^-, \varphi^+)$ in X by

X	0	a	b
φ^-	-0.5	-0.2	-0.2
φ^+	0.6	0.3	0.3

Then $\varphi = (X; \varphi^-, \varphi^+)$ is both a bipolar fuzzy ideal and a bipolar fuzzy subalgebra of X, but it is not a bipolar fuzzy a-ideal of X since

$$\varphi^-(b * a) = \varphi^-(a) = -0.2 > -0.5 = \max\{\varphi^-((a * 0) * (0 * b)), \varphi^-(0)\}$$

and/or

$$\varphi^+(b * a) = \varphi^+(a) = 0.3 < 0.6 = \min\{\varphi^+((a * 0) * (0 * b)), \varphi^+(0)\}.$$

(2) Consider a BCK-algebra, and hence a BCI-algebra, $X = \{0, a, b, c\}$ with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Define a bipolar fuzzy set $\varphi = (X; \varphi^-, \varphi^+)$ in X by

X	0	a	b	c
φ^-	−0.7	−0.7	−0.2	−0.7
φ^+	0.6	0.6	0.3	0.6

Then we can check that $\varphi = (X; \varphi^-, \varphi^+)$ is a bipolar fuzzy subalgebra of X. But it is not a bipolar fuzzy ideal of X since

$$\varphi^-(b) = -0.2 \not\leq -0.7 = \max\{\varphi^-(b * a), \varphi^-(a)\},$$

and hence it is not a bipolar fuzzy a-ideal of X.

(3) Consider a BCK-algebra, and so a BCI-algebra, $X = \{0, a, b, c, d\}$ with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	0	0
c	c	c	c	0	0
d	d	c	d	a	0

Define a bipolar fuzzy set $\varphi = (X; \varphi^-, \varphi^+)$ in X by

X	0	a	b	c	d
φ^-	−0.8	−0.7	−0.8	−0.7	−0.7
φ^+	0.7	0.2	0.7	0.2	0.2

Then we can check that $\varphi = (X; \varphi^-, \varphi^+)$ is both a bipolar fuzzy subalgebra of X and a bipolar fuzzy ideal of X. But it is not a bipolar fuzzy a-ideal of X since

$$\varphi^-(a * b) = -0.7 \not\leq -0.8 = \max\{\varphi^-((b * 0) * (0 * a)), \varphi^-(0)\}.$$

Lemma 3.10 ([4])**.** *Let $\varphi = (X; \varphi^-, \varphi^+)$ be a bipolar fuzzy ideal of X. If the inequality $x * y \leq z$ holds in X, then*

$$\begin{aligned} &\varphi^-(x) \leq \max\{\varphi^-(y), \varphi^-(z)\}, \\ &\varphi^+(x) \geq \min\{\varphi^+(y), \varphi^+(z)\}. \end{aligned} \tag{3.7}$$

Now we intend to present conditions under which a bipolar fuzzy ideal is a bipolar fuzzy a-ideal.

Theorem 3.11. *Let $\varphi = (X; \varphi^-, \varphi^+)$ be a bipolar fuzzy ideal of X. Then the following are equivalent*:

(1) $\varphi = (X; \varphi^-, \varphi^+)$ *is a bipolar fuzzy a-ideal of X.*
(2) $\varphi = (X; \varphi^-, \varphi^+)$ *satisfies the following assertions*:

$$\begin{aligned} &\varphi^-(y * (x * z)) \le \varphi^-((x * z) * (0 * y)), \\ &\varphi^+(y * (x * z)) \ge \varphi^+((x * z) * (0 * y)) \end{aligned} \tag{3.8}$$

for all $x, y, z \in X$.
(3) $\varphi = (X; \varphi^-, \varphi^+)$ *satisfies the following assertions*:

$$\begin{aligned} &\varphi^-(y * x) \le \varphi^-(x * (0 * y)), \\ &\varphi^+(y * x) \ge \varphi^+(x * (0 * y)) \end{aligned} \tag{3.9}$$

for all $x, y \in X$.

Proof. Assume that $\varphi = (X; \varphi^-, \varphi^+)$ is a bipolar fuzzy a-ideal of X and let $x, y, z \in X$. Using (3.5), (a1) and (3.3), we get

$$\begin{aligned} \varphi^-(y * (x * z)) &\le \max\{\varphi^-(((x * z) * 0) * (0 * y)), \varphi^-(0)\} \\ &= \varphi^-((x * z) * (0 * y)) \end{aligned}$$

and

$$\begin{aligned} \varphi^+(y * (x * z)) &\ge \min\{\varphi^+(((x * z) * 0) * (0 * y)), \varphi^+(0)\} \\ &= \varphi^+((x * z) * (0 * y)), \end{aligned}$$

which proves (2). Taking $z = 0$ in (2) and using (a1) induces (3). Suppose that (3) is valid. Note that

$$(x * (0 * y)) * ((x * z) * (0 * y)) \le x * (x * z) \le z$$

for all $x, y, z \in X$. It follows from (3.9) and Lemma 3.10 that

$$\varphi^-(y * x) \le \varphi^-(x * (0 * y)) \le \max\{\varphi^-((x * z) * (0 * y)), \varphi^-(z)\}$$

and

$$\varphi^+(y * x) \ge \varphi^+(x * (0 * y)) \ge \min\{\varphi^+((x * z) * (0 * y)), \varphi^+(z)\}.$$

Hence $\varphi = (X; \varphi^-, \varphi^+)$ is a bipolar fuzzy a-ideal of X. □

We give another conditions for a bipolar fuzzy ideal to be a bipolar fuzzy a-ideal.

Theorem 3.12. *Assume that X is associative, i.e., X satisfies the following identity*:

$$(\forall x, y, z \in X)\,((x * y) * z = x * (y * z).$$

Then every bipolar fuzzy ideal of X is a bipolar fuzzy a-ideal of X.

Proof. Let $\varphi = (X; \varphi^-, \varphi^+)$ be a bipolar fuzzy ideal of an associative BCI-algebra X. Since $0 * x = x$ for all $x \in X$, it follows that

$$y * x = (0 * y) * x = (0 * x) * y = x * y = x * (0 * y)$$

for all $x, y \in X$. Therefore

$$\varphi^-(y * x) = \varphi^-(x * (0 * y)) \text{ and } \varphi^+(y * x) = \varphi^+(x * (0 * y)).$$

Using Theorem 3.11, we conclude that $\varphi = (X; \varphi^-, \varphi^+)$ is a bipolar fuzzy a-ideal of X. □

Theorem 3.13. *Let $\varphi = (X; \varphi^-, \varphi^+)$ be a bipolar fuzzy a-ideal of X. Then the set*

$$\Omega = \{x \in X \mid \varphi^-(x) = \varphi^-(0),\ \varphi^+(x) = \varphi^+(0)\}$$

is an a-ideal of X.

Proof. Obviously, $0 \in \Omega$. Let $x, y, z \in X$ be such that $(x * z) * (0 * y) \in \Omega$ and $z \in \Omega$. Then

$$\varphi^-(0) \leq \varphi^-(y * x) \leq \max\{\varphi^-((x * z) * (0 * y)), \varphi^-(z)\} = \varphi^-(0)$$

and

$$\varphi^+(0) \geq \varphi^+(y * x) \geq \min\{\varphi^+((x * z) * (0 * y)), \varphi^+(z)\} = \varphi^+(0)$$

by using (3.3) and (3.5). It follows that $\varphi^-(y * x) = \varphi^-(0)$ and $\varphi^+(y * x) = \varphi^+(0)$, that is, $y * x \in \Omega$. Therefore Ω is an a-ideal of X. □

The following example shows that there exists a bipolar fuzzy set $\varphi = (X; \varphi^-, \varphi^+)$ in X such that

(1) $\varphi = (X; \varphi^-, \varphi^+)$ is not a bipolar fuzzy a-ideal of X,
(2) $\Omega = \{x \in X \mid \varphi^-(x) = \varphi^-(0),\ \varphi^+(x) = \varphi^+(0)\}$ is an a-ideal of X.

Example 3.14. Consider a BCI-algebra $X = \{0, 1, a, b, c\}$ with the following Cayley table:

$*$	0	1	a	b	c
0	0	0	a	b	c
1	1	0	a	b	c
a	a	a	0	c	b
b	b	b	c	0	a
c	c	c	b	a	0

Define a bipolar fuzzy set $\varphi = (X; \varphi^-, \varphi^+)$ in X by

X	0	1	a	b	c
φ^-	−0.7	−0.7	−0.5	−0.5	−0.2
φ^+	0.8	0.8	0.5	0.3	0.3

Then $\varphi = (X; \varphi^-, \varphi^+)$ is not a bipolar fuzzy a-ideal of X since

$$\varphi^-(b * a) = -0.2 \not\leq -0.5 = \max\{\varphi^-((a * b) * (0 * b)), \varphi^-(b)\}.$$

But we can check that $\Omega = \{0, 1\}$ is an a-ideal of X.

Theorem 3.15. *Let $\varphi = (X; \varphi^-, \varphi^+)$ be a bipolar fuzzy set in X. Then $\varphi = (X; \varphi^-, \varphi^+)$ is a bipolar fuzzy a-ideal of X if and only if for all $(s,t) \in [-1,0] \times [0,1]$, the nonempty negative s-cut $N(\varphi; s)$ and the nonempty positive t-cut $P(\varphi; t)$ are a-ideals of X.*

Proof. Assume that $\varphi = (X; \varphi^-, \varphi^+)$ is a bipolar fuzzy a-ideal of X and let $(s,t) \in [-1,0] \times [0,1]$ be such that $N(\varphi; s) \neq \emptyset$ and $P(\varphi; t) \neq \emptyset$. Obviously, $0 \in N(\varphi; s) \cap P(\varphi; t)$. Let $x, y, z \in X$ be such that $(x * z) * (0 * y) \in N(\varphi; s)$ and $z \in N(\varphi; s)$. Then $\varphi^-((x * z) * (0 * y)) \leq s$ and $\varphi^-(z) \leq s$. It follows from (3.5) that

$$\varphi^-(y * x) \leq \max\{\varphi^-((x * z) * (0 * y)), \varphi^-(z)\} \leq s$$

so that $y * x \in N(\varphi; s)$. Hence $N(\varphi; s)$ is an a-ideal of X. Now assume that $z \in P(\varphi; t)$ and $(x * z) * (0 * y) \in P(\varphi; t)$ for all $x, y, z \in X$. Using (3.5), we have

$$\varphi^+(y * x) \geq \min\{\varphi^+((x * z) * (0 * y)), \varphi^+(z)\} \geq t,$$

and so $y * x \in P(\varphi; t)$. Therefore $P(\varphi; t)$ is an a-ideal of X.

Conversely, suppose that the nonempty negative s-cut $N(\varphi; s)$ and the nonempty positive t-cut $P(\varphi; t)$ are a-ideals of X for every $(s,t) \in [-1,0] \times [0,1]$. If $\varphi^-(0) > \varphi^-(a)$ or $\varphi^+(0) < \varphi^+(b)$ for some $a, b \in X$, then $0 \notin N(\varphi; \varphi^-(a))$ or $0 \notin P(\varphi; \varphi^+(b))$. This is a contradiction. Thus $\varphi^-(0) \leq \varphi^-(x)$ and $\varphi^+(0) \geq \varphi^+(x)$ for all $x \in X$. Assume that

$$\varphi^-(b * a) > \max\{\varphi^-((a * c) * (0 * b)), \varphi^-(c)\} = s$$

for some $a, b, c \in X$. Then $(a * c) * (0 * b) \in N(\varphi; s)$ and $c \in N(\varphi; s)$, but $b * a \notin N(\varphi; s)$. This is impossible, and thus

$$\varphi^-(y * x) \leq \max\{\varphi^-((x * z) * (0 * y)), \varphi^-(z)\}$$

for all $x, y, z \in X$. If $\varphi^+(b * a) < \min\{\varphi^+((a * c) * (0 * b)), \varphi^+(c)\} = t$ for some $a, b, c \in X$, then $(a * c) * (0 * b) \in P(\varphi; t)$ and $c \in P(\varphi; t)$, but $b * a \notin P(\varphi; t)$. This is a contradiction since $P(\varphi; t)$ is an a-ideal of X. Therefore

$$\varphi^+(y * x) \geq \min\{\varphi^+((x * z) * (0 * y)), \varphi^+(z)\}$$

for all $x, y, z \in X$. Consequently $\varphi = (X; \varphi^-, \varphi^+)$ is a bipolar fuzzy a-ideal of X. □

Corollary 3.16. *If $\varphi = (X; \varphi^-, \varphi^+)$ is a bipolar fuzzy a-ideal of X, then the nonempty (s,t)-cut $C(\varphi; (s,t))$ of $\varphi = (X; \varphi^-, \varphi^+)$ is an a-ideal of X.*

Proof. Straightforward. □

Theorem 3.17. *Let $\varphi = (X; \varphi^-, \varphi^+)$ be a bipolar fuzzy set in X and*

$$\mathrm{Im}(\varphi) = \{(s_i, t_i) \in [-1,0] \times [0,1] \mid i = 0, 1, 2, \ldots, n\}$$

where $(s_i, t_i) < (s_j, t_j)$, that is, $s_i > s_j$ and $t_i < t_j$, whenever $i > j$. Let $\{I_k \mid k = 0, 1, 2, \ldots, n\}$ be a finite class of a-ideals of X such that

(i) $I_0 \subset I_1 \subset \cdots \subset I_n = X$,

(ii) $\varphi^-(I_k^*) = s_k$ *and* $\varphi^+(I_k^*) = t_k$ *where* $I_k^* = I_k \setminus I_{k-1}$ *and* $I_{-1} = \emptyset$ *for* $k = 0, 1, 2, \ldots, n$.

Then $\varphi = (X; \varphi^-, \varphi^+)$ *is a bipolar fuzzy a-ideal of* X.

Proof. Since $0 \in I_0$, we have $\varphi^-(0) = s_0 \leq \varphi^-(x)$ and $\varphi^+(0) = t_0 \geq \varphi^+(x)$ for all $x \in X$. Let $x, y, z \in X$. Then $(x * z) * (0 * y) \in I_k^*$ and $z \in I_r^*$ for some $k, r \in \{0, 1, 2, \ldots, n\}$. If $k \geq r$, then $(x*z)*(0*y) \in I_k$ and $z \in I_k$ since $I_k^* \subset I_k$ and $I_r^* \subset I_r \subseteq I_k$, and so $y * x \in I_k$ because I_k is an a-ideal of X. Thus

$$\varphi^-(y * x) \leq s_k = \max\{\varphi^-((x * z) * (0 * y)), \varphi^-(z)\}$$

and

$$\varphi^+(y * x) \geq t_k = \min\{\varphi^+((x * z) * (0 * y)), \varphi^+(z)\}.$$

If $k < r$, then $(x * z) * (0 * y) \in I_r$ and $z \in I_r$ since $I_r^* \subset I_r$ and $I_k^* \subset I_k \subset I_r$, and so $y * x \in I_r$ because I_r is an a-ideal of X. Thus

$$\varphi^-(y * x) \leq s_r = \max\{\varphi^-((x * z) * (0 * y)), \varphi^-(z)\}$$

and

$$\varphi^+(y * x) \geq t_r = \min\{\varphi^+((x * z) * (0 * y)), \varphi^+(z)\}.$$

Therefore $\varphi = (X; \varphi^-, \varphi^+)$ is a bipolar fuzzy a-ideal of X. □

Theorem 3.18. *If any bipolar fuzzy a-ideal $\varphi = (X; \varphi^-, \varphi^+)$ of X has a finite image, then every descending chain of a-ideals of X terminates at finite step.*

Proof. Suppose that there exists a strictly descending chain $I_0 \supset I_1 \supset I_2 \supset \cdots$ of a-ideals of X which does not terminate at finite step, where I_0 stands for X. Define a bipolar fuzzy set $\varphi = (X; \varphi^-, \varphi^+)$ in X by

$$\varphi^-(x) = \begin{cases} -\frac{n}{n+1} & \text{if } x \in I_n \setminus I_{n+1},\ n \in \mathbb{N} \cup \{0\}, \\ -1 & \text{if } x \in \bigcap_{n=0}^{\infty} I_n \end{cases}$$

and

$$\varphi^+(x) = \begin{cases} \frac{n}{n+1} & \text{if } x \in I_n \setminus I_{n+1},\ n \in \mathbb{N} \cup \{0\}, \\ 1 & \text{if } x \in \bigcap_{n=0}^{\infty} I_n. \end{cases}$$

We claim that $\varphi = (X; \varphi^-, \varphi^+)$ is a bipolar fuzzy a-ideal of X. Since $0 \in \bigcap_{n=0}^{\infty} I_n$, we have $\varphi^-(0) = -1 \leq \varphi^-(x)$ and $\varphi^+(0) = 1 \geq \varphi^+(x)$ for all $x \in X$. Let $x, y, z \in X$. Assume that $(x * z) * (0 * y) \in I_n \setminus I_{n+1}$ and $z \in I_k \setminus I_{k+1}$ for some $n, k \in \mathbb{N} \cup \{0\}$. Without loss of generality we may assume that $n \leq k$. Then $z \in I_n$ and so $y * x \in I_n$ because I_n is an a-ideal of X. Hence

$$\varphi^-(y * x) \leq -\tfrac{n}{n+1} = \max\{\varphi^-((x * z) * (0 * y)), \varphi^-(z)\}$$

and

$$\varphi^+(y * x) \geq \tfrac{n}{n+1} = \min\{\varphi^+((x * z) * (0 * y)), \varphi^+(z)\}.$$

If $(x * z) * (0 * y) \in \bigcap_{n=0}^{\infty} I_n$ and $z \in \bigcap_{n=0}^{\infty} I_n$, then $y * x \in \bigcap_{n=0}^{\infty} I_n$. Thus

$$\varphi^-(y * x) = -1 = \max\{\varphi^-((x * z) * (0 * y)), \varphi^-(z)\}$$

and

$$\varphi^+(y * x) = 1 = \min\{\varphi^+((x * z) * (0 * y)), \varphi^+(z)\}.$$

If $(x * z) * (0 * y) \notin \bigcap_{n=0}^{\infty} I_n$ and $z \in \bigcap_{n=0}^{\infty} I_n$, then there exists $k \in \mathbb{N} \cup \{0\}$ such that $(x * z) * (0 * y) \in I_k \setminus I_{k+1}$. It follows that $y * x \in I_k$ so that

$$\varphi^-(y * x) \le -\tfrac{k}{k+1} = \max\{\varphi^-((x * z) * (0 * y)), \varphi^-(z)\}$$

and

$$\varphi^+(y * x) \ge \tfrac{k}{k+1} = \min\{\varphi^+((x * z) * (0 * y)), \varphi^+(z)\}.$$

Finally suppose that $(x * z) * (0 * y) \in \bigcap_{n=0}^{\infty} I_n$ and $z \notin \bigcap_{n=0}^{\infty} I_n$. Then $z \in I_r \setminus I_{r+1}$ for some $r \in \mathbb{N} \cup \{0\}$. Hence $y * x \in I_r$ and so

$$\varphi^-(y * x) \le -\tfrac{r}{r+1} = \max\{\varphi^-((x * z) * (0 * y)), \varphi^-(z)\}$$

and

$$\varphi^+(y * x) \ge \tfrac{r}{r+1} = \min\{\varphi^+((x * z) * (0 * y)), \varphi^+(z)\}.$$

Consequently, we see that $\varphi = (X; \varphi^-, \varphi^+)$ is a bipolar fuzzy a-ideal of X and $\varphi = (X; \varphi^-, \varphi^+)$ has infinite number of different values. This is a contradiction, and the proof is complete. □

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