KYUNGPOOK Math. J. 51(2011), 293-300
http://dx.doi.org/10.5666/KMJ.2011.51.3.293

Path-Connectivity of Two-Interval MSF Wavelets

Divya Singh
Department of Mathematics, University of Allahabad, Allahabad - 211 002, India
e-mail : divya_allduniv@yahoo.com

Abstract. In this paper, we obtain that the space $\mathcal{W}_2$ of minimally supported frequency wavelets, the supports of whose Fourier transforms consist of two intervals, is path-connected.

1. Introduction

The study of topological notion of path-connectedness for the space of orthonormal wavelets was initiated by Dai and Larson [1]. In [9], a similar study has been made and it is obtained that the space $\mathcal{W}^M$ of MRA wavelets is path-connected. Latter, in [8], it has been proved that the space $\mathcal{W}^F$ of MSF wavelets (s-elementary wavelets) is path-connected. Further, path-connectedness of such spaces of wavelets have been considered for higher dimensions as well [5, 6, 8].

In this paper, we consider the set $\mathcal{W}_2$ of one-dimensional MSF wavelets, the supports of whose Fourier transforms consist of two intervals. These are known to be MRA wavelets [3]. Thus $\mathcal{W}_2$ being a subset of $\mathcal{W}^M$ derives induced topology on it. It has been observed that the path joining the Shannon wavelet ψ_0 and a member ψ of $\mathcal{W}_2$ not equal to ψ_0, described in [9], does not lie in $\mathcal{W}_2$. Thus, a question of path-connectivity of $\mathcal{W}_2$ arises. We answer this question in affirmative.

2. Pre-Requisites

Let $L^1(\mathbb{R})$ be the collection of all Lebesgue integrable functions on $\mathbb{R}$ and $L^2(\mathbb{R})$ be that of all Lebesgue square integrable functions on $\mathbb{R}$. With the usual addition and scalar multiplication of functions together with the inner-product $\langle f, g\rangle$ of $f, g \in L^2(\mathbb{R})$ defined by

$$\langle f, g\rangle = \int_{\mathbb{R}} f(x)\overline{g(x)}\, dx,$$

Received July 17, 2010; accepted March 7, 2011.
2010 Mathematics Subject Classification: 42C40.
Key words and phrases: Wavelet set, MSF wavelet, Multiresolution analysis.

$L^2(\mathbb{R})$ becomes a Hilbert space. The Fourier transform $\mathcal{F}$ is defined by

$$\hat{f}(s) \equiv (\mathcal{F}f)(s) = \int_{\mathbb{R}} f(t)e^{-ist}\,dt,$$

where $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. This Fourier transform can be extended uniquely to an operator on $L^2(\mathbb{R})$.

An *orthonormal wavelet* (or, a *wavelet*) in $L^2(\mathbb{R})$ is a function $\psi \in L^2(\mathbb{R})$ with unit norm such that the family $\left\{2^{n/2}\psi(2^n \cdot -l) \,:\, n, l \in \mathbb{Z}\right\}$ constitutes an orthonormal basis for $L^2(\mathbb{R})$.

One of the methods of constructing orthonormal wavelets is based on the existence of a family of closed subspaces of $L^2(\mathbb{R})$ satisfying certain properties. Such a family is called a *multiresolution analysis* or, simply an MRA.

Definition 2.1([7]). A sequence of closed subspaces $(V_j)_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R})$, together with a function $\varphi \in V_0$ is called a *multiresolution analysis* if it satisfies the following conditions:

(1) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$,

(2) $f \in V_j$ if and only if $f(2(\cdot)) \in V_{j+1}$ for all $j \in \mathbb{Z}$,

(3) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$,

(4) $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$,

(5) $\{\varphi(\cdot - k) : k \in \mathbb{Z}\}$ forms an orthonormal basis for V_0.

The function φ is called a *scaling function* of the given MRA. An MRA gives rise to a wavelet ψ which lies in the orthogonal complement of V_0 in V_1. A wavelet arising from an MRA is called an MRA *wavelet.* A scaling function φ for an MRA provides a 2π-periodic function m, known to be the *low-pass filter associated with* φ which satisfies the following:

(i) $\hat{\varphi}(2\xi) = m(\xi)\hat{\varphi}(\xi)$,

(ii) $|m(\xi)|^2 + |m(\xi+\pi)|^2 = 1$,

(iii) $\hat{\psi}(\xi) = e^{i\xi/2}\overline{m(\frac{\xi}{2}+\pi)}\hat{\varphi}(\frac{\xi}{2})$,

where the equalities above are all in the almost everywhere (a.e.) sense.

The following is a well-known characterization for a function $\varphi \in L^2(\mathbb{R})$ to be a scaling function.

Theorem 2.1([4]). *A function $\varphi \in L^2(\mathbb{R})$ is a scaling function for an MRA iff*

(i) $\sum_{k\in\mathbb{Z}} |\hat{\varphi}(\xi + 2k\pi)|^2 = 1$, *for a.e.* $\xi \in [-\pi, \pi)$,

(ii) $\lim_{j\to\infty} |\hat{\varphi}(2^{-j}\xi)| = 1$, *for a.e.* $\xi \in \mathbb{R}$,

(iii) *there exists a* 2π*-periodic function* $m \in L^2([-\pi,\pi))$ *such that* $\hat{\varphi}(2\xi) = m(\xi)\hat{\varphi}(\xi)$*, for a.e.* $\xi \in \mathbb{R}$.

From the characterization for a function $\varphi \in L^2(\mathbb{R})$ to be a scaling function stated as Theorem 2.1, it follows that if φ is a scaling function, then the function $\tilde{\varphi}$ whose Fourier transform is $|\hat{\varphi}|$, is also a scaling function.

Definition 2.2([9]). A measurable function ν is called a *functional wavelet multiplier* if the inverse Fourier transform of $\nu\hat{\psi}$ is a wavelet, whenever ψ is a wavelet.

The functional wavelet multipliers are characterized by the following theorem.

Theorem 2.2([9]). *A measurable function* ν *is a functional wavelet multiplier iff it is unimodular and* $\nu(2t)/\nu(t)$ *is a.e. equal to a* 2π*-periodic function.*

Notation ([9]).

(1) For an MRA wavelet ψ_0, S_{ψ_0} denotes the set

$$\left\{\psi \in L^2\left(\mathbb{R}\right) : \psi \text{ is an MRA wavelet and } |\hat{\varphi}_0(\xi)| = |\hat{\varphi}(\xi)|, \text{ a.e.}\right\},$$

where φ_0 and φ are scaling functions associated with ψ_0 and ψ, respectively.

(2) For a wavelet ψ_0, $\mathcal{M}_{\psi_0}$ denotes the set

$$\left\{\psi \in L^2\left(\mathbb{R}\right) : \hat{\psi} = \nu\hat{\psi}_0, \text{ where } \nu \text{ is a functional wavelet multiplier}\right\}.$$

Theorem 2.3([9]). *If* ψ_0 *is an MRA wavelet, then* $S_{\psi_0} = \mathcal{M}_{\psi_0}$.

The following result related with the path-connectivity of MRA wavelets can be found in [9].

Theorem 2.4. *The space* $\mathcal{W}^M$ *of all MRA wavelets is path-connected.*

The proof of the above result consists of two parts. In the first part, it is proved that for a wavelet ψ, the class $\mathcal{M}_\psi$ is path-connected. In the second part, the Shannon wavelet ψ_0 is considered. For an MRA wavelet ψ, on account of Theorem 2.1, an appropriate element $\psi_1 \in S_\psi$ is selected in such a way that it is associated with a scaling function φ_1 for which $\hat{\varphi}_1 \geq 0$. If m_1 is the low-pass filter associated with the scaling function φ_1, then

$$\hat{\psi}_1(\xi) = e^{i\xi/2}\overline{m_1\left(\frac{\xi}{2}+\pi\right)}\hat{\varphi}_1\left(\frac{\xi}{2}\right).$$

The Shannon wavelet ψ_0 satisfies

$$\hat{\psi}_0(\xi) = e^{i\xi/2}\overline{m_0\left(\frac{\xi}{2}+\pi\right)}\hat{\varphi}_0(\frac{\xi}{2}),$$

where $\hat{\varphi}_0(\xi) = \chi_{[-\pi,\pi)}(\xi)$ and m_0 is the low-pass filter associated with φ_0. Indeed, m_0 on $[-\pi, \pi)$ is given by

$$(2.1)\qquad m_0(\xi) = \frac{\hat{\varphi}_0(2\xi)}{\hat{\varphi}_0(\xi)} = \begin{cases} 1, & \text{if } \xi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right), \\ 0, & \text{if } \xi \in \left[-\pi, -\frac{\pi}{2}\right) \cup \left[\frac{\pi}{2}, \pi\right). \end{cases}$$

For $t \in [0,1]$, a function m_t on $[-\pi,\pi)$ is defined as follows:
(2.2)

$$m_t(\xi) = \begin{cases} (1-t)m_0(\xi) + tm_1(\xi), & \text{if } \xi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus \left[-(1-t)\frac{\pi}{2}, (1-t)\frac{\pi}{2}\right), \\ 1, & \text{if } \xi \in \left[-(1-t)\frac{\pi}{2}, (1-t)\frac{\pi}{2}\right), \\ \sqrt{1-m_t^2(\xi+\pi)}, & \text{if } \xi \in \left[-\pi, -\frac{\pi}{2}\right), \\ \sqrt{1-m_t^2(\xi-\pi)}, & \text{if } \xi \in \left[\frac{\pi}{2}, \pi\right). \end{cases}$$

Extension of m_t to a 2π-periodic function on $\mathbb{R}$ yields a low-pass filter. This family $\{m_t : t \in [0,1]\}$ of filters connecting m_0 to m_1 is used to define scaling functions $\{\varphi_t : t \in [0,1]\}$ from which the desired path-connectivity between ψ_0 and ψ_1, and hence between ψ_0 and ψ has been obtained.

3. Path-Connectedness of $\mathcal{W}_2$

In [4], it has been shown that if $\psi \in L^2(\mathbb{R})$ is a wavelet, then $|\text{supp}(\hat{\psi})| \geq 2\pi$. Wavelets, whose Fourier transforms have minimal support of measure 2π are called *Minimally Supported Frequency* (MSF) *Wavelets* by Fang and Wang [2]. Independent of Fang and Wang, Dai and Larson [1], studied the same class of wavelets and introduced the concept of wavelet sets.

A measurable set $W \subset \mathbb{R}$ is said to be a *wavelet set* if $|\hat{\psi}| = \chi_W$ for some wavelet ψ in $L^2(\mathbb{R})$. Such a wavelet ψ is called an *s-elementary wavelet* [1]. MSF wavelets are indeed those wavelets which are associated with wavelet sets. One of the earliest examples of wavelet sets is the Shannon or Littlewood-Paley wavelet set $[-2\pi, -\pi) \cup [\pi, 2\pi)$.

Ha, Kang, Lee and Seo [3], characterized wavelet sets in $\mathbb{R}$ having two intervals and proved that the wavelets arising from those two-interval wavelet sets are MRA wavelets. Such wavelets are termed to be *two-interval* MSF *wavelets*. The collection of all two-interval MSF wavelets will be denoted by $\mathcal{W}_2$. Wavelet sets with two intervals are precisely

$$[2a - 4\pi, a - 2\pi] \cup [a, 2a],$$

for some $0 < a < 2\pi$, which we denote by $W(a)$ ([3]). With no loss of generality we may exclude the right end points of the intervals constituting these wavelet sets.

Observation. Consider the Shannon wavelet ψ_0, where $|\hat{\psi_0}| = \chi_{W(\pi)}$ and a member ψ_1 in $\mathcal{W}_2$, for which $|\hat{\psi_1}| = \chi_{W(\frac{\pi}{2})}$. The low-pass filter m_0 for ψ_0 is given by (2.1), while that m_1 for ψ_1 is given by

$$m_1(\xi) = \begin{cases} 1, & \text{if } \xi \in \left[-\frac{3\pi}{4}, \frac{\pi}{4}\right), \\ 0, & \text{if } \xi \in \left[-\pi, -\frac{3\pi}{4}\right) \cup \left[\frac{\pi}{4}, \pi\right), \end{cases}$$

extended 2π-periodically to $\mathbb{R}$.

These wavelets are joined by a path $\sigma : [0, 1] \to \mathcal{W}^M$ such that the low-pass filter m_t of $\sigma(t) \equiv \psi_t$ is determined by (2.2).

For $t = \frac{1}{2}$,

$$m_{1/2}(\xi) = \begin{cases} 0, & \text{if } \xi \in \left[-\pi, -\frac{3\pi}{4}\right), \\ \frac{\sqrt{3}}{2}, & \text{if } \xi \in \left[-\frac{3\pi}{4}, -\frac{\pi}{2}\right), \\ 1, & \text{if } \xi \in \left[-\frac{\pi}{2}, \frac{\pi}{4}\right), \\ \frac{1}{2}, & \text{if } \xi \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right), \\ 0, & \text{if } \xi \in \left[\frac{\pi}{2}, \pi\right). \end{cases}$$

The Fourier transform of the corresponding scaling function $\varphi_{1/2}$ is given by

$$\hat{\varphi}_{1/2}(\xi) = \prod_{j=1}^{\infty} m_{1/2}(2^{-j}\xi)$$

$$= \begin{cases} 0, & \text{if } \xi \in \left(-\infty, -\frac{3\pi}{2}\right), \\ \frac{\sqrt{3}}{2}, & \text{if } \xi \in \left[-\frac{3\pi}{2}, -\pi\right), \\ 1, & \text{if } \xi \in \left[-\pi, \frac{\pi}{2}\right), \\ \frac{1}{2}, & \text{if } \xi \in \left[\frac{\pi}{2}, \pi\right), \\ 0, & \text{if } \xi \in [\pi, \infty), \end{cases}$$

and the Fourier transform of the wavelet $\psi_{1/2}$ is given by

$$\hat{\psi}_{1/2}(\xi) = e^{\frac{i\xi}{2}} \overline{m_{1/2}\left(\frac{\xi}{2} + \pi\right)} \hat{\varphi}_{1/2}\left(\frac{\xi}{2}\right)$$

$$= \begin{cases} 0, & \text{if } \xi \in (-\infty, -3\pi) \cup [2\pi, \infty), \\ \frac{\sqrt{3}}{2} e^{\frac{i\xi}{2}}, & \text{if } \xi \in [-3\pi, -2\pi), \\ e^{\frac{i\xi}{2}}, & \text{if } \xi \in \left[-2\pi, -\frac{3\pi}{2}\right), \\ \frac{1}{2} e^{\frac{i\xi}{2}}, & \text{if } \xi \in \left[-\frac{3\pi}{2}, -\pi\right), \\ 0, & \text{if } \xi \in \left[-\pi, \frac{\pi}{2}\right), \\ \frac{\sqrt{3}}{2} e^{\frac{i\xi}{2}}, & \text{if } \xi \in \left[\frac{\pi}{2}, \pi\right), \\ \frac{1}{2} e^{\frac{i\xi}{2}}, & \text{if } \xi \in [\pi, 2\pi). \end{cases}$$

Clearly, the wavelet $\psi_{1/2}$ does not lie in $\mathcal{W}_2$. Indeed, computation can be made to check that no point of the path σ other than $\sigma(0)$ and $\sigma(1)$ lies in $\mathcal{W}_2$.

Now we prove that $\mathcal{W}_2$ is path-connected. The technique of the proof involves adjustment of the intermediate filters which join the low-pass filter m_0 of the Shannon wavelet to the low-pass filter m_a^1 of a two-interval MSF wavelet ψ_a^1 with $\operatorname{supp}(\hat{\psi}_a^1) = [2a - 4\pi, a - 2\pi) \cup [a, 2a)$, keeping the path joining ψ_0 and ψ_a^1 in $\mathcal{W}_2$. Here it may be noticed that if ψ_0 is a two-interval MSF wavelet and $|\hat{\psi}_0| = \chi_{W(a)}$, where $a \in (0, 2\pi)$, then for each member ψ of $\mathcal{M}_{\psi_0}$, $|\hat{\psi}| = \chi_{W(a)}$.

Theorem 3.1. *The space* $\mathcal{W}_2$ *is path-connected.*

Proof. Let ψ_0 be the Shannon wavelet. Then the low-pass filter m_0 for the Shannon wavelet is given by (2.1).

Suppose ψ_a is a two-interval MSF wavelet given by

$$|\hat{\psi}_a(\xi)| = \chi_{W(a)}(\xi),$$

where $0 < a < 2\pi$. Then the modulus of the Fourier transform of the scaling function φ_a comes out to be

$$|\hat{\varphi}_a(\xi)| = \chi_{[a-2\pi,\, a)}(\xi).$$

Choose $\psi_a^1 \in S_{\psi_a} = \mathcal{M}_{\psi_a}$ such that it is associated with the scaling function φ_a^1, whose Fourier transform is given by

$$\hat{\varphi}_a^1 = |\hat{\varphi}_a| = \chi_{[a-2\pi,\, a)}.$$

Clearly $\hat{\varphi}_a^1 \geq 0$. The low-pass filter m_a^1 associated with φ_a^1 on $[a-2\pi,\, a)$ is given by

$$m_a^1(\xi) = \frac{\hat{\varphi}_a^1(2\xi)}{\hat{\varphi}_a^1(\xi)} = \begin{cases} 1, & \text{if } \xi \in \left[\frac{a-2\pi}{2},\, \frac{a}{2}\right), \\ 0, & \text{if } \xi \in \left[a-2\pi, \frac{a-2\pi}{2}\right) \cup \left[\frac{a}{2}, a\right), \end{cases}$$

which is then extended 2π-periodically to $\mathbb{R}$.
Also, we have

$$|\hat{\psi}_a^1(\xi)| = \chi_{W(a)}(\xi).$$

Now, for each $s \in [0,1]$, we define the function m_a^s on $[-\pi + (a-\pi)s, \pi + (a-\pi)s)$ as follows:

$$m_a^s(\xi) = \begin{cases} 1, & \text{if } \xi \in \left[-\frac{\pi}{2} + \frac{(a-\pi)s}{2},\, \frac{\pi}{2} + \frac{(a-\pi)s}{2}\right), \\ 0, & \text{if } \xi \in \left[-\pi + (a-\pi)s, -\frac{\pi}{2} + \frac{(a-\pi)s}{2}\right) \cup \left[\frac{\pi}{2} + \frac{(a-\pi)s}{2}, \pi + (a-\pi)s\right), \end{cases}$$

and extend it 2π-periodically to the whole of $\mathbb{R}$. For $\xi \in \mathbb{R}$, we set

$$\hat{\varphi}_a^s(\xi) = \prod_{j=1}^{\infty} m_a^s(2^{-j}\xi).$$

Since $0 \leq m_a^s(\xi) \leq 1$, this product is well-defined. For

$$\xi \in [2^j(\pi + (a-\pi)s), 2^{j+1}(\pi + (a-\pi)s)), \text{ with } j \geq 0,$$

we have

$$t \equiv 2^{-(j+1)}\xi \in \left[\frac{\pi}{2} + \frac{(a-\pi)s}{2}, \pi + (a-\pi)s\right),$$

and hence $m_a^s(t) = 0$. Thus

$$\begin{aligned} \hat{\varphi}_a^s(\xi) &= \hat{\varphi}_a^s(2^{j+1}t) = m_a^s(2^j t)\hat{\varphi}_a^s(2^j t) \\ &= m_a^s(2^j t) m_a^s(2^{j-1}t) \cdots m_a^s(t) \textstyle\prod_{l=1}^{\infty} m_a^s(2^{-l}t) = 0. \end{aligned}$$

Therefore, $\hat{\varphi}_a^s(\xi) = 0$ for all $\xi \in [\pi+(a-\pi)s, \infty)$. Similarly, we can show that if $\xi \in (-\infty, -\pi+(a-\pi)s)$, then $\hat{\varphi}_a^s(\xi) = 0$. Thus $\text{supp}(\hat{\varphi}_a^s) \subseteq [-\pi+(a-\pi)s, \pi+(a-\pi)s]$. In fact, $\hat{\varphi}_a^s = \chi_{[-\pi+(a-\pi)s,\, \pi+(a-\pi)s)}$. Now, we show that φ_a^s is a scaling function for an MRA. For this we use Theorem 2.1. By the definition of $\hat{\varphi}_a^s$, it is clear that $\hat{\varphi}_a^s(2\xi) = m_a^s(\xi)\hat{\varphi}_a^s(\xi)$ for all $\xi \in \mathbb{R}$. Further, $\hat{\varphi}_a^s(\xi) = 1$, when $\xi \in [-\pi+(a-\pi)s, \pi+(a-\pi)s)$ and $0 \le s \le 1$. Therefore, φ_a^s satisfies conditions (ii) and (iii). For condition (i), let $S = [-\pi+(a-\pi)s, \pi+(a-\pi)s)$. We easily see that the set $\{S - 2k\pi : k \in \mathbb{Z}\}$ partitions $\mathbb{R}$, and therefore for a.e. $\xi \in \mathbb{R}$, there exists exactly one $k \in \mathbb{Z}$ such that $\xi + 2k\pi \in S$. Thus condition (i) follows.

Using the relation

$$\hat{\psi}_a^s(\xi) = e^{i\xi/2}\overline{m_a^s\left(\frac{\xi}{2}+\pi\right)}\hat{\varphi}_a^s\left(\frac{\xi}{2}\right),$$

we have

$$|\hat{\psi}_a^s(\xi)| = \chi_{[-2\pi+2(a-\pi)s,\, -\pi+(a-\pi)s)\cup[\pi+(a-\pi)s,\, 2\pi+2(a-\pi)s)}(\xi),$$

with $0 < \pi + (a-\pi)s < 2\pi$.

Thus ψ_a^s is a two-interval MSF wavelet. The proof of the continuity of the map $s \mapsto \psi_a^s$ is drawn on almost the similar lines as that of $s \mapsto \psi_s$ described in [9], by observing the following:

(i) For a fixed $s \in [0,1]$ the set of points $\xi \in \mathbb{R}$, for which the map $t \mapsto m_a^t(2^{-j}\xi)$, $j \ge 1$ is not continuous at s, is countable.

(ii) For $0 < a \le \pi$, $\hat{\varphi}_a^t(\xi) = \hat{\varphi}_a^1(\xi)$ on $[-a, a)$, for all $t \in [0,1]$.

(iii) For $\pi \le a < 2\pi$, $\hat{\varphi}_a^t(\xi) = \hat{\varphi}_a^1(\xi)$ on $[-(2\pi-a), (2\pi-a))$, for all $t \in [0,1]$.

Thus, we have a path $\gamma_a : [0,1] \to \mathcal{W}_2$ defined by $\gamma_a(s) = \psi_a^s$, joining $\psi_0 \equiv \psi_a^0$ to ψ_a^1, and hence with ψ_a as each $\mathcal{M}_{\psi_a}$ is path-connected. $\square$

Acknowledgements The author thanks to Professor K. K. Azad for his help and encouragement.

References

[1] X. Dai and D. R. Larson, *Wandering vectors for unitary systems and orthogonal wavelets*, Mem. Amer. Math. Soc., **134**(1998), no. 640, MR 98m: 47067.

[2] X. Fang and X. Wang, *Construction of minimally-supported-frequency wavelets*, J. Fourier Anal. Appl., **2**(1996), 315-327.

[3] Y. Ha, H. Kang, J. Lee and J. K. Seo, *Unimodular wavelets for L^2 and the Hardy space H^2*, Michigan Math. J., **41**(1994), 345-361.

[4] E. Hernández and G. Weiss, A First Course on Wavelets, CRC Press, 1996.

[5] Z. Li, X. Dai, Y. Diao and W. Huang, *The Path-connectivity of MRA wavelets in* $L^2(\mathbb{R}^d)$, Illinois J. Math., to appear.

[6] Z. Li, X. Dai, Y. Diao and J. Xin, *Multipliers, phases and connectivity of MRA wavelets in* $L^2(\mathbb{R}^2)$, J. Fourier Anal. Appl., **16**(2010), 155-176.

[7] S. G. Mallat, *Multiresolution approximations and wavelet orthonormal bases of* $L^2(\mathbb{R})$, Trans. Amer. Math. Soc., **315**(1989), 69-87.

[8] D. Speegle, *The s-elementary wavelets are path-connected*, Proc. Amer. Math. Soc., **127**(1999), 223-233.

[9] The Wutam Consortium, *Basic properties of wavelets*, J. Fourier Anal. Appl., **4**(1998), 575-594.