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EXISTENCE OF PERIODIC SOLUTIONS FOR PLANAR HAMILTONIAN SYSTEMS AT RESONANCE

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Abstract. The existence of periodic solutions for the planar Hamiltonian systems with positively homogeneous Hamiltonian is discussed. The asymptotic expansion of the Poincaré map is calculated up to higher order and some sufficient conditions for the existence of periodic solutions are given in the case when the first order term of the Poincaré map is identically zero.

1. Introduction

Consider the existence of periodic solutions for the following planar Hamiltonian systems:

$$Ju' = \nabla H(u) + f(t), \tag{1}$$

where $' = d/dt$, $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the standard symplectic matrix, $f = (f_1, f_2) : \mathbb{R} \to L^1[0, 2\pi] \times L^1[0, 2\pi]$ is 2π-periodic, and $H \in C^3(\mathbb{R}^2, \mathbb{R})$ is positive for $u \neq 0$ and positively homogeneous of degree 2, that is, we have

$$H(\lambda u) = \lambda^2 H(u) \ \forall u \in \mathbb{R}^2,\ \lambda > 0 \tag{2}$$

and

$$\min_{\|u\|=1} H(u) > 0. \tag{3}$$

There have been many authors who have considered the system (1) with properties (2) and (3) (e.g., [7], [13]). Fonda [7] has discussed the existence of periodic solutions and unbounded solutions of (1) at the same time, however, Yang [13] has been inspired by the work of [7], and has focused only on the existence of unbounded solutions of (1).

It is well known that under the conditions (2) and (3), the origin is an isochronous center for the autonomous system

$$Ju' = \nabla H(u), \tag{4}$$

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that is, all solutions of (4) are periodic with the same minimal period, denoted by τ. Throughout this paper, the scalar product of two vectors a and b will be denoted by $\langle a,\, b\rangle$.

The system (1) is said to be *at resonance* if the period 2π of the forcing term $f(t)$ is an integral multiple of τ, that is, $\frac{2\pi}{\tau} \in \mathbb{N}$. Fonda [7] has already shown that if the system (1) with $f \in C(\mathbb{R},\, \mathbb{R}^2)$ is not at resonance, then (1) has a 2π-periodic solution for any 2π-periodic forcing term $f(t)$ (Theorem 1 in [7]) and that if the system (1) is at resonance, then there is a $f(t)$ such that all solutions of (1) are unbounded (Theorem 2 in [7]). Moreover, by taking a reference solution ϕ of the autonomous system (4) such that $H(\phi(t)) = \frac{1}{2}$ and defining the τ-periodic function $\Phi(\theta)$ by

$$\Phi(\theta) = \int_0^{2\pi} \langle f(t), \phi(t+\theta)\rangle dt$$

he has shown that if (1) is at resonance with $f \in C^6$ and $\Phi(\theta) \neq 0$ for all $\theta \in [0, \tau]$, then all solutions of (1) are bounded and hence (1) has a 2π-periodic solution (Theorem 3 in [7]), and also that if (1) is at resonance and $\Phi(\theta)$ has at least four simple zeros in $[0, \tau]$, then (1) has a 2π-periodic solution (Theorem 4 in [7]).

In this paper, it will turn out that the function $\Phi(\theta)$ is nothing but the first order term of the Poincaré mapping for the solutions of a certain equation equivalent to (1). Here, a question arises naturally. What if the first order term is identically zero? The purpose of this paper mainly aims at giving a partial answer to this question.

As in [7], an example of the system (1) with the properties (2) and (3) can be easily given by defining the Hamiltonian function on the unit circle and identifying the vector on S^1 with $e^{i\alpha}$. For instance, let

$$\alpha_1 < \alpha_2 < \cdots < \alpha_n < \alpha_{n+1} = \alpha_1 + 2\pi,$$

and $A_1,\, A_2, \ldots, A_n$ be symmetric positive definite matrices, and define the Hamiltonian function such that

$$\alpha \in [\alpha_k,\, \alpha_{k+1}] \quad \Rightarrow \quad H(e^{i\alpha}) = \frac{1}{2}\langle A_k e^{i\alpha},\, e^{i\alpha}\rangle, \quad k = 1, 2, \ldots, n,$$

where the matrices A_k are chosen so that ∇H is continuous. In this case, ∇H is piecewise linear. If u is a solution of the autonomous system (4) denoted by $u(t) = r(t)e^{i\alpha(t)}$, then we have

$$\alpha(t) \in [\alpha_k,\, \alpha_{k+1}] \quad \Rightarrow \quad \alpha'(t) = -\langle A_k e^{i\alpha(t)},\, e^{i\alpha(t)}\rangle.$$

Hence in this case, we have

$$\tau = \sum_{k=1}^{n} \int_{\alpha_k}^{\alpha_{k+1}} \frac{d\alpha}{\langle A_k e^{i\alpha},\, e^{i\alpha}\rangle}.$$

As a special case of the above example, let $n = 2$, $\alpha_1 = \pi/2$, $\alpha_2 = 3\pi/2$ and

$$A_1 = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} \mu & 0 \\ 0 & 1 \end{pmatrix},$$

with $\lambda > 0$, $\mu > 0$. Then the system (1) becomes

$$\begin{aligned} x' &= y + f_2(t), \\ y' &= -\lambda x^+ + \mu x^- - f_1(t), \end{aligned} \tag{5}$$

where $x^+ = \max\{x, 0\}$ is the positive part of x and $x^- = \max\{-x, 0\}$ is the negative part of x. Set $f_2(t) = 0$ and $f(t) = -f_1(t)$, then the system (5) is equivalent to the second order differential equation

$$x'' + \lambda x^+ - \mu x^- = f(t). \tag{6}$$

Let $S(t)$ be the solution of the following initial value problem:

$$x'' + \alpha x^+ - \beta x^- = 0, \quad x(0) = 0, \quad x'(0) = 1, \tag{7}$$

then, as is mentioned in the above, $S \in C^2(\mathbb{R})$ is τ-periodic with

$$\tau = \frac{\pi}{\sqrt{\alpha}} + \frac{\pi}{\sqrt{\beta}}. \tag{8}$$

In [3], Capietto and Wang have discussed the existence of 2π-periodic solutions of the following Liénard equation with asymmetric nonlinearities:

$$x'' + f(x)x' + \alpha x^+ - \beta x^- + g(x) = h(t), \tag{9}$$

where α and β are positive constants satisfying (8) with $\tau = \frac{2\pi}{n}$ for some $n \in \mathbb{N}$, and h is 2π-periodic and continuous, and f, $g : \mathbb{R} \to \mathbb{R}$ are continuous and have the following finite limits

$$\lim_{x \to \pm\infty} F(x) = F(\pm\infty),$$

$$\lim_{x \to \pm\infty} g(x) = g(\pm\infty),$$

where

$$F(x) = \int_0^x f(s)ds.$$

They have defined the functions $\lambda(\theta)$ and $\mu(\theta)$ by

$$\lambda(\theta) = 2n\left[\frac{g(+\infty)}{\alpha} - \frac{g(-\infty)}{\beta}\right] - \int_0^{2\pi} S(\theta + t)h(t)dt, \qquad \theta \in [0,\, 2\pi/n],$$

$$\mu(\theta) = 2n\left[F(+\infty) - F(-\infty)\right] - \int_0^{2\pi} S'(\theta + t)h(t)dt, \qquad \theta \in [0,\, 2\pi/n]$$

and have proved that:

Theorem A. *Eq.*(9) *has at least one* 2π*-periodic solution if either the function* $\lambda(\theta)$ *or the function* $\mu(\theta)$ *is of constant sign.*

Theorem B. *Eq.*(9) *has at least one* 2π*-periodic solution if*

(i) *The zeros of* $\lambda(\theta)$ *are simple and the zeros of* $\lambda(\theta)$ *and* $\mu(\theta)$ *are different*;

(ii) *The signs of* $\mu(\theta)$ *at the zeros of* $\lambda(\theta)$ *in* $[0,\, 2\pi/n]$ *do not change or change more than two times.*

Now, here again, we encounter the same question as before: what if the functions λ and/or μ are identically zero? In this case, it is natural to consider the higher order approximations of some relevant Poincaré mapping. By using this idea, which was also used before by [13], we will give some partial answers to this question in this paper.

2. Main results

Let $\phi(t)$ be the solution of the autonomous system (4) satisfying

$$H(\phi(t)) = 1/2, \quad \forall\, t \in \mathbb{R}. \tag{10}$$

Here we consider only the resonance case, that is, the case when $\frac{2\pi}{\tau} = n$ for some $n \in \mathbb{N}$.

The main results of this paper are the following two theorems:

Theorem 1. *Suppose* $\frac{2\pi}{\tau} = n$ *for some* $n \in \mathbb{N}$*. Define* τ*-periodic functions* $\lambda_1,\ \mu_2,\ \lambda_3$ *as follows*:

$$\begin{aligned} &\lambda_1(\theta) = \int_0^{2\pi} \langle \phi(\theta + t), f(t)\rangle\, dt, \\ &\mu_2(\theta) = -\int_0^{2\pi} \langle \phi''(\theta + t), f(t)\rangle \int_0^t \langle \phi(\theta + s), f(s)\rangle\, ds dt, \\ &\lambda_3(\theta) = \alpha(\theta) + \frac{1}{2}\int_0^{2\pi} \langle \phi''(\theta + t), f(t)\rangle \left(\int_0^t \langle \phi(\theta + s), f(s)\rangle\, ds\right)^2 dt, \end{aligned} \tag{11}$$

where

$$\alpha(\theta) = \lambda_1(\theta)\left[(\lambda_1'(\theta))^2 - \mu_2(\theta)\right].$$

Assume that $\lambda_1(\theta) \equiv 0$*. Then the system* (1) *has at least one* 2π*-periodic solution provided that one of the following conditions is satisfied*:

(i) $\lambda_3(\theta) \neq 0$ *for all* $\theta \in [0,\, \tau]$;

(ii) $\mu_2(\theta) \neq 0$ *for all* $\theta \in [0,\, \tau]$;

(iii) $\mu_2(\theta) \equiv 0$ *and* $\lambda_3'(\theta)) \neq 0$ *for all* $\theta \in [0,\, \tau]$.

Theorem 2. *Suppose* $\frac{2\pi}{\tau} = n$ *for some* $n \in \mathbb{N}$ *and let* $\lambda_1, \mu_2, \lambda_3$ *be defined as in* (11)*. Assume that* $\lambda_1(\theta) \equiv 0$*. Then the system* (1) *has at least one* 2π*-periodic solution if one of the following conditions holds*:

(I) $\lambda_3(\theta)$ *has an even number of zeros* $\{\theta_1, \theta_2, \ldots, \theta_{2M_1}\}$ *with* $M_1 \geq 1$ *and for each* $i = 1, 2, \ldots, 2M_1$,

$$\mu_2(\theta_i) \neq 0,$$

and

$$\lambda_3(\theta)\mu_2(\theta)(\theta - \theta_i) > 0$$

for $|\theta - \theta_i| > 0$ *small*;

(II) *$\mu_2(\theta) \equiv 0$ and $\lambda_3(\theta)$ has an even number of zeros $\{\theta_1, \theta_2, \dots, \theta_{2M_2}\}$ with $M_2 \geq 2$ and for each $i = 1, 2, \dots, 2M_2$,*

$$\lambda_3'(\theta_i) \neq 0,$$

and

$$\lambda_3(\theta)\lambda_3'(\theta)(\theta - \theta_i) < 0$$

for $|\theta - \theta_i| > 0$ small.

3. Asymptotic expansion of the Poincaré mapping

Since H is positively homogeneous of degree 2, we have the following Euler's identity:

$$\langle \nabla H(u),\, u\rangle = 2H(u) \quad \forall u \in \mathbb{R}^2. \tag{12}$$

Let $\phi(t)$ be a τ-periodic solution of (4) satisfying (10). Any solution of (1) with $u(t) \neq 0$ can be written as

$$u(t) = r(t)\phi(\theta(t)) \tag{13}$$

for $r(t) > 0$ and $\theta(t) \in \mathbb{R} \pmod{\tau}$. Then the map $T : (r, \theta) \to u$ is a diffeomorphism from the half plane $\{r > 0\}$ to $\mathbb{R}^2 \backslash \{(0, 0)\}$, the functions r and θ are of C^3 as far as $u(t)$ does not cross the origin. Substituting (13) into (1) yields

$$r'J\phi + r\theta'J\phi' = r\nabla H(\phi) + f. \tag{14}$$

By using the fact that for any $u \in \mathbb{R}^2$, $\langle Ju,\, u\rangle = 0$ and the equations (4) and (10), and the Euler's identity (12), a scalar product with ϕ in (14) yields

$$r\theta' = r + \langle \phi,\, f\rangle,$$

while a scalar product with ϕ' in (14) yields

$$-r' = \langle \phi',\, f\rangle.$$

Therefore for $u(t) = r(t)\phi(\theta(t)) \neq 0$, we have the following system equivalent to (1):

$$\begin{aligned} r' &= -\langle \phi', f\rangle, \\ \theta' &= 1 + r^{-1}\langle \phi, f\rangle. \end{aligned} \tag{15}$$

Let $(r(t; r_0, \theta_0)), \theta(t; , r_0, \theta_0))$ be the solution of (15) with initial value $(r_0,\, \theta_0)$. Then by the boundedness of ϕ, ϕ' and the assumption $f \in L^1(0,\, 2\pi) \times L^1(0,\, 2\pi)$, we get for $r_0 \gg 1$ and $t \in [0, 2\pi]$,

$$r(t) = r_0 + O(1), \quad r^{-1}(t) = r_0^{-1} + O(r_0^{-2}), \quad \theta(t) = \theta_0 + t + O(r_0^{-1}). \tag{16}$$

Lemma 1. *For $r_0 \gg 1$, the Poincaré mapping*

$$P : (r_0,\, \theta_0) \to (r_1,\, \theta_1) = (r(2\pi; r_0, \theta_0),\, \theta(2\pi; r_0, \theta_0))$$

of the solution of (15) *with initial value* (r_0, θ_0) *has the following asymptotic expression*

$$\begin{aligned} r_1 &= r_0 - \lambda_1'(\theta_0) + \mu_2(\theta_0) r_0^{-1} + (\alpha'(\theta_0) - \lambda_3'(\theta_0))\, r_0^{-2} + O(r_0^{-3}), \\ \theta_1 &= \theta_0 + 2\pi + \lambda_1(\theta_0) r_0^{-1} + \lambda_1(\theta_0)\lambda_1'(\theta_0) r_0^{-2} + \lambda_3(\theta_0) r_0^{-3} + O(r_0^{-4}), \end{aligned} \tag{17}$$

where $\lambda_1,\ \mu_2,\ \lambda_3$ *and* α *are given by* (11).

Proof. Substituting (16) into (15) and integrating over $[0,\, t] \subset [0,\, 2\pi]$ yields

$$\begin{aligned} r(t) &= r_0 + \mu_1(\theta_0, t) + O(r_0^{-1}), \\ r^{-1}(t) &= r_0^{-1} - \mu_1(\theta_0, t) r_0^{-2} + O(r_0^{-3}), \\ \theta(t) &= \theta_0 + t + \lambda_1(\theta_0, t) r_0^{-1} + O(r_0^{-2}), \end{aligned} \tag{18}$$

where

$$\begin{aligned} \lambda_1(\theta, t) &= \textstyle\int_0^t \langle \phi(\theta + s), f(s) \rangle\, ds, \\ \mu_1(\theta, t) &= -\textstyle\int_0^t \langle \phi'(\theta + s), f(s) \rangle\, ds. \end{aligned} \tag{19}$$

Substituting (18) into (15) and integrating over $[0,\, t] \subset [0,\, 2\pi]$ yields

$$\begin{aligned} r(t) &= r_0 + \mu_1(\theta_0, t) + \mu_2(\theta_0, t) r_0^{-1} + O(r_0^{-2}), \\ r^{-1}(t) &= r_0^{-1} - \mu_1(\theta_0, t) r_0^{-2} + \left[\mu_1^2(\theta_0, t) - \mu_2(\theta_0, t)\right] r_0^{-3} + O(r_0^{-4}), \\ \theta(t) &= \theta_0 + t + \lambda_1(\theta_0, t) r_0^{-1} + \lambda_2(\theta_0, t) r_0^{-2} + O(r_0^{-3}), \end{aligned} \tag{20}$$

where

$$\begin{aligned} \lambda_2(\theta, t) = &\int_0^t \langle \phi'(\theta + s), f(s)) \rangle\, \lambda_1(\theta, s) ds \\ &- \int_0^t \langle \phi(\theta + s), f(s) \rangle\, \mu_1(\theta, s) ds, \\ \mu_2(\theta, t) = &- \int_0^t \langle \phi''(\theta + s), f(s) \rangle\, \lambda_1(\theta, s) ds. \end{aligned} \tag{21}$$

Substituting (20) into (15) again and integrating over $[0,\, t] \subset [0,\, 2\pi]$ yields

$$\begin{aligned} r(t) &= r_0 + \mu_1(\theta_0, t) + \mu_2(\theta_0, t) r_0^{-1} + \mu_3(\theta_0, t) r_0^{-2} + O(r_0^{-3}), \\ \theta(t) &= \theta_0 + t + \lambda_1(\theta_0, t) r_0^{-1} + \lambda_2(\theta_0, t) r_0^{-2} + \lambda_3(\theta_0, t) r_0^{-3} + O(r_0^{-4}), \end{aligned} \tag{22}$$

where
(23)

$$\lambda_3(\theta,t)=\int_0^t \langle \phi'(\theta+s), f(s)\rangle\, \lambda_2(\theta,s)ds+\frac{1}{2}\int_0^t \langle \phi''(\theta+s), f(s)\rangle\, \lambda_1^2(\theta,s)ds$$

$$-\int_0^t \langle \phi'(\theta+s), f(s))\rangle\, \lambda_1(\theta,s)\mu_1(\theta,s)ds$$

$$+\int_0^t \langle \phi(\theta+s), f(s)\rangle \left(\mu_1^2(\theta,s)-\mu_2(\theta,s)\right) ds;$$

$$\mu_3(\theta,t)=-\int_0^t \langle \phi''(\theta+s), f(s)\rangle\, \lambda_2(\theta,s)ds-\frac{1}{2}\int_0^t \langle \phi'''(\theta+s), f(s)\rangle\, \lambda_1^2(\theta,s)ds.$$

Substituting (19) and (21) into (23) and letting $\lambda_k(\theta)=\lambda_k(\theta,2\pi)$, $\mu_k(\theta)=\mu_k(\theta,2\pi)$ for $k=1$, 2, 3, and after some calculations and simplifications, we obtain (17) and the following relations:

$$\begin{aligned}\mu_1(\theta)&=-\lambda_1'(\theta);\\ \lambda_2(\theta)&=\lambda_1(\theta)\lambda_1'(\theta);\\ \mu_3(\theta)&=\alpha'(\theta)-\lambda_3'(\theta),\end{aligned}\tag{24}$$

where $\alpha(\theta)=\lambda_1(\theta)\left[(\lambda_1'(\theta))^2-\mu_2(\theta)\right]$. □

4. Planar mappings and rotation numbers

For $\sigma>0$, let B_σ be the open ball centered at the origin and of radius σ and let E_σ be its exterior, that is, $E_\sigma=\mathbb{R}^2-B_\sigma$. Let $S^1=\mathbb{R}/(2\pi\mathbb{Z})$. Then the points θ in S^1 are defined by

$$\theta=\bar{\theta}+2k\pi,\quad k\in\mathbb{Z},\quad \bar{\theta}\in\mathbb{R}$$

and the norm of θ in S^1 is defined by

$$\|\theta\|=\min\{|\bar{\theta}+2k\pi|\,|k\in\mathbb{Z}\}.$$

Let $P:E_\sigma\to\mathbb{R}^2$ be a one-to-one and continuous mapping, which can be expressed in the following form:

$$\begin{aligned}\theta_1&=\theta+2\pi+\lambda(\theta)r^{-k}+F(r,\theta),\\ r_1&=r+\mu(\theta)r^{-m}+G(r,\theta),\end{aligned}\tag{25}$$

where $r\geq R_0\gg 1$ and $\theta\in S^1$, and λ, $\mu\in C(S^1,\mathbb{R})$, $k\geq 1$, $m\geq 0$, and $F=o(r^{-k})$, $G=o(r^{-m})$ are continuous and τ-periodic in θ. Given a point $(\theta_0,r_0)\in E_\sigma$, let $\{(\theta_k,r_k)\}_{k\in I}$ be the unique solution of the initial value problem for the difference equation

$$(\theta_{k+1},r_{k+1})=P(\theta_k,r_k)$$

defined in a maximal interval

$$I = \{k \in Z | k_a < k < k_b\},$$

where k_a and k_b are certain numbers in the set $\mathbb{Z} \cup \{+\infty, -\infty\}$ satisfying

$$-\infty \le k_a < 0 < k_b \le +\infty.$$

Lemma 2. *Assume that* $\tau = \frac{2\pi}{n}$ *for some* $n \in \mathbb{N}$ *and that the Poincaré mapping of the solutions of* (15) *has the form of* (25). *Suppose that either* $\lambda(\theta) \neq 0$ *or* $\mu(\theta) \neq 0$ *for all* $\theta \in S^1$. *Then* (1) *has at least one* 2π*-periodic solution.*

Proof. The proof is similar to the proof of Theorem A in [3], so we omit it for brevity. □

Consider the mapping (25) and note that by assumption F and G satisfy

$$|F(r, \theta)| r^k + |G(r, \theta)| r^m \to 0 \text{ as } r \to +\infty \tag{26}$$

uniformly with respect to $\theta \in [0, \tau]$.

Now consider the case when $\lambda(\theta)$ has a finite number of zeros in $[0, \tau)$. Since the function $\lambda(\theta)$ is τ-periodic, it must have an even number of zeros in $[0, \tau)$. Suppose that for all zeros $\{\theta_i\}$ of $\lambda(\theta)$, $i = 1, 2, \ldots, 2M$, we have

$$\mu(\theta_i) \neq 0, \tag{27}$$

and

$$\lambda(\theta)\mu(\theta)(\theta - \theta_i) > 0 \tag{28}$$

for $|\bar{\theta} - \bar{\theta}_i| > 0$ and small.

Now let us consider the curve $\Gamma_r : [0, \tau] \to \mathbb{R}^2$ defined by

$$\Gamma_r(\theta) = (\theta_1 - \theta, r_1 - r), \tag{29}$$

which is easily seen to be closed. Assume that if $r > 0$ is large enough, then $\Gamma_r(\theta) \neq (0, 0)$ for every $\theta \in [0, \tau]$. Hence, we may look for the number of rotations around the origin performed by the vector $\Gamma_r(\theta)$ while θ varies from 0 to τ. If we denote that number by d_{Γ_r}, then for $r \gg 1$, we have the following result.

Lemma 3. *Consider the mapping* (25). *Assume that* $\tau = \frac{2\pi}{n}$ *for some* $n \in \mathbb{N}$ *and that* $|\bar{\theta} - \bar{\theta}_i| > 0$ *and small for all* $i = 1, 2, \ldots, 2M$, *and that* (27) *and* (28) *are satisfied. Then for* $r \gg 1$, *we have*

$$d_{\Gamma_r} = 1 - M.$$

Proof. Let $V_r(\theta) = (2\pi + \lambda(\theta) r^{-k}, \mu(\theta) r^{-m})$. Then it follows from (25) that

$$\Gamma_r(\theta) = V_r(\theta) + (F(r, \theta), G(r, \theta)),$$

where

$$F(r, \theta) = o(r^{-k}), \quad G(r, \theta) = o(r^{-m}).$$

From the assumption $\Gamma_r(\theta) \neq (0, 0)$ for all $\theta \in [0, \tau]$ and (27), we see that for $r \gg 1$, $V_r(\theta) \neq (0, 0)$ for all $\theta \in [0, \tau]$ and $d_{\Gamma_r} = d_{V_r}$. Therefore, we need only to compute d_{V_r} for $r \gg 1$. From the expression of $V_r(\theta)$, we see easily that $d_{V_r} = 1 + d_\phi$, where d_ϕ is the rotation number of the vector $(\lambda(\theta)r^{-k},\ \mu(\theta)r^{-m})$.

Let $0 \le \theta_1 < \theta_2 < \cdots < \theta_{2M} < \tau$ be the zeros of $\lambda(\theta)$ in $[0,\ \tau)$. Then we get from (28), $\lambda(\theta)\mu(\theta) < 0$ for $\theta < \theta_i$ and $|\bar\theta - \bar\theta_i| > 0$ small, and $\lambda(\theta)\mu(\theta) > 0$ for $\theta > \theta_i$ and $|\bar\theta - \bar\theta_i| > 0$ small, $i = 1, 2, \ldots, 2M$. From the above analysis, we obtain

$$\begin{aligned}
d_\phi &= \frac{1}{2\pi}\int_0^\tau \left(\arctan\left(\frac{\mu(\theta)r^{k-m}}{\lambda(\theta)}\right)\right)' d\theta \\
&= -\frac{1}{2\pi}\sum_{i=1}^{2M}\int_{\theta_i^-}^{\theta_i^+}\left(\arctan\left(\frac{\mu(\theta)r^{k-m}}{\lambda(\theta)}\right)\right)' d\theta \\
&= -\frac{1}{2\pi}\sum_{i=1}^{2M}\left[\lim_{\theta\to\theta_i^+}\arctan\left(\frac{\mu(\theta)r^{k-m}}{\lambda(\theta)}\right) - \lim_{\theta\to\theta_i^-}\arctan\left(\frac{\mu(\theta)r^{k-m}}{\lambda(\theta)}\right)\right] \\
&= -\frac{1}{2\pi}\sum_{i=1}^{2M}[\arctan(+\infty) - \arctan(-\infty)] \\
&= -\frac{1}{2\pi}\sum_{i=1}^{2M}\left[\frac{\pi}{2} + \frac{\pi}{2}\right] \\
&= -M.
\end{aligned}$$

Hence $d_{\Gamma_r} = d_{V_r} = 1 - M$ for $r \gg 1$. $\square$

Similarly, we can prove the following lemma:

Lemma 4. *Suppose that* (27) *holds for all* $i = 1, 2, \ldots, 2M$ *and that for* $|\bar\theta - \bar\theta_i| > 0$ *small and* $r \gg 1$, *one has*

$$\lambda(\theta)\mu(\theta)(\theta - \theta_i) < 0.$$

Then

$$d_{\Gamma_r} = 1 + M.$$

5. Proofs of theorems

Proof of Theorem 1. In the case of (i) and (ii), since $\lambda_1(\theta) \equiv 0$, it follows from Lemma 1 that the Poincaré mapping of the solutions of (15) takes the form of (25) with

$$\lambda(\theta) = \lambda_3(\theta),\ \mu(\theta) = \mu_2(\theta) \quad \text{and} \quad k = 3,\ m = 1. \tag{30}$$

Similarly, in the case of (iii), Lemma 1 implies that the Poincaré mapping takes the form of (25) with

$$\lambda(\theta) = \lambda_3(\theta),\ \mu(\theta) = \mu_3(\theta) = -\lambda_3'(\theta) \quad \text{and} \quad k = 3,\ m = 2. \tag{31}$$

Now Theorem 1 follows from Lemma 2. $\square$

Proof of Theorem 2. It follows from the proof of Theorem 1 that the Poincaré mapping of the solutions of (15) takes the form of (25) with (30) in the case of (I) and with (31) in the case of (II). Let $B_r \subset \mathbb{R}^2$ be the ball of radius $r > 0$. Define the set D_r by

$$D_r = \{u \in C^1[0,\, 2\pi] :\, u(t) \in B_r,\;\; t \in [0,\, 2\pi]\}.$$

Then in this case, the Brouwer degree of the map $I - P$ with respect to the set B_r at the origin is precisely the number of rotations around the origin by the vector Γ_r. Hence, for $r \gg 1$, in the case (I) and (II), we get $\deg(I-P,\, B_r,\, 0) = 1 - M \neq 0$, which implies that the Poincaré mapping P has at leat one fixed point, and hence (1) has at least one 2π-periodic solution. $\square$

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