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Bayesian reliability estimation in a stress-strength system

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Abstract

We consider the problem of estimating the system reliability using noninformative priors when both stress and strength follow generalized gamma distributions with index, scale, and shape parameters. We first derive group-ordering reference priors using the reparametrization. We next provide the sufficient condition for propriety of posterior distributions and provide marginal posterior distributions under those noninformative priors. Finally, we provide and compare estimated values of the system reliability based on the simulated values of parameter of interest in some special cases.

Keywords: system reliability, generalized gamma distribution, reparametrization, reference prior.

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1. Introduction

Suppose a system, made up of k identical components, functions if r or more of the k components simultaneously operate. We assume that the strengths of these components, $Y_1 \cdots, Y_n$ are identically and independently distributed(i.i.d.) random variables with a common cumulative distribution function(c.d.f.), $G(y)$. We suppose that this system is subject to a stress, say X, which is a random variable with c.d.f. $F(x)$. We define the system reliability, say R

$$R = P(X < Y) = \int_{-\infty}^{\infty} [1 - G(x)] dF(x). \tag{1.1}$$

The present paper focuses exclusively on Bayesian inference for R when $F(x)$ and $G(y)$ are c.d.f.'s of the generalized gamma distribution, $GG(\beta, \eta_1, p)$and $GG(\beta, \eta_2, p)$, respectively, with corresponding density functions

$$f(x) = \frac{\beta}{\Gamma(p)} \eta_1^{-p\beta} x^{p\beta - 1} e^{\left(\frac{-x}{\eta_1}\right)^{\beta}}, \; x > 0$$

and

$$g(y) = \frac{\beta}{\Gamma(p)} \eta_2^{-p\beta} y^{p\beta - 1} e^{\left(\frac{-y}{\eta_2}\right)^{\beta}}, \; y > 0,$$

with $\eta_1 > 0$, $\eta_2 > 0$, $\beta > 0$, and $p > 0$. In this situation, the system reliability R in (1.1) reduces, after some manipulation, to

$$R = \frac{\Gamma(p_1 + p_2)}{\Gamma(p_1)\Gamma(p_2)} \int_0^{\frac{\theta_1}{1+\theta_1}} t^{p_1 - 1} (1 - t)^{p_2 - 1} dt, \tag{1.2}$$

where $\theta_1 = \left(\frac{\eta_2}{\eta_1}\right)^{\beta}$.

In the generalized gamma distribution $GG(\eta, \beta, p)$, η, β, p are, respectively, called the scale parameter, the shape parameter, and the index parameter. This distribution includes many interesting distributions as special cases : exponential distribution $(p = \beta = 1)$, Raleigh distribution $(p = 1, \beta = 2)$, Weibull distribution $(p = 1)$, Maxwell distribution $(p = 3/2, \beta = 2)$, half-normal distribution $(p = 1/2, \beta = 2)$, gamma distribution $(\beta = 1)$.
In this paper, since R depends only θ_1 and p, the emphasis is on noninformative priors for (θ_1, p). The most frequently used noninformative prior is Jeffrey's(1961), which is

proportional to the positive square root of the determinant of the Fisher information matrix. In spite of its success in one-parameter problem, Jeffreys' prior frequently runs into serious difficulties in the presence of nuisance parameters. As an alternative, we use the method of Peers(1965) to find priors which require the frequentist coverage probability of the posterior region of a real-values parametric function to match the nominal level with a remainder of $O(n^{-1})$. Tibshirani(1989) reconsidered the case when the real-valued parameter is orthogonal to the nuisance parameter vector in the sense of Cox and Reid(1987). These priors, as usually referred to as matching priors, were further studied in Datta and Ghosh(1995).

On the other hand Berger and Bernard(1989, 1992) and Datta and Ghosh(1995) extended Bernardo's(1979) reference prior approach, giving a general algorithm to derive a reference priors by splitting the parameters in several groups according to their order of inferential importance. This approach is very successful in various practical problems. Thompson and Basu(1993) derived reference priors when the stress and strength are both exponentially distribution. It turns out that in such cases, the reference priors agree with Jeffreys' prior. Sun, et al. (1998) derived Jeffreys' prior, reference priors and matching priors when the stress and strength have both Weibull distribution. It turns out that none of the Jeffreys' prior and the reference priors is a matching prior. Their study shows that the matching prior performs better than Jeffreys' prior and reference priors in meeting the target coverage probabilities.

In this paper we consider the problem of estimating the system reliability R using noninformative priors when both stress and strength follow generalized gamma distributions. In section 2 we derive the Fisher information matrix using reparimatization from $(\beta, \eta_1, \eta_2, p)$ to $(\Theta_1, \Theta_2, \Theta_3, \Theta_4)$ when (Θ_1, Θ_2) as the parameters of interest and (Θ_3, Θ_4) is nuisance parameter. In section 3 we derive Jeffreys' prior π_J and the reference prior distributions $\pi_1, \pi_2, \pi_3, \pi_4$ for different group ordering for $((\Theta_1, \Theta_2), \Theta_3, \Theta_4)$ when (Θ_1, Θ_2) is the parameters of interest. In section 4 we provide the sufficient condition for propriety of posterior distributions and provide marginal posterior distributions under those noninformative priors. Finally, in section 5 we provide and compare estimated values of the system reliability R based on the simulated values of parameter of interest in some special cases.

2. Fisher Information Matrix

Suppose that $X_1 \cdots, X_m$ are i.i.d as the generalized gamma distribution, $GG(\eta_1, \beta, p)$ and independently $Y_1 \cdots, Y_n$ are i.i.d as the generalized gamma distribution, $GG(\eta_2, \beta, p)$.

Then the likelihood function of $(\eta_1, \eta_2, \beta, p)$ is

$$L(p, \eta_1, \eta_2, \beta \mid \underline{x}, \underline{y}) = \beta^{m+n}[\Gamma(p)]^{-(m+n)} \eta_1^{-mp\beta} \eta_2^{-np\beta} \left(\prod_{i=1}^{m} x_i \prod_{j=1}^{n} y_j\right)^{p\beta-1} e^{-\sum_{i=1}^{m}\left(\frac{x_i}{\eta_1}\right)^{\beta} - \sum_{j=1}^{n}\left(\frac{y_j}{\eta_2}\right)^{\beta}}.$$

The log-likelihood function of $(\eta_1, \eta_2, \beta, p)$ is

$$l(p, \eta_1, \eta_2, \beta) = (m+n)\log\beta - (m+n)\log\Gamma(p) - mp\beta\log\eta_1$$
$$- np\beta\log\eta_2 + (p\beta-1)\left(\sum_{i=1}^{m}\log x_i + \sum_{j=1}^{n}\log y_j\right) - \sum_{i=1}^{m}\left(\frac{x_i}{\eta_1}\right)^{\beta} - \sum_{j=1}^{n}\left(\frac{y_j}{\eta_2}\right)^{\beta}.$$

Lemma 2.1 The Fisher information matrix for $(\eta_1, \eta_2, \beta, p)$ is

$$I(p, \eta_1, \eta_2, \beta) = \begin{pmatrix} (m+n)\psi'(p) & \frac{m\beta}{\eta_1} & \frac{n\beta}{\eta_2} & -\frac{m+n}{\beta\Gamma(p)}\gamma_1^* \\ \frac{m\beta}{\eta_1} & \frac{mp\beta^2}{\eta_1^2} & 0 & -\frac{m\gamma_1}{\eta_1\Gamma(p)} \\ \frac{n\beta}{\eta_2} & 0 & \frac{np\beta^2}{\eta_2^2} & -\frac{n\gamma_1}{\eta_2\Gamma(p)} \\ -\frac{m+n}{\beta\Gamma(p)}\gamma_1^* & -\frac{m\gamma_1}{\eta_1\Gamma(p)} & -\frac{m\gamma_1}{\eta_2\Gamma(p)} & \frac{m+n}{\beta^2}\left(1+\frac{\gamma_2}{\Gamma(p)}\right) \end{pmatrix},$$

where $\gamma_i = \int_0^{\infty} (\log z)^i z^p e^{-z} dz,\ i=1,2,\quad \gamma_i^* = \int_0^{\infty} (\log z) z^{p-1} e^{-z} dz = \Gamma'(p),$

$$\psi'(p) = \frac{d}{dp^2}[\log\Gamma(p)] = \int_0^{\infty} \frac{te^{pt}}{1-e^{-t}}\, dt.$$

Proof. The results easily follows from the following identities;

$$\frac{\partial^2 l}{\partial\eta_1^2} = \frac{mp\beta}{\eta_1^2} - \beta(\beta+1)\sum_{i=1}^{m}\frac{x_i^{\beta}}{\eta_1^{\beta+2}}$$
$$\frac{\partial^2 l}{\partial\eta_1\partial\eta_2} = 0,$$
$$\frac{\partial^2 l}{\partial\eta_1\partial\beta} = -\frac{mp}{\eta_1} + \frac{1}{\eta_1}\sum_{i=1}^{m}\left[\left(\frac{x_i}{\eta_1}\right)^{\beta} + \beta\left(\frac{x_i}{\eta_1}\right)^{\beta}\log\left(\frac{x_i}{\eta_1}\right)\right]$$
$$\frac{\partial^2 l}{\partial\eta_2^2} = \frac{np\beta}{\eta_2^2} - \beta(\beta+1)\sum_{i=1}^{n}\frac{y_i^{\beta}}{\eta_2^{\beta+2}},$$
$$\frac{\partial^2 l}{\partial\eta_2\partial\beta} = -\frac{np}{\eta_2} + \frac{1}{\eta_2}\sum_{j=1}^{1}\left[\left(\frac{y_i}{\eta_2}\right)^{\beta} + \beta\left(\frac{y_i}{\eta_2}\right)^{\beta}\log\left(\frac{y_i}{\eta_2}\right)\right]$$
$$\frac{\partial^2 l}{\partial\beta^2} = -\frac{m+n}{\beta^2} - \sum_{i=1}^{m}\left(\frac{x_i}{\eta_1}\right)^{\beta}\left(\log\left(\frac{x_i}{\eta_1}\right)\right)^2 - \sum_{j=1}^{n}\left(\frac{y_i}{\eta_2}\right)^{\beta}\left(\log\left(\frac{y_i}{\eta_2}\right)\right)^2,$$

and

$$E(X_i^{\beta}) = p\eta_1^{\beta}, \qquad E(Y_j^{\beta}) = p\eta_2^{\beta},$$

$$E(X_i^{\beta}\log X_i) = p\eta_1^{\beta}(\log\eta_1) + \frac{1}{\beta\Gamma(p)}\eta_1^{\beta}\gamma_1,$$

$$E(Y_j^{\beta}\log Y_j) = p\eta_2^{\beta}(\log\eta_2) + \frac{1}{\beta\Gamma(p)}\eta_2^{\beta}\gamma_1,$$

$$E(X_i^{\beta}(\log X_i)^2) = p\eta_1^{\beta}(\log\eta_1)^2 + \frac{2}{\beta\Gamma(p)}\eta_1^{\beta}(\log\eta_1)\gamma_1 + \frac{1}{\beta^2\Gamma(p)}\eta_1^{\beta}\gamma_2,$$

$$E(Y_j^{\beta}(\log Y_j)^2) = p\eta_2^{\beta}(\log\eta_2)^2 + \frac{2}{\beta\Gamma(p)}\eta_2^{\beta}(\log\eta_2)\gamma_1 + \frac{1}{\beta^2\Gamma(p)}\eta_2^{\beta}\gamma_2.$$

and

$$-E\left(\frac{\partial^2\ell}{\partial p\,\partial\beta}\right) = m\log\eta_1 + n\log\eta_2 - mE(\log X) - nE(\log Y) = -\frac{(m+n)}{\beta\Gamma(p)}r_1^*$$

$$-E\left(\frac{\partial^2\ell}{\partial\eta_1^2}\right) = -\frac{mp\beta}{\eta_1^2} + \beta(\beta+1)\frac{m}{\eta_1^{\beta+2}}E(X^{\beta}) = -\frac{mp\beta^2}{\eta_1^2}$$

$$-E\left(\frac{\partial^2\ell}{\partial\eta_1\,\partial\beta}\right) = \frac{mp}{\eta_1} - \frac{m}{\eta_1^{\beta+1}}E(X^{\beta}) - \frac{\beta m}{\eta_1^{\beta+1}}E(X^{\beta}\log X) + \frac{m\beta(\log\eta_1)}{\eta_1^{\beta+1}}E(X^{\beta}) = -\frac{m}{\eta_1\Gamma(p)}r_1$$

$$-E\left(\frac{\partial^2\ell}{\partial\eta_2^2}\right) = -\frac{np\beta}{\eta_2^2} + \frac{\beta(\beta+1)n}{\eta_1^{\beta+2}}E(Y^{\beta}) = -\frac{np\beta^2}{\eta_2^2}$$

$$-E\left(\frac{\partial^2\ell}{\partial\eta_2\partial\beta}\right) = \frac{np}{\eta_2} - \frac{n}{\eta_2^{\beta+1}}E(Y^{\beta}) - \frac{\beta n}{\eta_2^{\beta+1}}E(Y^{\beta}\log Y) + \frac{n\beta(\log\eta_1)}{\eta_2^{\beta+1}}E(Y^{\beta}) = -\frac{n}{\eta_2\Gamma(p)}r_1$$

$$-E\left(\frac{\partial^2\ell}{\partial\beta^2}\right) = \frac{m+n}{\beta^2} + \frac{m}{\eta_1^{\beta}}E[X^{\beta}(\log X - \log\eta_1)^2] + \frac{n}{\eta_2^{\beta}}E[Y^{\beta}(\log Y - \log\eta_2)^2] = \frac{m+n}{\beta^2}\left(1 + \frac{1}{\Gamma(p)}r_2\right).$$

Since the system reliability R in (1.2) depends on $\Theta_1 = \left(\frac{\eta_2}{\eta_1}\right)^{\beta}$, and $\Theta_2 = p$ we consider (Θ_1, Θ_2) as the parameters of interest.

Now, consider the following transformation from $(\alpha, \eta_1, \eta_2, p)$ to $(\Theta_1, \Theta_2, \Theta_3, \Theta_4)$:

$$\Theta_1 = \left(\frac{\eta_2}{\eta_1}\right)^{\beta}, \quad \Theta_2 = p, \quad \Theta_3 = \eta_1^{\frac{m}{m+n}}\eta_2^{\frac{n}{m+n}}, \quad \Theta_4 = \beta.$$

Then the Fisher information matrix for $(\Theta_1, \Theta_2, \Theta_3, \Theta_4)$ becomes, after lengthy calculations,

$$I_2(\theta_1,\theta_2,\theta_3,\theta_4)$$
$$=\begin{pmatrix} \frac{mn}{m+n}\theta_1^{-2}\theta_2 & 0 & 0 & -\frac{mn}{m+n}\theta_1^{-2}\theta_2(\log\theta_1) \\ 0 & (m+n)\psi'(\theta_2) & \frac{(m+n)\theta_4}{\theta_3} & -\frac{(m+n)\gamma_1^*}{\theta_4\Gamma(\theta_2)} \\ 0 & \frac{(m+n)\theta_4}{\theta_3} & \frac{(m+n)\theta_2\theta_4^2}{\theta_3^2} & -\frac{(m+n)\gamma_1}{\theta_3\Gamma(\theta_2)} \\ -\frac{mn}{(m+n)}\theta_1^{-1}\theta_2(\log\theta_1) & -\frac{(m+n)\gamma_1}{\theta_4\Gamma(\theta_2)} & -\frac{(m+n)\gamma_1}{\theta_3\Gamma(\theta_2)} & \frac{\theta_4^{-2}}{m+n}A \end{pmatrix}$$

where $A=A(\theta_1,\theta_2)=mn\theta_2(\log\theta_1)^2+(m+n)^2\left(1+\frac{\gamma_2}{\Gamma(\theta_2)}\right)$,

$\gamma_i=\int_0^\infty(\log z)^i z^{\theta_2}e^{-z}dz,\ i=1,2,\ \ \gamma_1^*=\int_0^\infty(\log z)z^{\theta_2-1}e^{-z}dz,\ \ \psi'(\theta_2)=\frac{\partial^2}{\partial\theta_2^2}\log\Gamma(\theta_2).$

3. Noninformative priors

In this section, we provide Jeffreys' prior and group-ordering reference priors.

Theorem 3.1. The Jeffreys' prior for $(\theta_1,\theta_2,\theta_3,\theta_4)$ is

$$\pi_J(\theta_1,\theta_2,\theta_3,\theta_4)\propto|I_2(\theta_1,\theta_2,\theta_3,\theta_4)|^{\frac{1}{2}}$$
$$\propto\theta_1^{-1}\theta_2\theta_3^{-1}h^{\frac{1}{2}}(\theta_2),$$

where $h(\theta_2)=\left[1+\theta_2\psi'(\theta_2)-\frac{1}{\theta_2}\right]\left[\psi'(\theta_2)-\frac{1}{\theta_2}\right]-\frac{1}{\theta_2^2}$.

The following theorem gives the reference prior distributions for different groups of ordering for $(\theta_1,\theta_2,\theta_3,\theta_4)$ when (θ_1,θ_2) is the parameters of interest.

Theorem 3.2. If (θ_1,θ_2) is the parameters of interest, then the reference prior distributions for different group ordering for $((\theta_1,\theta_2),\theta_3,\theta_4)$ are:

Group ordering	Reference prior

(1) $\{(\theta_1,\theta_2),(\theta_3,\theta_4)\}$,

$$\pi_1=\theta_1^{-1}\theta_2\theta_3^{-1}h(\theta_2)^{\frac{1}{2}}\cdot\left[mn\theta_2(\log\theta_1)^2+(m+n)^2\left(1+\frac{\gamma_2}{\Gamma(\theta_2)}-\frac{\gamma_1^2}{\theta_2\Gamma(\theta_2)^2}\right)\right]^{\frac{-1}{2}}$$

(2) $\{(\theta_1,\theta_2,\theta_3),\theta_4\}$,

$$\pi_2 = \theta_1^{-1}\theta_2\,\theta_3^{-1}\theta_4^{-1}\,h(\theta_2)^{\frac{1}{2}} \cdot \left[mn\theta_2(\log\theta_1)^2 + (m+n)^2\left(1+\frac{\gamma_2}{\Gamma(\theta_2)}\right)\right]^{\frac{-1}{2}}$$

(3) $\{(\theta_1, \theta_2, \theta_4), \theta_3\}$, $\quad \pi_3 = \theta_1^{-1}\theta_2^{\frac{1}{2}}\,\theta_3^{-1}\theta_4^{-1}\,h(\theta_2)^{\frac{1}{2}}$

(4) $\{(\theta_1, \theta_2), \theta_3, \theta_4\}$, $\{(\theta_1, \theta_2), \theta_4, \theta_3\}\}$

$$\pi_4 = \theta_1^{-1}\theta_2^{\frac{1}{2}}\,h(\theta_2)^{\frac{1}{2}} \cdot \left[mn\theta_2(\log\theta_1)^2 + (m+n)^2\left(1+\frac{\gamma_2}{\Gamma(\theta_2)} - \frac{\gamma_1^2}{\theta_2\Gamma(\theta_2)^2}\right)\right]^{\frac{-1}{2}}$$

where, $h(\theta_2) = \left[1+\frac{\gamma_2}{\Gamma(\theta_2)} - \frac{\gamma_1^2}{\theta_2\Gamma(\theta_2)^2}\right]\left[\psi'(\theta_2) - \frac{1}{\theta_2}\right] - \frac{1}{\theta_2^2}$,

$$\gamma_i = \int_0^\infty (\log z)^i z^{\theta_2} e^{-z}dz,\ i=1,2, \quad \psi'(\theta_2) = \int_0^\infty \frac{te^{-\theta_2 t}}{1-e^{-t}}dt = \frac{\partial^2}{\partial\theta_2^2}\log\Gamma(\theta_2).$$

Proof. We only prove the result for prior π_4. From Berger and Bernardo(1992), we have the group-ordering reference priors.

Step1.

(1) $h_1(\underline{\theta}) = \frac{|I_{[\sim 00]}|}{|I_{[\sim 11]}|} = \frac{|I(\underline{\theta})|}{|I_{[\sim 11]}(\underline{\theta})|} = \frac{|I_{22}||I_{11} - I_{12}I_{22}^{-1}I_{21}|}{|I_{22}|} = |I_{11} - I_{12}I_{22}^{-1}I_{21}|$

(2) $h_2(\underline{\theta}) = \frac{|I_{[\sim 11]}(\underline{\theta})|}{|I_{[\sim 22]}(\underline{\theta})|} = \frac{(m+n)\theta_4^2}{\theta_3^2}\left(\theta_2 - \frac{(m+n)^2\gamma_1^2}{\Gamma(\theta_2)^2}A(\theta_1, \theta_2)\right)$

(3) $h_3(\underline{\theta}) = \frac{|I_{[\sim 22]}|}{|I_{[\sim 33]}|} = |I_{[\sim 22]}(\underline{\theta})| = \frac{\theta_4^{-2}}{m+n}A(\theta_1, \theta_2)$

Step2.

(1) $$\pi^i_{\underline{\theta},3}(\underline{\theta}_{[\sim 2]}\ \underline{\theta}_{[2]}) = \pi^i_{\underline{\theta},3}(\underline{\theta}_{(3)}\ \underline{\theta}_{[2]}) = \frac{h_3(\underline{\theta})^{0.5}}{\int_{\frac{1}{i}}^{i} h_3(\underline{\theta})^{0.5}d\theta_4}$$

$$= \frac{\left(\frac{\theta_4^{-2}}{m+n}A(\theta_1, \theta_2)\right)^{0.5}}{\int_{\frac{1}{i}}^{i}\left(\frac{\theta_4^{-2}}{m+n}A(\theta_1, \theta_2)\right)^{0.5}d\theta_4} = \frac{\theta_4^{-1}}{2\log i}$$

(2) $$\pi^i_{\underline{\theta},2}(\underline{\theta}_{[\sim 1]}\ \underline{\theta}_{[1]}) = \pi^i_{\underline{\theta},2}(\underline{\theta}_{(2)}, \underline{\theta}_{(3)}\ \underline{\theta}_{[1]})$$

$$= \pi^i_{\underline{\theta},3}(\underline{\theta}_{[2]}, \underline{\theta}_{[2]}) \cdot \frac{e^{\frac{1}{2}E_{\underline{\theta},2}[(\log|h_2(\underline{\theta})|)|\underline{\theta}_{[2]}]}}{\int_{\frac{1}{i}}^{i} e^{\frac{1}{2}E_{\underline{\theta},2}[(\log|h_2(\underline{\theta})|)|\underline{\theta}_{[2]}]}d\theta_3} = \frac{\theta_3^{-1}\theta_4^{-1}}{(2\log i)^2}$$

(3) $\pi^{i}_{\underline{\theta},1}(\underline{\theta}_{[\sim 0]}|\underline{\theta}_{[0]}) = \pi^{i}_{\underline{\theta},2}(\underline{\theta}_{[\sim 1]}|\underline{\theta}_{[1]}) \dfrac{e^{-\frac{1}{2}E^{i}_{\underline{\theta},1}[(\log|h_1(\underline{\theta}))|\underline{\theta}_{[1]}]}}{\int_{\frac{1}{i}}^{i}\int_{\frac{1}{i}}^{i} e^{-\frac{1}{2}E^{i}_{\underline{\theta},1}[(\log|h_1(\underline{\theta}))|\underline{\theta}_{[1]}]d\theta_1 d\theta_2}}$

$$\cdot \frac{\left[\dfrac{\theta_1^{-2}\theta_2 h(\theta_2)}{mm\theta_2(\log\theta_1)^2+(m+n)^2(1+\frac{\gamma_2}{\Gamma(\theta_2)}-\frac{\gamma_1^2}{e_2\Gamma(\theta_2)^2}}\right]^{0.5}}{\int_{\frac{1}{i}}^{i}\int_{\frac{1}{i}}^{i}\left[\dfrac{\theta^{-2}\theta_2 h(\theta_2)}{mm\theta_2(\log\theta_1)^2+(m+n)^2\left(1+\frac{\gamma_2}{\Gamma(\theta_2)}-\frac{\gamma_1^2}{\theta_2\Gamma(\theta_2)^2}\right)}\right]^{0.5} d\theta_1 d\theta_2}.$$

Then we have group-ordering reference prior for $((\theta_1,\theta_2),\theta_3,\theta_4)$ as follows:

$$\pi_R(\underline{\theta}) = \lim_{i\to\infty}\frac{\pi^{i}_{\underline{\theta},1}(\underline{\theta}_{[\sim 0]}|\underline{\theta}_{[0]})}{\pi^{i}_{\underline{\theta},0,1}(\underline{\theta}_{0[\sim 0]}|\underline{\theta}_{0[0]})}$$
$$\propto \theta_1^{-1}\theta_2^{\frac{1}{2}}h(\theta_2)^{-\frac{1}{2}}\left[mm\theta_2(\log\theta_1)^2+(m+n)^2\left(1+\frac{\gamma_2}{\Gamma(\theta_2)}-\frac{\gamma_1^2}{\theta_2\Gamma(\theta_2)^2}\right)\right]^{-\frac{1}{2}}.$$

4. Posterior Distributions

The posterior density of $(\theta_1,\theta_2,\theta_3,\theta_4)$ under a prior π is

$$\pi(\theta_1,\theta_2,\theta_3,\theta_4|\underline{x},\underline{y}) \propto L(\theta_1,\theta_2,\theta_3,\theta_4|\underline{x},\underline{y})\pi(\theta_1,\theta_2,\theta_3,\theta_4)$$

$$=\theta_1^{-1}\theta_2^{b}\theta_2^{-1-(m+n)\theta_2\theta_4}\theta_4^{d+m+n}h(\theta_2)^{-\frac{1}{2}}A(\theta_1,\theta_2)^{e}A(\theta_1,\theta_2)^{f}\Gamma(\theta_2)^{-(m+n)}$$

$$\cdot\left(\prod_{i=1}^{m}x_i\prod_{j=1}^{n}y_i\right)^{\theta_2\theta_4^{-1}} e^{-\sum_{i=1}^{m}\left(\frac{x_i}{\theta_1\theta_3-\frac{n}{m+n}\theta_4^{-1}}\right)^{\theta_4}-\sum_{j=1}^{n}\left(\frac{y_j}{\theta_1\theta_3+\frac{m}{m+n}\theta_4^{-1}}\right)^{\theta_4}}. \qquad (4.1)$$

We first provide the sufficient condition under which the posteriors are proper under $\pi_1,\pi_2,\pi_3,\pi_4,\pi_j$. Note that for almost samples from a continuous distribution. observations are distinct.

Theorem 4.1 All the posterior under π_J and π_1,π_2,π_3,π_4 in (4.1) are proper if $d > -(m+n)$.

Proof. We show that integration is finite.

$$\int_0^\infty\int_0^\infty\int_a^b\int_0^\infty \Theta_1^{-1}\Theta_2^b\Theta_2^{-1-(m+n)\Theta_2\Theta_4}\Theta_4^{d+m+n}h(\Theta_2)^{\frac{1}{2}}A(\Theta_1,\Theta_2)^e A(\Theta_1,\Theta_2)^f\Gamma(\Theta_2)^{-(m+n)}$$

$$\cdot\left(\prod_{i=1}^m x_i\prod_{j=1}^n y_j\right)^{\Theta_2\Theta_4^{-1}} e^{-\sum_{i=1}^m\left(\frac{x_i}{\Theta_1\Theta_3-\frac{n}{m+n}\Theta_4^{-1}}\right)^{\Theta_4}-\sum_{j=1}^n\left(\frac{y_j}{\Theta_1\Theta_3+\frac{m}{m+n}\Theta_4^{-1}}\right)^{\Theta_4}} d\Theta_1 d\Theta_2 d\Theta_3 d\Theta_4$$

$$=\int_a^b\int_0^\infty\int_0^\infty\int_0^\infty u_1^{-1}u_2^b\left(u_1^{\frac{n}{m+n}u_4^{-1}}u_3^{u_4^{-1}}\right)u_4^{m+n+d}h^{\frac{1}{2}}(u_2)$$

$$\cdot\ e^{-\left(u_1^{\frac{n}{m+n}u_4^{-1}}u_3^{u_4^{-1}}\right)^{-u_4}\left[\sum_i\left(\frac{x_i^{u_4}}{u_1^{-\frac{n}{n+m}}}\right)+\sum_j\left(\frac{y_j^{u_4}}{u_1^{\frac{n}{n+m}}}\right)\right]} u_1^{\frac{n}{m+n}u_4^{-1}}u_3^{u_4^{-1}}u_4^{-1}\,du_3\,du_1\,du_4\,du_2$$

$$=\int_a^b\int_0^\infty\int_0^\infty\int_0^\infty u_1^{-1-nu_2}u_2^b u_3^{-(m+n)u_2-1}u_4^{m+n+d-1}$$

$$\cdot\ A^e A_1^f h^{\frac{1}{2}}(u_2)[\Gamma(u_2)]^{-(m+n)}\left(\prod_{i=1}^m x_i\prod_{j=1}^n y_j\right)^{u_2u_4-1}$$

$$\cdot\ e^{-u_1^{-\frac{n}{m+n}}u_3^{-1}\left[\sum_i\left(\frac{x_i^{u_4}}{u_1^{-\frac{n}{m+n}}}\right)+\sum_j\left(\frac{y_j^{u_4}}{u_1^{\frac{m}{m+n}}}\right)\right]} du_3\,du_1\,du_4\,du_2. \qquad (4.2)$$

And,

$$\int_0^\infty u_1^{-1-nu_2}\left[\sum_i x_i^{u_4}+\sum_j\frac{y_j^{u_4}}{u_1}\right]^{-(m+n)u_2}[A(u_1,u_2)]^e[A_1(u_1,u_2)]^f du_1$$

$$\le\int_0^\infty u_1^{-1-nu_2}\left[\sum_i x_i^{u_1}+\sum_j\frac{y_j^{u_4}}{u_1}\right]^{-(m+n)u_2}du_1$$

$$=\int_0^\infty\left(\frac{\sum_j y_j^{u_4}}{\sum_i x_i^{u_4}}w\right)^{-1+nu_2}[1+w]^{-(m+n)u_2}\cdot\frac{\sum_j y_j^{u_4}}{\sum_i x_i^{u_4}}\left(\sum_j y_j^{u_4}\right)^{-(m+n)u_2}du_1$$

$$=\left(\frac{\sum_j y_j^{u_4}}{\sum_i x_i^{u_4}}\right)^{\mu_2}\left(\sum_j y_j^{u_4}\right)^{-(m+n)u_2}\int_0^1\left(\frac{v}{1-v}\right)^{\mu_2-1}\left(\frac{1}{1-v}\right)^{-(m+n)u_2}\frac{1}{(1-v)^2}\,dv$$

$$=\left(\sum_i x_i^{u_4}\right)^{-mu_2}\left(\sum_j y_j^{u_4}\right)^{-nu_2}\cdot\frac{\Gamma(mu_2)\Gamma(nu_2)}{\Gamma((m+n)u_2)}.$$

Hence, we have

$$(4.2)=\int_a^b\int_0^\infty u_2^b u_5^{m+n+d-1}h^{\frac{1}{2}}(u_2)[\Gamma(u_2)]^{-(m+n)}\left(\prod x_i\prod y_j\right)^{u_2u_4-1}$$
$$\left(\sum_i x_i^{u_4}\right)^{-mu_2}\left(\sum_j y_j^{u_4}\right)^{-nu_2}\Gamma(mu_2)\Gamma(nu_2)\,du_4\,du_2. \qquad (4.3)$$

And,

$$\int_0^\infty u_4^{m+n+d-1}\left(\prod x_i\prod y_j\right)^{u_2u_4-1}\left(\sum_i x_i^{u_4}\right)^{-mu_2}\left(\sum_j y_j^{u_4}\right)^{-nu_2}du_4$$

$$< \left(\prod x_i \prod y_j\right)^{-1} \int_0^\infty u_4^{m+n+d-1} \left[\frac{x_k}{\max x_i} \frac{y_l}{\max y_j} \right]^{u_2 u_4} du_4$$

$$= \left(\prod x_i \prod y_j\right)^{-1} \int_0^\infty u_4^{m+n+d-1} e^{-u_4\left[-u_2 \log\left(\frac{x_k}{\max x_i} \frac{y_l}{\max y_j}\right)\right]} du_4$$

$$= \left(\prod x_i \prod y_j\right)^{-1} \Gamma[m+n+d] e^{-u_4\left[-u_2 \log\left(\frac{x_k}{\max x_i} \frac{y_l}{\max y_j}\right)\right]^{-(m+n)-d}}, \quad \text{if } d > -(m+n).$$

Therefore,

$$(4.3) \quad < \int_a^b u_2^{b-(m+n)-d} \left(\prod x_i \prod y_j\right)^{-1} h^{-\frac{1}{2}}(u_2) \frac{\Gamma(mu_2)\Gamma(nu_2)}{[\Gamma(u_2)]^{m+n}} \cdot \Gamma(d+m+n) \left[-\log\left(\frac{x_k}{\max x_i} \frac{y_l}{\max y_j}\right)\right]^{-(m+n)-d} du_2$$

$$= \left(\prod x_i \prod y_i\right)^{-1} \left[-\log\left(\frac{x_k}{\max x_i} \frac{y_l}{\max y_j}\right)\right]^{-(m+n)-d} \Gamma(d+m+n) \cdot \int_a^b u_2^{b-(m+n)-d} h^{-\frac{1}{2}}(u_2) \frac{\Gamma(mu_2)\Gamma(nu_2)}{[\Gamma(u_2)]^{m+n}} du_2 < \infty, \quad \text{if } d > -(m+n).$$

Hence, the proof is complete.

Finally, under the priors $\pi_1, \pi_2, \pi_3, \pi_4, \pi_j$, the marginal posterior densities of (Θ_1, Θ_2) are, respectively, given by

$$\pi(\Theta_1, \Theta_2 | \underline{x}, \underline{y}) \propto \Theta_2^b h(\Theta_2)^{-\frac{1}{2}} A(\Theta_1, \Theta_2)^e A_1(\Theta_1, \Theta_2)^f \Gamma(\Theta_2)^{-(m+n)}$$

$$\cdot \int_0^\infty \Theta_1^{1-\frac{n(-2)}{(m+n)u_4}-n\Theta_2} u_4^{-d+m+n-1} \left(\prod_{i=1}^m x_i \prod_{j=1}^n y_j\right)^{\Theta_2 u_4}$$

$$\cdot \left(\sum_{i=1}^m x_i^{u_4} + \frac{1}{\Theta_1} \sum_{j=1}^n y_j^{u_4}\right)^{\frac{2}{u_4}-(m+n)\Theta_2+1} \Gamma\left[(m+n)\Theta_2 + \frac{-2}{u_4}\right] du_4.$$

5. Simulation Study and Discussion

In this section, we compare with Jeffreys' prior π_j and reference priors$\pi_1, \pi_2, \pi_3, \pi_4$in finding the estimated values for system reliability R, in (1.3).

First, Table 1 gives true values and the corresponding estimated values of when $m = n = 3$. Table 2 provides true values and the corresponding estimated values of when $m = n = 5$. Table 3 gives true values and the corresponding estimated values of when $m = n = 10MMMMMMsMMMMgc$.

For most of the cases presented in Table1, Table 2, and Table 3, we see that the reference prior π_4 is the most appealing reference in the sense of the asymptotic frequentist coverage probability in estimating the system reliability R.

<표 1> m=n=3인 경우

m	n	η_1	η_2	β	p	R	R_{π_1}	R_{π_2}	R_{π_3}	R_{π_4}	R_{π_J}
3	3	2	3	1	0.5	0.5640942	0.6763159	0.9303459	0.7072762	0.9891236	0.8068397
3	3	2	3	1	1	0.6	0.7320398	0.8887196	0.7637387	0.7662798	0.8066162
3	3	2	3	1	1.5	0.62647	0.8157985	0.975333	0.7947148	0.7738208	0.8057292
3	3	2	3	1	2	0.648	0.8915453	0.8857595	0.8100853	0.6896189	0.9977942
3	3	2	4	1	0.5	0.6081734	0.7919005	0.9149504	0.9368465	0.9993618	0.8631885
3	3	2	4	1	1	0.6666667	0.8037823	0.9986874	0.9072696	0.8106848	0.7129784
3	3	2	4	1	1.5	0.7082087	0.9354458	0.9346502	0.7987184	0.992343	0.8327531
3	3	2	4	1	2	0.7407407	0.7794441	0.9117454	0.822892	0.7152262	0.9980103
3	3	3	3	1	0.5	0.5	0.7350874	0.7667848	0.8547061	0.6515795	0.7041093
3	3	3	3	1	1	0.5	0.7305751	0.8435472	0.7426292	0.6994637	0.8317172
3	3	3	3	1	1.5	0.5000001	0.8183573	0.8880482	0.9794997	0.933676	0.8229178
3	3	3	3	1	2	0.5	0.771468	0.9549122	0.7848341	0.7107413	0.7708204
3	3	2	3	2	0.5	0.6256659	0.7693039	0.8486542	0.8219729	0.7496999	0.7767322
3	3	2	3	2	1	0.6923077	0.9806806	0.9743205	0.7655623	0.7845934	0.961524
3	3	2	3	2	1.5	0.7386755	0.9285827	0.9370584	0.916118	0.845535	0.8757116
3	3	2	3	2	2	0.7742376	0.935032	0.9209068	0.990694	0.9930293	0.8800817
3	3	2	4	2	0.5	0.7048328	0.9474486	0.9329281	0.936776	0.8399856	0.997086
3	3	2	4	2	1	0.8	0.8106132	0.997562	0.9998063	0.9899002	0.9985567
3	3	2	4	2	1.5	0.8576216	0.9661866	0.9990467	0.8904041	0.8882407	0.9705585
3	3	2	4	2	2	0.896	0.8859092	0.9947093	0.9476928	0.9162166	0.9685488
3	3	3	3	2	0.5	0.5	0.8264827	0.966162	0.7403909	0.8049553	0.7557124
3	3	3	3	2	1	0.5	0.8870768	0.8915157	0.8050747	0.7769243	0.8981415
3	3	3	3	2	1.5	0.5000001	0.8230363	0.9159304	0.983555	0.883652	0.8868418
3	3	3	3	2	2	0.5	0.8900847	0.8960143	0.8673937	0.9229703	0.8922516
3	3	2	3	4	0.5	0.7337501	0.9024726	0.8473498	0.9077516	0.8998373	0.8818556
3	3	2	3	4	1	0.8350515	0.8366942	0.9322558	0.9217282	0.8756923	0.9837059
3	3	2	3	4	1.5	0.8920763	0.999827	0.9975522	0.992601	0.8345219	0.9999804
3	3	2	3	4	2	0.9273519	0.9999565	0.9655504	0.9726307	0.9333059	0.979955
3	3	2	4	4	0.5	0.8440417	0.9200995	0.9269008	0.967668	0.8284924	0.9423876
3	3	2	4	4	1	0.9411765	0.9603596	0.995928	0.968487	0.989428	0.9949647
3	3	2	4	4	1.5	0.976212	0.9996892	0.9873865	0.9998506	0.9996895	0.982117
3	3	2	4	4	2	0.9900265	0.9998313	0.9976572	0.997525	0.9993221	0.9995647
3	3	3	3	4	0.5	0.5	0.7791746	0.872985	0.9996454	0.8068887	0.870426
3	3	3	3	4	1	0.5	0.9206929	0.8991258	0.89782	0.8444409	0.9104857
3	3	3	3	4	1.5	0.5000001	0.8480261	0.8410351	0.891418	0.9266263	0.9948316
3	3	3	3	4	2	0.5	0.907096	0.9261687	0.9278558	0.9439784	0.8751398

<표 2> m=n=5인 경우

m	n	η_1	η_2	β	p	R	R_{π_1}	R_{π_2}	R_{π_3}	R_{π_4}	R_{π_J}
5	5	2	3	1	0.5	0.5640942	0.7138311	0.9520714	0.775351	0.6763472	0.8442714
5	5	2	3	1	1	0.6	0.8610336	0.8420171	0.7230919	0.8651229	0.6977819
5	5	2	3	1	1.5	0.62647	0.8385476	0.8556645	0.7715734	0.7096474	0.7705347
5	5	2	3	1	2	0.648	0.8705392	0.9876841	0.7964402	0.7302701	0.8452668
5	5	2	4	1	0.5	0.6081734	0.7704975	0.8083388	0.7933741	0.7436597	0.6984719
5	5	2	4	1	1	0.6666667	0.7807283	0.8416024	0.8089087	0.7690397	0.7783236
5	5	2	4	1	1.5	0.7082087	0.7713932	0.9552702	0.8543788	0.858318	0.9076503
5	5	2	4	1	2	0.7407407	0.8550945	0.9162533	0.8561232	0.9028517	0.9944325
5	5	3	3	1	0.5	0.5	0.8626274	0.899558	0.6824657	0.679969	0.7364923
5	5	3	3	1	1	0.5	0.7311528	0.896216	0.7961504	0.6909574	0.9540556
5	5	3	3	1	1.5	0.5000001	0.8813158	0.6976831	0.7397367	0.7407978	0.6990024
5	5	3	3	1	2	0.5	0.7586675	0.7751647	0.7796921	0.8988693	0.7960397
5	5	2	3	2	0.5	0.6256659	0.8029925	0.8189167	0.7930246	0.8035396	0.9842605
5	5	2	3	2	1	0.6923077	0.9008463	0.9692164	0.8752083	0.8277525	0.8509223
5	5	2	3	2	1.5	0.7386755	0.9178708	0.963217	0.9271723	0.8468131	0.9590322
5	5	2	3	2	2	0.7742376	0.9680273	0.9143512	0.9574344	0.811945	0.8500265
5	5	2	4	2	0.5	0.7048328	0.8312635	0.9471247	0.7964894	0.6711075	0.7655572
5	5	2	4	2	1	0.8	0.9170735	0.9939561	0.8281642	0.8422077	0.97282
5	5	2	4	2	1.5	0.8576216	0.9915162	0.9660986	0.925377	0.9856775	0.8363102
5	5	2	4	2	2	0.896	0.9809097	0.9570868	0.9708009	0.9994161	0.9962482
5	5	3	3	2	0.5	0.5	0.7217885	0.8249169	0.7511006	0.813546	0.7215128
5	5	3	3	2	1	0.5	0.8394517	0.7713073	0.9935671	0.7213605	0.9304447
5	5	3	3	2	1.5	0.5000001	0.8361466	0.9419938	0.815464	0.7602452	0.846441
5	5	3	3	2	2	0.5	0.8004836	0.7609048	0.7781604	0.8047272	0.8525886
5	5	2	3	4	0.5	0.7337501	0.7877088	0.9013226	0.9128516	0.7903254	0.9931733
5	5	2	3	4	1	0.8350515	0.8983503	0.8108206	0.9744678	0.8396952	0.9720256
5	5	2	3	4	1.5	0.8920763	0.8524645	0.9474888	0.9829517	0.9614674	0.971106
5	5	2	3	4	2	0.9273519	0.9911672	0.9376804	0.989575	0.9991259	0.9704803
5	5	2	4	4	0.5	0.8440417	0.8410482	0.9401067	0.883204	0.982437	0.9613316
5	5	2	4	4	1	0.9411765	0.983475	0.9925347	0.9410714	0.9971679	0.9974766
5	5	2	4	4	1.5	0.976212	0.9858154	0.9682848	0.9942526	0.9941802	0.9928115
5	5	2	4	4	2	0.9900265	0.999804	0.9985424	0.9997529	0.9977974	0.9844516
5	5	3	3	4	0.5	0.5	0.81871	0.8213902	0.8232214	0.7821314	0.73662
5	5	3	3	4	1	0.5	0.7415672	0.8066439	0.7518998	0.750507	0.8241635
5	5	3	3	4	1.5	0.5000001	0.847434	0.8959736	0.8164705	0.885639	0.8308881
5	5	3	3	4	2	0.5	0.8699632	0.8936967	0.8270409	0.777907	0.8369532

<표 3> m=n=10인 경우

m	n	η_1	η_2	β	p	R	R_{π_1}	R_{π_2}	R_{π_3}	R_{π_4}	R_{π_J}
10	10	2	3	1	0.5	0.5640942	0.903501	0.9450703	0.9867327	0.9571915	0.9475679
10	10	2	3	1	1	0.6	0.9432705	0.83113	0.8987408	0.8758291	0.9032654
10	10	2	3	1	1.5	0.62647	0.9284782	0.9454032	0.9325997	0.939437	0.9019711
10	10	2	3	1	2	0.648	0.914529	0.9514746	0.8733395	0.900304	0.9087344
10	10	2	4	1	0.5	0.6081734	0.9407826	0.9445631	0.9405827	0.9456175	0.950064
10	10	2	4	1	1	0.6666667	0.9286306	0.969512	0.966197	0.8412887	0.9853377
10	10	2	4	1	1.5	0.7082087	0.969329	0.8495778	0.924983	0.9946606	0.901559
10	10	2	4	1	2	0.7407407	0.9396492	0.8885324	0.8907007	0.9541818	0.9543072
10	10	3	3	1	0.5	0.5	0.9238762	0.917644	0.8529576	0.9188132	0.9337285
10	10	3	3	1	1	0.5	0.953039	0.9219325	0.9529463	0.899834	0.9549323
10	10	3	3	1	1.5	0.5000001	0.9643788	0.902766	0.9475112	0.9164376	0.9006962
10	10	3	3	1	2	0.5	0.8906137	0.9004815	0.9030595	0.9170621	0.8721407
10	10	2	3	2	0.5	0.6256659	0.8631316	0.919817	0.9656724	0.9670031	0.960507
10	10	2	3	2	1	0.6923077	0.876314	0.9188632	0.9213576	0.8971003	0.8432176
10	10	2	3	2	1.5	0.7386755	0.9741861	0.9220065	0.931117	0.909229	0.9566228
10	10	2	3	2	2	0.7742376	0.9390027	0.8676227	0.9538281	0.9819643	0.9950951
10	10	2	4	2	0.5	0.7048328	0.9458886	0.8944235	0.984629	0.976169	0.924509
10	10	2	4	2	1	0.8	0.9930665	0.8974163	0.9992181	0.9692454	0.9655957
10	10	2	4	2	1.5	0.8576216	0.8801858	0.8506843	0.9944486	0.9686476	0.9952531
10	10	2	4	2	2	0.896	0.9568383	0.8926308	0.996705	0.9988799	0.986991
10	10	3	3	2	0.5	0.5	0.8901642	0.8924577	0.9064287	0.8862207	0.9518913
10	10	3	3	2	1	0.5	0.9201129	0.9425036	0.8730165	0.9161049	0.979195
10	10	3	3	2	1.5	0.5000001	0.8943501	0.9421084	0.9692358	0.9255642	0.902188
10	10	3	3	2	2	0.5	0.8552568	0.9291671	0.8775429	0.874215	0.856066
10	10	2	3	4	0.5	0.7337501	0.8712832	0.9496198	0.9260478	0.962545	0.9135493
10	10	2	3	4	1	0.8350515	0.9980975	0.9872926	0.9934905	0.99792	0.9737439
10	10	2	3	4	1.5	0.8920763	0.9903227	0.9843516	0.9957782	0.989355	0.9982992
10	10	2	3	4	2	0.9273519	0.993965	0.941704	0.9996602	0.9998974	0.9999798
10	10	2	4	4	0.5	0.8440417	0.9494341	0.9847138	0.9576175	0.9744882	0.9858268
10	10	2	4	4	1	0.9411765	0.9965184	0.9978916	0.9951128	0.9963409	0.9992464
10	10	2	4	4	1.5	0.976212	0.9998456	0.9995296	0.999562	0.9999603	0.9982855
10	10	2	4	4	2	0.9900265	0.9998845	0.9993998	0.9998344	0.9999987	0.999746
10	10	3	3	4	0.5	0.5	0.8178472	0.9261997	0.931591	0.8769788	0.9241865
10	10	3	3	4	1	0.5	0.9158736	0.8740084	0.9114375	0.9181079	0.8593685
10	10	3	3	4	1.5	0.5000001	0.93088	0.8867447	0.8701955	0.8533307	0.8448448
10	10	3	3	4	2	0.5	0.8159069	0.8246307	0.8850831	0.9154994	0.8147906

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