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HEIGHT BOUND AND PREPERIODIC POINTS FOR JOINTLY REGULAR FAMILIES OF RATIONAL MAPS

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Abstract. Silverman [14] proved a height inequality for a jointly regular family of rational maps and the author [10] improved it for a jointly regular pair. In this paper, we provide the same improvement for a jointly regular family: let $h : \mathbb{P}^n_{\overline{\mathbb{Q}}} \to \mathbb{R}$ be the logarithmic absolute height on the projective space, let $r(f)$ be the D-ratio of a rational map f which is defined in [10] and let $\{f_1, \ldots, f_k \mid f_l : \mathbb{A}^n \to \mathbb{A}^n\}$ be a finite set of polynomial maps which is defined over a number field K. If the intersection of the indeterminacy loci of $f_1, \ldots, f_k$ is empty, then there is a constant C such that

$$\sum_{l=1}^{k} \frac{1}{\deg f_l} h\big(f_l(P)\big) > \left(1 + \frac{1}{r}\right) f(P) - C \quad \text{for all } P \in \mathbb{A}^n$$

where $r = \max_{l=1,\ldots,k}\,(r(f_l))$.

1. Introduction

Let K be a number field and let $h : \mathbb{P}^n_{\overline{K}} \to \mathbb{R}$ be the logarithmic absolute height function on the projective space. If $f : \mathbb{P}^n_K \to \mathbb{P}^n_K$ is a morphism defined over a number field K, then we can make a good estimate of $h(P)$ with $h\big(f(P)\big)$. Define the degree of f to be the number induced by the linear operator f^* on $\mathrm{Pic}(\mathbb{P}^n) = \mathbb{Z}$:

$$f^*H = \deg f \cdot H \ \text{ on } \ \mathrm{Pic}(\mathbb{P}^n).$$

Then, the functorial property of the Weil height machine will prove the Northcott's theorem. The author refers [16, Theorem B.3.2] to the reader for the details of the Weil height machine.

Theorem 1.1 (Northcott [12]). *If $f : \mathbb{P}^n_K \to \mathbb{P}^n_K$ is a morphism defined over a number field K, then there are two constants C_1 and C_2, which are independent of point P, such that*

$$\frac{1}{\deg f} h\big(f(P)\big) + C_1 > h(P) > \frac{1}{\deg f} h\big(f(P)\big) - C_2$$

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for all $P \in \mathbb{P}^n_K$.

If f is not a morphism but a rational map, then the functoriality of the Weil height machine breaks down: the two height functions $h_{f^*H}(P)$ and $h_H\big(f(P)\big)$ are not equivalent. Hence, Northcott's Theorem is not valid for rational maps (However, we still have $h(P) > \frac{1}{\deg f} h\big(f(P)\big) - C_2$ by the triangular inequality. See [16, Proposition B.7.1]).

For example, consider a polynomial map, one of popular objects in complex dynamics. Define

$$f := (f_1, \ldots, f_n) : \mathbb{A}^n_K \to \mathbb{A}^n_K,$$

where $f_1, \ldots, f_n$ are homogeneous polynomials of degree d. We may consider f as a rational map on $\mathbb{P}^n_K$: let $P \in \mathbb{P}^n_K$. We define $f(P)$ to be the following limit value if it exists.

$$f(P) := \lim_{\substack{Q \to P \\ Q \in \mathbb{A}^n_K}} f(Q).$$

We call it the meromorphic extension of f. In general, the meromorphic extension of f is not a morphism. So, we need other way to find an upper bound of $h(P)$.

Silverman suggested a way of constructing an upper bound of $h(P)$ when we have a special family of polynomial maps.

Definition 1.2. Let $S = \{f_1, \ldots, f_k \mid f_l : \mathbb{P}^n_K \dashrightarrow \mathbb{P}^n_K\}$ be a finite set of rational maps defined over a number field K and let $I(f)$ be the indeterminacy locus of f. We say that S is *jointly regular* when

$$\bigcap_{l=1}^{k} I(f_l) = \emptyset.$$

We also say that a finite set of polynomial maps $S' = \{g_1, \ldots, g_k \mid g_l : \mathbb{A}^n_K \to \mathbb{A}^n_K\}$ is *jointly regular* if the set of rational maps

$$S = \{f_l : \mathbb{P}^n \dashrightarrow \mathbb{P}^n \mid f_l \text{ is the meromorphic extension of } g_l \in S'\}$$

is jointly regular.

Theorem 1.3 ([14], Theorem 3)**.** *Let* $\{f_1, \ldots, f_k \mid f_l : \mathbb{A}^n_K \to \mathbb{A}^n_K\}$ *be a jointly regular family of polynomial maps defined over a number field* K*. Then, there is a constant* C *satisfying*

$$\sum_{l=1}^{k} \frac{1}{\deg f_l} h\big(f_l(P)\big) > h(P) - C$$

for all $P \in \mathbb{A}^n_K$.

In this paper, we will improve Theorem 1.3 using the D-ratio. The D-ratio requires new concepts to be defined so that we will state the main theorem without the definition of the D-ratio first and will introduce the D-ratio in Definition 2.12 later.

Theorem 1.4. *Let H be a hyperplane of $\mathbb{P}^n$, let $\mathbb{A}^n = \mathbb{P}^n \setminus H$, let $S = \{f_1, \dots, f_k \mid f_l : \mathbb{A}^n_K \to \mathbb{A}^n_K\}$ be a jointly regular family of polynomial maps defined over a number field K, let $r(f)$ be the D-ratio of f and let $r = \max_{l=1,\dots,k}(r(f_l))$. Suppose that S has at least two elements. Then, there is a constant C satisfying*

$$\sum_{l=1}^{k} \frac{1}{\deg f_l} h\big(f_l(P)\big) > \left(1 + \frac{1}{r}\right) h(P) - C$$

for all $P \in \mathbb{A}^n_K$.

This theorem improves Silverman's result for preperiodic points [14, Theorem 4], which is exactly same with Theorem 1.5 except the description of δ_S.

Theorem 1.5. *Let $S = \{f_1, \dots, f_k \mid f_l : \mathbb{A}^n_K \to \mathbb{A}^n_K\}$ be a jointly regular family of polynomial maps, let $f(f_l)$ be the D-ratio of f_l and let Φ be the monoid of polynomial maps generated by S. Define*

$$\delta_S := \left(\frac{1}{1 + 1/r}\right) \sum_{l=1}^{k} \frac{1}{\deg f_l},$$

where $r = \max_{l=1,\dots,k}\big(r(f_l)\big)$.

If $\delta_S < 1$, then

$$\mathrm{Preper}(\Phi) := \bigcap_{f \in \Phi} \mathrm{Preper}(f) \subset \mathbb{A}^n_K$$

is a set of bounded height.

From now on, we will let K be a number field, let H be a hyperplane on $\mathbb{P}^n$ and let $\mathbb{A}^n = \mathbb{P}^n \setminus H$ be an affine space. We also let $f : \mathbb{A}^n \to \mathbb{A}^n$ be a polynomial map and let $I(f)$ be the indeterminacy locus of f unless stated otherwise.

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2. Preliminaries

We need two main ingredients, the resolution of indeterminacy and the D-ratio of polynomial maps. For details, the author refers [1] and [3, II.7] for blowups and the resolution of indeterminacy, and [10] for the D-ratio.

2.1. Blowup and the resolution of indeterminacy

We have the general theorem of the resolution of indeterminacy, which is a corollary of the theorem of the resolution of singularity.

Theorem 2.1 (Resolution of indeterminacy)**.** *Let $f : V \dashrightarrow W$ be a rational map between proper varieties such that V is nonsingular. Then there is a proper nonsingular variety $\widetilde{V}$ with a birational morphism $\pi : \widetilde{V} \to V$ such that $\phi = f \circ \pi : \widetilde{V} \to W$ is a morphism*:

$$\begin{array}{ccc} \widetilde{V} & & \\ {\scriptstyle \pi}\downarrow & \searrow^{\phi} & \\ V & \underset{f}{\dashrightarrow} & W \end{array}$$

For notational convenience, we will define the following.

Definition 2.2. Let $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ be a rational map and let V be a blowup of $\mathbb{P}^n$ with a birational morphism $\pi : V \to \mathbb{P}^n$. We say that a pair (V, π) is *a resolution of indeterminacy of f* if

$$f \circ \pi : V \to \mathbb{P}^n$$

is extended to a morphism. And we call the extended morphism $\phi := f \circ \pi$ a *resolved morphism of f.*

Using Hironaka's Theorem (Theorem 2.5), we will observe the relation between the resolution of indeterminacy and the indeterminacy locus of f.

Definition 2.3. Let $\pi : V \to \mathbb{P}^n$ be a birational morphism. Then, we say that a closed subscheme $\mathfrak{I}$ of $\mathbb{P}^n$ is *the center scheme of* π if the ideal sheaf $\mathcal{S}$ corresponding to $\mathfrak{I}$ generates V:

$$V = \operatorname{Proj}\left(\bigoplus_{d \geq 0} \mathcal{S}^d\right).$$

Definition 2.4. Let $\pi : W \to V$ be a birational morphism. We say that π is a *monoidal transformation* if its center scheme is a smooth irreducible subvariety of V. We say that W *is a successive blowup of* V if the corresponding birational map $\pi : W \to V$ is a composition of monoidal transformations.

Theorem 2.5 (Hironaka)**.** *Let $f : X \dashrightarrow Y$ be a rational map between proper varieties such that V is nonsingular. Then, there is a finite sequence of proper varieties $X_0, \dots, X_r$ such that*

(1) *$X_0 = X$,*
(2) *$\rho_i : X_i \to X_{i-1}$ is a monoidal transformation,*
(3) *If T_i is the center scheme of π_i, then $\rho_0 \circ \cdots \circ \rho_i(T_i) \subset I(f)$ on X,*
(4) *f is extended to a morphism $\widetilde{f} : X_r \to Y$ on X_r,*
(5) *Consider the composition of all monoidal transformation $\rho : X_m \to X$. Then, the underlying subvariety of the center scheme T of ρ, a subvariety made by the zero set of the ideal sheaf corresponding to T, is exactly $I(f)$.*

Proof. See [4, Question (E) and Main Theorem II]. □

In § 2.2, we will find a basis of $\mathrm{Pic}(V)$ when (V, π) is a resolution of indeterminacy. Especially, we need a basis consisting of irreducible divisors. However, pullbacks of divisors may not be irreducible because of the exceptional part. So, we define the proper transformation, which is usually irreducible.

Definition 2.6. Let $\pi : \widetilde{V} \to V$ be a birational morphism with center scheme $\mathfrak{I}$ and let D be an irreducible divisor on V. We define *the proper transformation of D by π* to be

$$\pi^{\#} D = \overline{\pi^{-1}(D \cap U)},$$

where $U = V \setminus Z(\mathfrak{I})$ and $Z(\mathfrak{I})$ is the underlying subvariety made by the zero set of the ideal corresponding to $\mathfrak{I}$.

2.2. The $\mathbb{A}^n$-effectiveness and the D-ratio

The main question of this paper is to find an upper bound of $h(P)$ using $h\big(f_l(P)\big)$ for jointly regular family $\{f_1, \ldots, f_k \mid f_l : \mathbb{A}^n \to \mathbb{A}^n\}$. So, we will consider $\mathbb{A}^n$ as a dense open subset of $\mathbb{P}^n$ and fix the hyperplane $H = \mathbb{P}^n \setminus \mathbb{A}^n$ to find a basis of the Picard group of a blowup of $\mathbb{P}^n$ and use a special kind of divisors on a blowup of $\mathbb{P}^n$ to measure the height values of $P \in \mathbb{A}^n$. First of all, we need to clarify how to get such basis of $\mathrm{Pic}(V)$.

Proposition 2.7. *Let V be a successive blowup of $\mathbb{P}^n$ with a birational morphism $\pi : V \to \mathbb{P}^n$: there are monoidal transformations $\pi_i : V_i \to V_{i-1}$ such that $V_r = V$ and $V_0 = \mathbb{P}^n$. Let H be a hyperplane on $\mathbb{P}^n$, let F_i be the exceptional divisor of the blowup $\pi_i : V_i \to V_{i-1}$, let $\rho_i = \pi_{i+1} \circ \cdots \circ \pi_r$ and let $E_i = \rho_i^{\#} F_i$. Then, $\mathrm{Pic}(V)$ is a free $\mathbb{Z}$-module with a basis*

$$\{H_V = \pi^{\#} H, E_1, \ldots, E_r\}.$$

Proof. [3, Exer.II.7.9] shows that

$$\mathrm{Pic}(\widetilde{X}) \simeq \mathrm{Pic}(X) \oplus \mathbb{Z}$$

if $\pi : \widetilde{X} \to X$ is a monoidal transformation. More precisely,

$$\mathrm{Pic}(\widetilde{X}) = \{\pi^{\#} D + nE \mid D \in \mathrm{Pic}(X)\},$$

where E is the exceptional divisor of π on $\widetilde{X}$. Suppose that $X = V_{i-1}$ and $\widetilde{X} = V_i$ and get the desired result. □

Now, we define the special kind of divisors, the $\mathbb{A}^n$-effective divisors.

Definition 2.8. Let V be a successive blowup of $\mathbb{P}^n$ with a birational morphism $\pi : V \to \mathbb{P}^n$, let H be a fixed hyperplane of $\mathbb{P}^n$ and let

$$\mathrm{Pic}_{\mathbb{Q}}(V) = \mathbb{Q}H_V \oplus \mathbb{Q}E_1 \oplus \cdots \oplus \mathbb{Q}E_r$$

with the basis described in Proposition 2.7. We define the $\mathbb{A}^n$*-effective cone* to be

$$\mathrm{AFE}(V) := \mathbb{Q}^{\geq 0} H_V \oplus \mathbb{Q}^{\geq 0} E_1 \oplus \cdots \oplus \mathbb{Q}^{\geq 0} E_r,$$

where $\mathbb{Q}^{\geq 0}$ is the set of nonnegative rational numbers. We say a divisor D of V is $\mathbb{A}^n$*-effective* if the linear equivalence class of D is contained in $\mathrm{AFE}(V)$ and denote it by

$$D \succ 0.$$

Moreover, on $\mathrm{Pic}_{\mathbb{Q}}(V)$, we write

$$D_1 \succ D_2$$

if $D_1 - D_2$ is $\mathbb{A}^n$-effective.

The next proposition will explain why we define the "$\mathbb{A}^n$-effectiveness". Namely, the height functions corresponding to $\mathbb{A}^n$-effective divisors will have nice properties on $\mathbb{A}^n$.

Proposition 2.9. *Let V be a successive blowup of $\mathbb{P}^n$ with a birational morphism $\pi : V \to \mathbb{P}^n$ and let D, D_i be divisors on V.*

(1) (Effectiveness) *If D is $\mathbb{A}^n$-effective, then D is effective.*

(2) (Boundedness) *If D is $\mathbb{A}^n$-effective, then $h_D(P)$ is bounded below on $V \setminus (H_V \cup (\bigcup_{i=1}^r E_i))$.*

(3) (Transitivity) *If $D_1 \succ D_2$ and $D_2 \succ D_3$, then $D_1 \succ D_3$.*

(4) (Funtoriality) *If $\rho : W \to V$ is a monoidal transformation and $D_1 \succ D_2$, then $\rho^* D_1 \succ \rho^* D_2$.*

Proof. See [10, Proposition 3.3]. □

In Section 1, we introduce the main theorem without the definition of the D-ratio because it requires the $\mathbb{A}^n$-effectiveness. Now, we are ready to define the D-ratio, one of main ingredients of this paper.

Definition 2.10. Let $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ be a rational map such that $I(f) \subset H$, let (V, π_V) be a resolution of indeterminacy of f and let ϕ_V be a resolved morphism so that the following diagram commutes:

$$\begin{array}{ccc} V & & \\ {\scriptstyle \pi_V}\downarrow & \searrow^{\phi_V} & \\ \mathbb{P}^n & \overset{f}{\dashrightarrow} & \mathbb{P}^n \end{array}$$

Suppose that

$$\pi_V^* H = a_0 H_V + \sum_{i=1}^r a_i E_i \quad \text{and} \quad \phi_V^* H = b_0 H_V + \sum_{i=1}^r b_i E_i,$$

where a_i, b_i are nonnegative integers. If $b_i \neq 0$ for all i satisfying $a_i \neq 0$, we define *the D-ratio* of ϕ_V to be

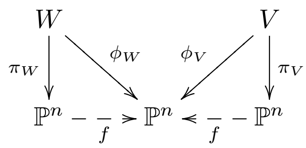
$$r(\phi_V) = \deg \phi_V \cdot \max_{a_i \neq 0} \left(\frac{a_i}{b_i} \right).$$

If there is an index i satisfying $a_i \neq 0$ and $b_i = 0$, define

$$r(\phi_V) = \infty.$$

The readers might concern if the D-ratio is only defined for resolved morphisms. The following lemma will allow us to define the D-ratio for the rational maps.

Lemma 2.11. *Let (V, π_V) and (W, π_V) be resolutions of indeterminacy of f with resolved morphisms $\phi_V = f \circ \pi_V$ and $\phi_W = f \circ \pi_W$ respectively:*



Then, we have

$$r(\phi_V) = r(\phi_W).$$

Proof. See [10, Lemma 4.3]. □

Definition 2.12. Let $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ be a rational map with $I(f) \subset H$. Then, we define *the D-ratio of f* to be

$$r(f) = r(\phi_V)$$

for any resolution of indeterminacy (V, π_V) of f with resolved morphism ϕ_V.

Proposition 2.13. *Let $f, g : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ be rational maps such that $I(f), I(g) \subset H$. Then,*

(1) $r(f) = 1$ *if and only if f is a morphism.*
(2) $r(f) \in [1, \infty]$.
(3) $\frac{r(f)}{\deg f} \cdot \frac{r(g)}{\deg g} \geq \frac{r(g \circ f)}{\deg(g \circ f)}$.

Proof. See [10, Proposition 4.5, Theorem 5.2]. □

Example 2.14. Let $f : \mathbb{A}^n \to \mathbb{A}^n$ be a polynomial automorphism with the inverse map $f^{-1} : \mathbb{A}^n \to \mathbb{A}^n$. Then, $r(f) = \deg f \times \deg f^{-1}$ (For details, see [9]). For example, a Hénon map

$$f_H(x, y, z) = (z, x + z^2, y + x^2)$$

is a regular polynomial automorphism with the inverse map

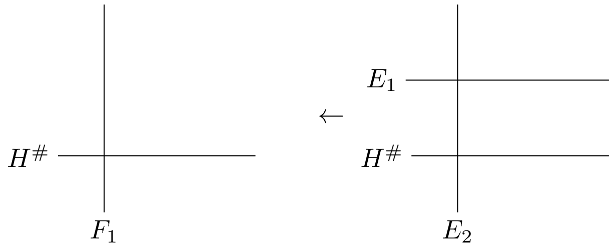
$$f_H^{-1}(x, y, z) = (y - x^2, z - (y - x^2)^2, x).$$

Thus,

$$r(f_H) = r(f_H^{-1}) = \deg f_H \times \deg f_H^{-1} = 2 \times 4 = 8.$$

Example 2.15. Let $f[x,y,z] = [x^2, yz, z^2]$. Then, the indeterminacy locus of f consists of one point $P = [0,1,0]$. Then, the blowup V along closed scheme corresponding ideal sheaf (z, x^2) will resolve indeterminacy, which is a successive blowup along P and $H^{\#} \cap F_1$, where F_1 is the exceptional divisor of the first blowup.

Let E_1 be the proper transformation of F_1 and let E_2 be the exceptional divisor of the second blowup:



Then, the following intersection numbers are easily calculated:

$E_2^2 = -1,\ E_1^2 = -2,\ (H^{\#})^2 = -1,\ H^{\#} \cdot E_1 = 0$ and $H^{\#} \cdot E_2 = E_1 \cdot E_2 = 1$.

Furthermore, by the projection formula and the exact calculation of ϕ_*, we get

$$H^{\#} \cdot \phi^* H = \phi_* H^{\#} \cdot H = 0,$$
$$E_1 \cdot \phi^* H = \phi_* E_1 \cdot H = 0,$$
$$E_2 \cdot \phi^* H = \phi_* E_2 \cdot H = 1.$$

Since $\mathrm{Pic}(V) = \langle H^{\#}, E_1, E_2 \rangle$, we may assume that

$$\phi^* H = a H^{\#} + b E_1 + c E_2$$

for some integers a, b and c. Then, by previous facts,

$$\phi^* H \cdot H^{\#} = -a + c = 0, \quad \phi^* H \cdot E_1 = a - 2b = 0.$$

Therefore,

$$\phi^* H = 2H^{\#} + E_1 + 2E_2, \quad \pi^* H = H^{\#} + E_1 + 2E_2$$

and hence

$$r(f) = 2 \times 1 = 2.$$

3. Jointly regular families of rational maps

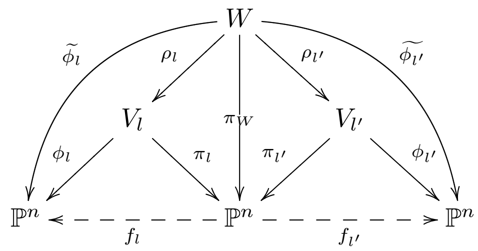
Proof of Theorem 1.4. For notational convenience, let

- $d_l = \deg f_l$.
- $r_l = r(f_l)$.
- (V_l, π_l) be a resolution of indeterminacy of f_l constructed by Theorem 2.5: assume that π_l is a composition of monoidal transformations and $\{\pi_l^{\#} H = H_{V_l}, E_{l1}, \ldots, E_{ls_l}\}$ is the basis of $\mathrm{Pic}(V_l)$ given by Proposition 2.7.

- ϕ_l be the resolved morphism of f_l on V_l.
- $$\pi_l^* H = a_0 H_{V_l} + \sum_{i=1}^{s_l} a_{li} E_{li} \quad \text{and} \quad \phi_l^* H = b_0 H_{V_l} + \sum_{i=1}^{s_l} b_{li} E_{li}$$
  in $\mathrm{Pic}(V_l) = \mathbb{Z}\pi_l^{\#} H \oplus \mathbb{Z}E_{l1} \oplus \cdots \oplus \mathbb{Z}E_{ls_l}$.

We can easily check that $a_0 = 1$ and $b_0 = d_l$ from $\pi_{l*}\pi_l^* H = H$ and $\pi_{l*}\phi_l^* H = \deg \phi_l \cdot H$. For details, see [10, Proposition 4.5(2)].

Let T_l be the center scheme of blowup for V_l and let W be the blowup of $\mathbb{P}^n$ whose center scheme is $\sum T_l$. Then, W is a blowup of V_l for all l. Furthermore, since the underlying set of T_l is exactly $I(f_l)$, the underlying set of $\sum T_l = \cup I(f_l)$. Let $\rho_l : W \to V_l$, π_W be a composition of monoidal transformations:



Then, still W is a blowup of $\mathbb{P}^n$ and hence $\mathrm{Pic}(W)$ is generated by $\pi_W^{\#} H$ and the irreducible components F_j of the exceptional divisor:

$$\mathrm{Pic}(W) = \mathbb{Z}\pi_W^{\#} H \oplus \mathbb{Z}F_1 \oplus \cdots \oplus \mathbb{Z}F_s.$$

Thus, we can represent $\pi_W^* H$ as follows.

$$\pi_W^* H = \pi_W^{\#} H + \sum_{j=1}^{s} \alpha_j F_j.$$

To describe $\phi_l^* H$ precisely, define

$$\mathcal{I}_l = \{1 \le j \le s \mid \pi_W(F_j) \subset I(f_l)\} \quad \text{and} \quad \mathcal{I}_l^c = \{1 \le j \le s \mid \pi_W(F_j) \not\subset I(f_l)\}.$$

By definition, it is clear that

$$\mathcal{I}_l \cup \mathcal{I}_l^c = \{1, \ldots, s\} \quad \text{and} \quad \mathcal{I}_l \cap \mathcal{I}_l^c = \emptyset.$$

Thus, we can say

$$\widetilde{\phi_l}^* H = d_l \pi_W^{\#} H + \sum_{j=1}^{s} \beta_{lj} F_j = d_l \pi_W^{\#} H + \sum_{j \in \mathcal{I}_l^c} \beta_{lj} F_j + \sum_{j \in \mathcal{I}_l} \beta_{lj} F_j.$$

We have the following lemmas which clarify the relation between coefficients of F_j's.

Lemma 3.1.

$$\bigcup_{l=1}^{k} \mathcal{I}_l = \bigcup_{l=1}^{k} \mathcal{I}_l^c = \{1, \dots, s\}.$$

Proof. $\bigcup_l \mathcal{I}_l = \{1, \dots, s\}$ is clear; because the underlying set of the center scheme of W is $\cup I(f_l)$, $\cup \pi_W(F_j) = \pi_W(\cup F_j) = \cup I(f_l)$.

Suppose $\bigcup_l \mathcal{I}_l^c \subsetneq \{1, \dots, s\}$. Then, there is an index l_0 satisfying $\pi_W(F_{l_0}) \subset I(f_l)$ for all l. This implies $\pi_W(F_{l_0}) \subset I(f_l)$ for all l and hence $\emptyset \neq \pi_W(F_{l_0}) \subset \bigcap_l I(f_l)$ which contradicts to the assumption that S is jointly regular. $\square$

Lemma 3.2. *Let α_j and β_{lj} be the coefficients of F_j in $\pi_V^* H$ and $\widetilde{\phi}_l^* H$ respectively. Then,*

$$d_l \frac{\alpha_j}{\beta_{lj}} \leq r_l.$$

Especially, if $j \in \mathcal{I}_l^c$, then

$$d_l \alpha_j = \beta_{lj}.$$

Proof. By definition of the D-ratio, the first inequality is clear:

$$r_l = d_l \cdot \max_i \left(\frac{\alpha_i}{\beta_{li}} \right) \geq d_l \cdot \frac{\alpha_j}{\beta_{lj}}.$$

Now, suppose that

$$\rho_l^* \pi_l^{\#} H = \kappa_{l0} \pi_W^{\#} H + \sum_{j=1}^{s} \kappa_{lj} F_j = \kappa_{lj} \pi_W^{\#} H + \sum_{j \in \mathcal{I}_l^c} \kappa_{lj} F_j + \sum_{j \in \mathcal{I}_l} \kappa_{lj} F_j,$$

$$\rho_l^* E_{li} = \lambda_{li0} \pi_W^{\#} H + \sum_{j=1}^{s} \lambda_{lij} F_j = \lambda_{li0} \pi_W^{\#} H + \sum_{j \in \mathcal{I}_l^c} \lambda_{lij} F_j + \sum_{j \in \mathcal{I}_l} \lambda_{lij} F_j.$$

We can easily calculate some of $\kappa_{lj}, \lambda_{lij}$:

(a) $\lambda_{li0} = 0$ for all $i = 1, \dots, s_l$.
Because $\pi_W\big(\rho_l^* E_i\big) \subset \cup I(f_l)$, we get $(\pi_W)_*\big(\rho_l^* E_i\big) = 0$. On the other hand, π_W* eliminates all F_j so that

$$\pi_{W*} \left(\lambda_{li0} \pi_W^{\#} H + \sum_{j=1}^{s} \lambda_{lij} F_j \right) = \lambda_{li0} H.$$

Hence, $\lambda_{li0} = 0$.

(b) $\kappa_{l0} = 1$.
We have

$$\pi_{W*}\big(\pi_W^* H\big) = H$$

because π_W is one-to-one outside of the center of blowup of W. Therefore,

$$\pi_{W*}\big(\rho_l^* \pi_l^{\#} H\big) = \pi_{W*} \left(\pi_W^* H - \sum_{j=1}^{s} a_{li} \rho_l^* E_{li} \right) = H.$$

On the other hand, π_* eliminates all F_j so that

$$\pi_{W*}\left(\kappa_{l0}\pi_W^{\#}H+\sum_{j=1}^{s}\kappa_{lj}F_j\right)=\kappa_{l0}H.$$

Hence, $\kappa_{l0}=1$.

(c) $\lambda_{lij}=0$ for all $j\in\mathcal{I}_l^c$.
Because $\pi_l(E_{li})\subset I(f_l)$ and $\pi_W(F_j)\not\subset I(f_l)$ for any $j\in\mathcal{I}_l^c$, the multiplicity of $\rho_l(F_j)$ on E_l is zero and hence $\gamma_{lij}=0$. Thus, we can say

$$\rho_l^*E_{li}=\sum_{j\in\mathcal{I}_l}\lambda_{lij}F_j.$$

Let's complete the proof of Lemma 3.2. Since $\widetilde{\phi}_l=\rho_l\circ\phi_l$ and $\pi_W=\rho_l\circ\pi_l$, we can use (a), (b) and (c) to get the description of π_{W*} and $\widetilde{\phi}_l^*H$:

$$\begin{aligned}
\pi_W^*H &= \rho_l^*\pi_l^*H\\
&= \rho_l^*\left(\pi_l^{\#}H+\sum_{i=1}^{s_l}a_{li}E_{li}\right)\\
&= \left(\kappa_{l0}\pi_W^{\#}H+\sum_{j\in\mathcal{I}_l}\kappa_{lj}F_j+\sum_{j\in\mathcal{I}_l^c}\kappa_{lj}F_j\right)\\
&\quad+\sum_{i=1}^{s_l}a_{li}\left(\lambda_{li0}\pi_W^{\#}H+\sum_{j\in\mathcal{I}_l^c}\lambda_{lij}F_j+\sum_{j\in\mathcal{I}_l}\lambda_{lij}F_j\right)\\
&= \left(\pi_W^{\#}H+\sum_{j\in\mathcal{I}_l}\kappa_{lj}F_j+\sum_{j\in\mathcal{I}_l^c}\kappa_{lj}F_j\right)+\sum_{i=1}^{s_l}a_{li}\left(\sum_{j\in\mathcal{I}_l}\lambda_{lij}F_j\right)\\
&= \pi_W^{\#}H+\sum_{j\in\mathcal{I}_l^c}\kappa_{lj}F_j+\sum_{j\in\mathcal{I}_l}\left(\sum_{i=0}^{s_l}\kappa_{lj}+a_{li}\lambda_{lij}\right)F_j
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{\phi}_l^*H &= \rho_l^*\phi_l^*H\\
&= \rho_l^*\left(d_l\pi_l^{\#}H+\sum_{i=1}^{s_l}b_{li}E_{li}\right)\\
&= d_l\left(\kappa_{l0}\pi_W^{\#}H+\sum_{j\in\mathcal{I}_l}\kappa_{lj}F_j+\sum_{j\in\mathcal{I}_l^c}\kappa_{lj}F_j\right)\\
&\quad+\sum_{i=1}^{s_l}b_{li}\left(\lambda_{li0}\pi_W^{\#}H+\sum_{j\in\mathcal{I}_l^c}\lambda_{lij}F_j+\sum_{j\in\mathcal{I}_l}\lambda_{lij}F_j\right)
\end{aligned}$$

$$= d_l \left(\pi_W^\# H + \sum_{j \in \mathcal{I}_l} \kappa_{lj} F_j + \sum_{j \in \mathcal{I}_l^c} \kappa_{lj} F_j \right) + \sum_{i=1}^{s_l} b_{li} \left(\sum_{j \in \mathcal{I}_l} \lambda_{lij} F_j \right)$$

$$= d_l \pi_W^\# H + \sum_{j \in \mathcal{I}_l^c} d_l \kappa_{lj} F_j + \sum_{j \in \mathcal{I}_l} \left(\sum_{i=0}^{s_l} d_l \kappa_{lj} + b_{li} \lambda_{lij} \right) F_j.$$

Therefore,

$$d_l \alpha_j = d_l \sum_{j \in \mathcal{I}_l^c} \kappa_{lj} = \beta_j \quad \text{for all } j \in \mathcal{I}_l^c.$$

□

We now complete the proof of Theorem 1.4. Let $r = \max_{l=1,\ldots,k} r_l$. Note that

$$p_0 \pi_W^\# H + \sum_{j=1}^{s} p_j F_j \succ q_0 \pi_W^\# H + \sum_{j=1}^{s} q_j F_j$$

if $p_j \geq q_j$ for all $j = 0, \ldots, s$. Thus,

$$\sum_{l=1}^{k} \frac{1}{d_l} \widetilde{\phi}_l^* H = \sum_{l=1}^{k} \left[\pi_W^\# H + \sum_{j \in \mathcal{I}_l^c} \left(\frac{\beta_{lj}}{d_l} F_j \right) + \sum_{j \in \mathcal{I}_l} \left(\frac{\beta_{lj}}{d_l} F_j \right) \right]$$

$$\succ \sum_{l=1}^{k} \pi_W^\# H + \sum_{l=1}^{k} \sum_{k \in \mathcal{I}_l^c} \alpha_j F_j + \sum_{l=1}^{k} \left(\sum_{j \in \mathcal{I}_l} \frac{\alpha_j}{r_l} F_j \right) \quad (\because \text{ Lemma 3.2})$$

$$\succ k \pi_W^\# H + \sum_{l=1}^{k} \sum_{j \in \mathcal{I}_l^c} \alpha_j F_j + \sum_{l=1}^{k} \left(\sum_{j \in \mathcal{I}_l} \frac{\alpha_j}{r} F_j \right) \quad (\because \ r \geq r_l)$$

$$\succ k \pi_W^\# H + \sum_{j=1}^{s} \alpha_j F_j + \frac{1}{r} \sum_{j=1}^{s} \alpha_j F_j \quad (\because \text{ Lemma 3.1})$$

$$\succ \left(1 + \frac{1}{r} \right) \pi_W^* H \quad (\because \ k > 1, r \geq r_l \geq 1)$$

and hence

$$D = \sum_{l=1}^{k} \frac{1}{d_l} \widetilde{\phi}_l^* H - \left(1 + \frac{1}{r} \right) \pi_W^* H$$

is an $\mathbb{A}^n$-effective divisor.

So, by Proposition 2.9, h_D is bounded below on $\pi_W^{-1} \mathbb{A}^n$. Therefore, there is a constant C such that

$$\begin{aligned} h_D(Q) &= \sum_{l=1}^{k} \frac{1}{d_l} h_{\widetilde{\phi}_l^* H}(Q) - \left(1 + \frac{1}{r} \right) h_{\pi_W^* H}(Q) \\ &= \sum_{l=1}^{k} \frac{1}{d_l} h_{{}^* H}\big(\widetilde{\phi}_l(Q)\big) - \left(1 + \frac{1}{r} \right) h_H\big(\pi_W(Q)\big) > C \end{aligned}$$

for all $Q \in \pi_W^{-1}(\mathbb{A}^n)(\overline{K})$. Finally, for $P = \pi_W(Q)$, we have $\widetilde{\phi}_l(Q) = f(P)$ and hence we obtain

$$\sum_{l=1}^{k} \frac{1}{d_l} h_H(P) - \left(1 + \frac{1}{r}\right) h_H(P) > C.$$

□

Example 3.3. Let

$$f_1 = (z, y + z^2, x + (y + z^2)^2), \quad f_2 = (x, y^2, z), \quad \text{and } f_3 = (x^3, x + y, y + z^2).$$

Their indeterminacy loci in $\mathbb{P}^3$ are

$$I(f_1) = \{[x, y, 0, 0]\}, \qquad I(f_2) = \{[x, 0, z, 0]\}, \quad \text{and } I(f_3) = \{[0, y, z, 0]\}.$$

Then, the $r(f_1) = 8$, $r(f_2) = 2$ and $r(f_3) = 3/2$ (For details of the D-ratio calculation, see [10]). Therefore,

$$\begin{aligned} &h\big((z, y + z^2, x + (y + z^2)^2)\big) + h\big((x, y^2, z)\big) + h\big((z^3, x + y, y + z^2)\big) \\ \geq &\left(1 + \frac{1}{8}\right) h\big((x, y, z)\big) - C \end{aligned}$$

for some constant C.

Corollary 3.4. *Let S be a jointly regular set of affine morphisms. Then,*

$$\kappa(S) := \liminf_{\substack{P \in \mathbb{A}^n(\overline{\mathbb{Q}}) \\ h(P) \to \infty}} \sum_{f \in S} \frac{1}{\deg f} \frac{h\big(f(P)\big)}{h(P)} \geq 1 + \frac{1}{r},$$

where $r = \max_{f \in S} r(f)$.

Remark 3.5. Corollary 3.4 may not be the exact limit infimum values. For example, if there is a subset $S' \subset S$ such that S' is still jointly regular and $\max_{f \in S'} r(f) < \max_{f \in S} r(f)$, then

$$\kappa(S) \geq \kappa(S') \geq 1 + \frac{1}{r'} > 1 + \frac{1}{r}.$$

Example 3.6. We have the following examples for $\kappa(S) = 1 + \min_{f \in S} \left(\frac{1}{r(f)}\right)$.

(1) $S = \{f, g\}$ where f, g are morphisms.
If f, g are morphisms, then $r(f) = r(g) = 1$. Therefore,

$$\frac{1}{\deg f} h\big(f(P)\big) + \frac{1}{\deg g} h\big(g(P)\big) = h(P) + h(P) + O(1).$$

(2) $S = \{f, f^{-1}\}$ where f is a regular affine automorphism and f^{-1} is the inverse of f.
It is proved by Kawaguchi. See [6].

4. An application to arithmetic dynamics

The purpose of this section is to prove Theorem 1.5. This result is a generalization of [14, Section 4]. The proof is almost the same except one: the only difference is that we have an improved height inequality for a jointly regular family.

Fix an integer $m \geq 1$ and let $S = \{f_1, \ldots, f_k : \mathbb{A}^n_K \to \mathbb{A}^n_K\}$ be a jointly regular family defined over a number field K. For each $m \geq 0$, let W_m be the collection of ordered m-tuples chosen from $\{1, \ldots, k\}$,

$$W_m = \big\{(i_1, \ldots, i_m) \mid i_j \in \{1, \ldots, k\}\big\}$$

and let

$$W_* = \bigcup_{m \geq 0} W_m.$$

Thus W_* is the collection of words of r symbols.

For any $I = (i_1, \ldots, i_m) \in W_m$, let f_I denote the composition of corresponding polynomial maps in S:

$$f_I := f_{i_1} \circ \cdots \circ f_{i_m}.$$

Definition 4.1. We denote *the monoid of rational maps generated by* $S = \{f_1, \ldots, f_k\}$ *under composition* by

$$\Phi_S = \Phi := \{\phi = f_I \mid I \in W_*\}.$$

Let $P \in \mathbb{A}^n$. *The* Φ-*orbit of* P is defined to be

$$\Phi(P) = \{\phi(P) \mid \phi \in \Phi\}.$$

The set of (*strongly*) Φ-*preperiodic points* is the set

$$\mathrm{Preper}(\Phi) = \{P \in \mathbb{A}^n \mid \Phi(P) \text{ is finite}\}.$$

Proof of Theorem 1.5. By Theorem 1.4, we have a constant C such that

$$0 \leq \left(\frac{1}{1+\frac{1}{r}}\right) \sum_{l=1}^{k} \frac{1}{d_l} h\big(f_l(Q)\big) - h(Q) + C \quad \text{for all } Q \in \mathbb{A}^n. \tag{1}$$

Note that if $r = \infty$, then $\left(\frac{1}{1+\frac{1}{r}}\right) = 1$, then it is done because of [14, Theorem 4]. Thus, we may assume that r is finite.

We define a map $\mu : W_* \to \mathbb{Q}$ by the following rule:

$$\mu_I = \mu_{(i_1, \ldots, i_m)} = \prod d_l^{p_{I,l}},$$

where $p_{I,l} = -|\{t \mid i_t = l\}|$. Then, by definition of δ_S and μ_I, the following is true:

$$\delta_S^m = \left[\left(\frac{r}{r+1}\right) \sum_{l=1}^{k} \frac{1}{d_l}\right]^m = \left(\frac{r}{r+1}\right)^m \sum_{I \in W_m} \frac{1}{\deg f_{i_1} \cdots \deg f_{i_m}}$$

$$= \left(\frac{r}{r+1}\right)^m \sum_{I\in W_m} \mu_I.$$

Let $P \in \mathbb{A}^n(\overline{\mathbb{Q}})$. Then, (1) holds for $f_I(P)$ for all $I \in W_m$:

$$0 \leq \left(\frac{r}{r+1}\right) \sum_{l=1}^{k} \frac{1}{d_l} h\big(f_l(f_I(P))\big) - h(f_I(P)) + C.$$

Hence

(2)

$$0 \leq \sum_{m=0}^{M} \sum_{I\in W_m} \mu_I \left(\frac{r}{r+1}\right)^m \left[\sum_{l=1}^{k} \frac{1}{d_l} h\big(f_l(f_I(P))\big) - \left(1+\frac{1}{r}\right) h(f_I(P)) + C\right].$$

The main difficulty of the inequality is to figure out the constant term. From the definition of δ_S, we have

$$\sum_{m=0}^{M-1} \left(\frac{r}{r+1}\right)^m \sum_{I\in W_m} \mu_I = \sum_{m=1}^{M} \delta_S^m \leq \frac{1}{1-\delta_S}.$$

Now, do the telescoping sum and most terms in (2) will be canceled:

$$\begin{aligned}
&\left(\sum_{m=0}^{M-1} \sum_{I\in W_m} \left(\frac{r}{r+1}\right)^m \mu_I \sum_{l=1}^{k} \frac{1}{d_k} h\big(f_l f_I(P)\big)\right) \\
&\quad - \left(\sum_{m=1}^{M} \sum_{I\in W_m} \left(\frac{r}{r+1}\right)^{m-1} \mu_I h\big(f_I(P)\big)\right) \\
&= \left(\sum_{m=0}^{M-1} \sum_{I\in W_m} \left(\frac{r}{r+1}\right)^m \sum_{l=1}^{k} \frac{\mu_I}{d_l} h\big(f_l f_I(P)\big)\right) \\
&\quad - \left(\sum_{m=0}^{M-1} \sum_{I\in W_m} \sum_{l=1}^{k} \left(\frac{r}{r+1}\right)^m \frac{\mu_I}{d_l} h\big(f_l f_I(P)\big)\right) \\
&= 0.
\end{aligned}$$

Therefore, the remaining terms in (2) are the first term when $m = M$ and the last term when $m = 0$. Thus, we get

$$\begin{aligned}
0 &\leq \left[\sum_{I\in W_M} \left(\frac{r}{r+1}\right)^M \mu_I \sum_{l=1}^{K} \frac{1}{d_l} h\big(f_l(f_I(P))\big)\right] - h(P) + \sum_{I\in W_M} \left(\frac{r}{r+1}\right)^M \mu_I C \\
&\leq \left[\sum_{I\in W_M} \left(\frac{r}{r+1}\right)^M \mu_I \sum_{l=1}^{k} \frac{1}{d_l} h\big(f_l(f_I(P))\big)\right] - h(P) + \frac{1}{1-\delta_S} C.
\end{aligned}$$

Let P be a Φ-periodic point and define the height of the images of P by the monoid Φ to be

$$h(\Phi(P)) = \sup_{R\in\Phi(P)} h(R).$$

Since

$$\sum_{I \in W_M} \left(\frac{r}{r+1}\right)^M \mu_I \sum_{l=1}^{k} \frac{1}{d_l} = \left(\frac{r}{r+1}\right)^M \sum_{I \in W_{M+1}} \mu_I = \left(1 + \frac{1}{r}\right) \delta_S^{M+1}$$

and

$$h(\Phi(P)) \geq h\bigl(g(P)\bigr) \text{ for all } g \in \Phi,$$

we get

$$\begin{aligned} h(P) &\leq \left[\sum_{I \in W_M} \left(\frac{r}{r+1}\right)^M \mu_I \sum_{l=1}^{k} \frac{1}{d_l}\right] h\bigl(\Phi(P)\bigr) + \frac{1}{1-\delta_S} C \\ &\leq \left(1 + \frac{1}{r}\right) \delta_S^{M+1} h\bigl(\Phi(P)\bigr) + \frac{1}{1-\delta_S} C. \end{aligned}$$

By assumption, $\delta_S < 1$ and $h\bigl(\Phi(P)\bigr)$ is finite, so letting $M \to \infty$ shows that $h(P)$ is bounded by a constant that depends only on S. □

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