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Certain Class of Multidimensional Convolution Integral Equations Involving a Generalized Polynomial Set

Jamal Mohammed Shenan and Tariq Omar Salim*
Department of Mathematics, Al-Azhar University-Gaza, P. O. Box 1277, Gaza, Palestine
e-mail : shenanjm@yahoo.com *and* trsalim@yahoo.com

Abstract. The aim of this paper is to obtain a solution of a certain multidimensional convolution integral equation of Fredholm type whose kernel involves a generalized polynomial set. A number of results follow as special cases from the main theorem by specifying the parameters of the generalized polynomial set.

1. Introduction

Agrawal and Chaubey [1, p.1155]; see also Srivastava and Manocha [11, p.447] studied the following general class of polynomials

$$\begin{aligned}(1.1)\qquad & R_n^{\alpha,\beta}[x;A,B,C,D;p,r;q,c;\omega(x)]\\ &=\frac{(Ax^p+B)^{-\alpha}(Cx^r+D)^{-\beta}}{K_n\omega(x)}\times\ T_{k,l}^n\left\{(Ax^p+B)^{\alpha+qn}(Cx^r+D)^{\beta+cn}\omega(x)\right\}\end{aligned}$$

with differential operator $T_{k,l}^n$ being defined as

$$T_{k,l}\equiv x^l(k+xD_x), \tag{1.2}$$

where $D_x\equiv d/dx$.

In (1.1), $\{K_n\}_{n=0}^{\infty}$ is sequence of constants, and $\omega(x)$ is independent of n and differentiable an arbitrary number of times.

Assuming that $K_n=1$ and $\omega(x)=1$. Moreover, setting $p=D=1$, $C=-\tau$ in (1.1) and replacing β by β/τ therein, we arrive at the following generalized polynomial set

$$\begin{aligned}(1.3)\qquad & S_n^{\alpha,\beta,\tau}[x;r,c,q,A,B,k,l]\\ &=(Ax+B)^{-\alpha}(1-\tau x^r)^{-\beta/\tau}\ T_{k,l}^n\left\{(Ax+B)^{\alpha+qn}(1-\tau x^r)^{\beta/\tau+cn}\right\}.\end{aligned}$$

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This polynomial set has been studied by Raizada [8].
It may be remarked here that the generalized polynomial set is very general in nature and it unifies and extends a number of classical polynomials introduced and studied by various researches such as Chatterjea [2], Gould and Hopper [4], Krall and Frink [5], Srivastava and Singhal [14], etc. Some of special cases of (1.3) are given by Riazada in tabular form [8; p.65], see also Saigo , Goyal and Saxena [9].

Motivating essentially by the method used earlier by Srivastava and Panda [12,13], the main object of this paper is to present an exact solution of the following multidimensional convolution integral equations of Fredholm type

$$\int_0^\infty \cdots \int_0^\infty y_1^{-1} \dots y_m^{-1}\ u(\frac{x_1}{y_1}, \dots, \frac{x_m}{y_m}) f(y_1, \dots, y_m)\ dy_1 \dots dy_m = g(x_1, \dots, x_m), \tag{1.4}$$

where $x_i > 0, \forall\ i = 1, \dots, m$, g is a prescribed function, f is an unknown function to be determined, and the kernel u is given by

$$\begin{aligned} u(x_1, \dots, x_m) &= \prod_{i=1}^{m} (A_i x_i + B_i)^{\alpha_i} (1 - \tau_i x_i^{r_i})^{\beta_i/\tau_i} S_{n_i}^{\alpha_i, \beta_i, \tau_i} [x_i; r_i, c_i, q_i, A_i, B_i, k_i, l_i] \\ &= \prod_{i=1}^{m} [x_i^{l_i} (k_i + x_i D_{x_i})]^{n_i} \left\{ (A_i x_i + B_i)^{\alpha_i + q_i n_i} (1 - \tau_i x_i^{r_i})^{\beta_i/\tau_i + c_i n_i} \right\}. \end{aligned} \tag{1.5}$$

Throughout this paper, we assume $l_i, n_i, \alpha_i + q_i n_i$, $(i = 1, 2, \dots, m)$ to be non-negative integers.

Our method of solution of the integral (1.4) with kernel given by (1.5) would depend on the theory of multidimensional Mellin transform defined by [12, partI. p.125, eq.(3.5.1)]

$$M\{f(x_1, \dots, x_m); s_1, \dots, s_m\} = \int_0^\infty \cdots \int_0^\infty x_1^{s_1 - 1} \dots\ x_m^{s_m - 1} f(x_1, \dots, x_m)\ \ dx_1 \dots dx_m \tag{1.6}$$

provided that the multiple integral exists.

In particular, if $f(x_1, \dots, x_m) = \prod_{i=1}^{m} f_i(x_i)$, then

$$M\{f(x_1, \dots, x_m); s_1, \dots, s_m\} = \prod_{i=1}^{m} M_1\{f_i(x_i); s_i\}, \tag{1.7}$$

where M_1 is the one-dimensional Mellin transform.
Also the multidimensional Mellin convolution of two functions $f(x_1, \dots, x_m)$ and $g(x_1, \dots, x_m)$ is defined by

$$(f * g)(x_1, \dots, x_m) = \int_0^\infty \cdots \int_0^\infty y_1^{-1} \dots y_m^{-1}\ f(\frac{x_1}{y_1}, \dots, \frac{x_m}{y_m})\ g(y_1, \dots, y_m)\ dy_1 \dots dy_m, \tag{1.8}$$

provided that the multiple integral exists.

2. Multidimensional Mellin transform of $u(x_1,...,x_m)$

In order to solve the multidimensional integral equation (1.4), the following result is needed.

Lemma: *Let* $U(s_1,...,s_m) = M\{u(x_1,...,x_m); s_1,...,s_m\}$, *where* $u(x_1,...,x_m)$ *is defined by* (1.5), *then*

$$(2.1)\qquad U(s_1,...,s_m) = \prod_{i=1}^{m}\left\{\sum_{e_i=0}^{n_i}\sum_{p_i=0}^{\alpha_i+q_in_i}\frac{(-n_i)_{e_i}(-\alpha_i-q_in_i)_{p_i}}{e_i!p_i!}(-1)^{p_i}\ k_i^{n_i-e_i}\ A_i^{p_i}\right.$$
$$\times B_i^{\alpha_i+q_in_i-p_i}\ l_i^{e_i}\frac{(-\tau_i)^{-(s_i+l_ie_i+p_i)/r_i}}{|r_i|}\Gamma\left(\frac{s_i+l_ie_i+p_i}{r_i}\right)\left(-\frac{s_i+l_ie_i+p_i}{l_i}\right)_{e_i}$$
$$\left.\times\Gamma\left(\frac{-\beta_i}{\tau_i}-c_in_i-\frac{s_i+l_ie_i+p_i}{r_i}\right)\left\{\Gamma\left(\frac{-\beta_i}{\tau_i}-c_in_i\right)\right\}^{-1}\right\}$$

provided that $|arg\tau_i<\pi|$, $0<Re(s_i+l_ie_i+p_i)<r_iRe\left(\frac{-\beta_i}{\tau_i}-c_in_i\right)$, *when* $r_i>0$; $r_iRe\left(\frac{-\beta_i}{\tau_i}-c_in_i\right)<Re(s_i+l_ie_i+p_i)<0$ *when* $r_i<0$, *for* $(i=1,...,m), m\in N_0$.

Proof. Making use of the binomial expansion for $[x_i^{l_i}(k_i+x_iDx_i)]^{n_i}$, we find that

$$(2.2)\qquad u(x_1,...,x_m) = \prod_{i=1}^{m}\left\{\sum_{e_i=0}^{n_i}\sum_{p_i=0}^{\alpha_i+q_in_i}\frac{(-n_i)_{e_i}(-\alpha_i-q_in_i)_{p_i}}{e_i!p_i!}(-1)^{e_i+p_i}\ k_i^{n_i-e_i}\ A_i^{p_i}\right.$$
$$\left.\times B_i^{\alpha_i+q_in_i-p_i}x_i^{l_i(n_i-e_i)}(x_i^{l_i+1}Dx_i)^{e_i}\left[x_i^{p_i}\left\{1-\frac{\tau_i}{x_i^{-r_i}}\right\}^{\frac{\beta_i}{\tau_i}+c_in_i}\right]\right\}.$$

Applying the Multidimensional Mellin transform to both sides of (2.2), making use of (1.7) and applying the following known formulas [15, p.14, eq. (23); 1, p. 307, eq. (7)]

$$(2.3)\qquad M_1\{(x_i^{l_i+1}Dx_i)^{n_i}f(x_i); s_i\} = l_i^{n_i}\left(-\frac{s_i+l_in_i}{l_i}\right)_{n_i} M_1\{f(x_i); s_i+l_in_i\}$$

and

$$(2.4)\qquad M_1\{x_i^{\mu_i}f(x_i); s_i\} = M_1\{f(x_i); s_i+\mu_i\}$$

provided that the Mellin transforms of (2.3) and (2.4) exist, we get

$$u(x_1,...,x_m) = \prod_{i=1}^{m}\left\{\sum_{e_i=0}^{n_i}\sum_{p_i=0}^{\alpha_i+q_in_i}\frac{(-n_i)_{e_i}(-\alpha_i-q_in_i)_{p_i}}{e_i!p_i!}(-1)^{e_i+p_i}\,k_i^{n_i-e_i}\,A_i^{p_i} \times B_i^{\alpha_i+q_in_i-p_i}l_i^{e_i}\left(-\frac{s_i+l_ie_i}{l_i}\right)_{e_i} M_1\left[\left\{1-\frac{\tau_i}{x_i^{-r_i}}\right\}^{\frac{\beta_i}{\tau_i}+c_in_i}; s_i+l_ie_i+p_i\right]\right\}. \tag{2.5}$$

Again making use of (2.4) and the following result [3, p. 311, eq. (30)],

$$M_1\{(1+a_ix_i^{r_i})^{-\nu_i}; s_i\} = \frac{1}{|r_i|}\,a_i^{\frac{-s_i}{r_i}}\,\frac{\Gamma(\frac{s_i}{r_i})\Gamma(\nu_i-\frac{s_i}{r_i})}{\Gamma(\nu_i)}, \tag{2.6}$$

where $|arga_i| < \pi$, $0 < Res_i < r_iRe\nu_i$, when $r_i > 0$; $r_iRe\nu_i < Res_i < 0$, when $r_i < 0$, for $(i = 1,...,m), m \in N_0$. The result readily follows.

3. Solution of the integral equation (1.4)

Theorem: *Let the multidimensional Mellin transforms* $F(s_1,...,s_m)$, $G(s_1,...,s_m)$ *and* $U(s_1,...,s_m)$ *of the functions* $f(x_1,...,x_m), g(x_1,...,x_m)$ *and* $u(x_1,...,x_m)$ *defined by (1.5) exist and be analytic in some infinite strip* $\zeta_i < Res_i < \eta_i$ *of the complex s-plane. Also suppose that for a fixed* $\gamma_i \in (\zeta_i, \eta_i)$, $\bar{u}$ *is defined by*

$$\bar{u}(x_1,...,x_m) = M^{-1}\{\bar{U}(s_1,...,s_m); x_1,...,x_m\} = \frac{1}{(2\pi i)^m}\int_{\gamma_1-i\infty}^{\gamma_1+i\infty}\cdots\int_{\gamma_m-i\infty}^{\gamma_m+i\infty} x_1^{s_1}\ldots x_m^{s_m}\bar{U}(s_1,...,s_m)\,ds_1...ds_m, \tag{3.1}$$

where

$$\bar{U}(s_1,...,s_m) = \prod_{i=1}^{m}\left[\mu_i^{l_i}\frac{\Gamma\left(-\frac{s_i}{\mu_i}\right)}{\Gamma\left(-l_i-\frac{s_i}{\mu_i}\right)}\sum_{e_i=0}^{n_i}\sum_{p_i=0}^{\alpha_i+q_in_i}\frac{(-n_i)_{e_i}(-\alpha_i-q_in_i)_{p_i}}{e_i!p_i!}(-1)^{p_i}\,k_i^{n_i-e_i} A_i^{p_i}\,B_i^{\alpha_i+q_in_i-p_i}l_i^{e_i}\frac{(-\tau_i)^{-(s_i+\mu_il_i+\lambda_i+l_ie_i+p_i)/r_i}}{|r_i|}\Gamma\left(1+\frac{s_i+\mu_il_i+\lambda_i+l_ie_i}{l_i}\right) \frac{\Gamma\left(\frac{s_i+\mu_il_i+\lambda_i+l_ie_i+p_i}{r_i}\right)\Gamma\left(\frac{-\beta_i}{\tau_i}-c_in_i-\frac{s_i+\mu_il_i+\lambda_i+l_ie_i+p_i}{r_i}\right)}{\Gamma\left(1+\frac{s_i+\mu_il_i+\lambda_i}{l_i}\right)\Gamma\left(\frac{-\beta_i}{\tau_i}-c_in_i\right)}\right]^{-1} \tag{3.2}$$

provided that $|arg\tau_i < \pi|$, $0 < Re(s_i+\mu_il_i+\lambda_i+l_ie_i+p_i) < r_iRe\left(\frac{-\beta_i}{\tau_i}-c_in_i\right)$, *when* $r_i > 0$; $r_iRe\left(\frac{-\beta_i}{\tau_i}-c_in_i\right) < Re(s_i+\mu_il_i+\lambda_i+l_ie_i+p_i) < 0$ *when* $r_i < 0$, *for*

$(i = 1, ..., m), m \in N_0$. *Then the integral equation (1.4) has its solution given by*

(3.3)
$$f(x_1, ..., x_m) = x_1^{-\mu_1 l_1 - \lambda_1} ... x_m^{-\mu_m l_m - \lambda_m} \int_0^\infty ... \int_0^\infty y_1^{-1} ... y_m^{-1} \; \bar{u}(\frac{x_1}{y_1}, ..., \frac{x_m}{y_m}) (y_1^{\mu_1+1} Dy_1)^{l_1} ... (y_m^{\mu_m+1} Dy_m)^{l_m} \; \left\{ y_1^{\lambda_1} ... y_m^{\lambda_m} g(y_1, ..., y_m) \right\} \; dy_1 ... dy_m$$

provided that the integral exist.

Proof. Making use of (1.8), then (1.4) can be written as

(3.4)
$$(U * F)(x_1, ..., x_m) = g(x_1, ..., x_m).$$

Now, multidimensional Mellin transform of (3.4) yields

(3.5)
$$U(s_1, ..., s_m) F(s_1, ..., s_m) = G(s_1, ..., s_m),$$

where $U(s_1, ..., s_m)$ is given by (2.1).
Replacing s_i by $s_i + \mu_i l_i + \lambda_i$, we obtain

(3.6)
$$F(s_1 + \mu_1 l_1 + \lambda_1, ..., s_m + \mu_m l_m + \lambda_m) = \bar{U}(s_1, ..., s_m) \times M \left\{ (x_1^{\mu_1+1} Dx_1)^{l_1} ... (x_m^{\mu_m+1} Dx_m)^{l_m} \{ x_1^{\lambda_1} ... x_m^{\lambda_m} g(x_1, ..., x_m) \} \right\}$$

Again using the multidimensional Mellin convolution theorem and (2.4), we find that

(3.7)
$$M\{x_1^{\mu_1 l_1 + \lambda_1} ... x_m^{\mu_m l_m + \lambda_m} f(x_1, ..., x_m); s_1, ..., s_m\} = M \left\{ \int_0^\infty ... \int_0^\infty y_1^{-1} ... y_m^{-1} \; \bar{u}(\frac{x_1}{y_1}, ..., \frac{x_m}{y_m}) \{ y_1^{-1} ... y_m^{-1} g(y_1, ..., y_m) \} \left(y_1^{\mu_1+1} Dy_1 \right)^{l_1} ... \left(y_m^{\mu_m+1} Dy_m \right)^{l_m} \; dy_1 ... dy_m; s_1, ..., s_m \right\}$$

Inverting both sides of (3.7) by using the Mellin inversion theorem [3, p. 307, eq. (1)], we arrive at the required solution (3.3).

4. Applications

Since the generalized polynomial set defined by (1.3) has a large number of special cases, one can obtain the solutions of a number of multidimensional integral equations of the type (1.4) with the kernels involving products of Laguerre polynomials, Hermite polynomials, Bessel polynomials, $H_n^{(r)}(x, \alpha, \beta)$ polynomials defined by Gould and Hopper [4], $F_n^{(r)}(x, \alpha, q, \beta)$ polynomials defined by Chatterjea

[2], $G_n^{(r)}(x,\alpha,\beta,l)$ polynomials defined by Srivastava and Singhal [14], and several other polynomials. We mention in what follows some of these special cases.

If we put $A_i = 1,\ B_i = 0$ and $k_i = 0$, then we get the following result contained in the following corollary.

Corollary 2: *Under the hypothesis of Theorem, the multidimensional integral equation*

$$\int_0^\infty ... \int_0^\infty y_1^{-1}...y_m^{-1}\ u_2(\frac{x_1}{y_1},...,\frac{x_m}{y_m})f(y_1,...,y_m)\ dy_1...dy_m = g(x_1,...,x_m), \tag{4.1}$$

where $x_i > 0, \forall\ i = 1,...,m,$ *and*

$$\begin{aligned} u_2(x_1,...,x_m) &= \prod_{i=1}^m x_i^{\alpha_i}(1-\tau_i x_i^{r_i})^{\beta_i/\tau_i} S_{n_i}^{\alpha_i,\beta_i,\tau_i}[x_i;r_i,c_i,q_i,1,0,0,l_i] \\ &= \prod_{i=1}^m (x_i^{l_i+1}D_{x_i})^{n_i}\left\{x_i^{\alpha_i+q_i n_i}(1-\tau_i x_i^{r_i})^{\beta_i/\tau_i+c_i n_i}\right\}, \end{aligned} \tag{4.2}$$

has its solution given by

$$\begin{aligned} f(x_1,...,x_m) = x_1^{-\mu_1 l_1-\lambda_1}...x_m^{-\mu_m l_m-\lambda_m}\int_0^\infty ... \int_0^\infty y_1^{-1}...y_m^{-1}\ \bar{u_2}(\frac{x_1}{y_1},...,\frac{x_m}{y_m}) \\ (y_1^{\mu_1+1}Dy_1)^{l_1}...(y_m^{\mu_m+1}Dy_m)^{l_m}\ \left\{y_1^{\lambda_1}...y_m^{\lambda_m}g(y_1,...,y_m)\right\}\ dy_1...dy_m, \end{aligned} \tag{4.3}$$

provided the integral exist. where $\bar{u_2}(x_1,...,x_m)$ *is the the multidimensional Mellin inverse transform of*

$$\begin{aligned} &\bar{U_2}(s_1,...,s_m) \\ &= \prod_{i=1}^m \left[\frac{|r_i|}{\mu_i^{l_i} l_i^{n_i}}(-\tau_i)^{-(s_i+\mu_i l_i+n_i(l_i+q_i)+\lambda_i+\alpha_i)/r_i}\frac{\Gamma\left(\frac{-\beta_i}{\tau_i}-c_i n_i\right)\Gamma\left(-l_i-\frac{s_i}{\mu_i}\right)}{\Gamma\left(-\frac{s_i}{\mu_i}\right)\Gamma\left(\frac{-s_i-\mu_i l_i+\lambda_i}{l_i}\right)}\right. \\ &\left.\frac{\Gamma\left(-n_i-\frac{s_i+\mu_i l_i+\lambda_i}{l_i}\right)}{\Gamma\left(\frac{s_i+\mu_i l_i+\lambda_i+n_i(l_i+q_i)+\alpha_i}{r_i}\right)\Gamma\left(-c_i n_i-\frac{\beta_i}{\tau_i}-\frac{s_i+\mu_i l_i+\lambda_i+n_i(l_i+q_i)+\alpha_i}{r_i}\right)}\right], \end{aligned} \tag{4.4}$$

provided that $|arg\tau_i < \pi|,\ 0 < Re(s_i+\mu_i l_i+\lambda_i+n_i(l_i+q_i)+\alpha_i) < r_i Re\left(\frac{-\beta_i}{\tau_i}-c_i n_i\right)$,*when* $r_i > 0$; $r_i Re\left(\frac{-\beta_i}{\tau_i}-c_i n_i\right) < Re(s_i+\mu_i l_i+\lambda_i+n_i(l_i+q_i)+\alpha_i) < 0$ *when* $r_i < 0$, *for* $(i = 1,...,m), m \in N_0$.

Now, applying the Mellin inversion formula (3.1) and replacing s_i by -s_i w get

$$\bar{u}(x_1,...,x_m) = \prod_{i=1}^{m}\left[\frac{|r_i|}{\mu_i^{l_i} l_i^{n_i}}(-\tau_i)^{(\mu_i l_i + n_i(l_i+q_i)+\lambda_i+\alpha_i)/r_i}\right. \Gamma\left(\frac{-\beta_i}{\tau_i} - c_i n_i\right)\frac{1}{(2\pi i)^m}\int_{\gamma_1-i\infty}^{\gamma_1+i\infty}...\int_{\gamma_m-i\infty}^{\gamma_m+i\infty}\varphi_1(s_1)...\varphi_m(s_m) \left.\left(\frac{x_1}{(-\tau_1)^{1/r_1}}\right)^{s_1}...\left(\frac{x_m}{(-\tau_m)^{1/r_m}}\right)^{s_m} ds_1...ds_m\right], \tag{4.5}$$

where

$$\varphi_i(s_i) = \frac{\Gamma\left(-l_i + \frac{s_i}{\mu_i}\right)\Gamma\left(-n_i - \frac{-s_i+\mu_i l_i+\lambda_i}{l_i}\right)}{\Gamma\left(\frac{s_i}{\mu_i}\right)\Gamma\left(\frac{s_i-\mu_i l_i+\lambda_i}{l_i}\right)\Gamma\left(\frac{-s_i+\mu_i l_i+\lambda_i+n_i(l_i+q_i)+\alpha_i}{r_i}\right)} \frac{1}{\Gamma\left(-c_i n_i - \frac{\beta_i}{\tau_i} - \frac{-s_i+\mu_i l_i+\lambda_i+n_i(l_i+q_i)+\alpha_i}{r_i}\right)}. \tag{4.6}$$

Under various restrictions on the non-zero constants r_i, l_i, μ_i, $i = 1, 2, ..., m$, the contour integral in (4.5) can be expressed in terms of multivariable H-functions or a product of m Foxe's H-functions (see Srivastava et al. [10, ch.2]).For example, the solution of the the multidimensional integral equation (4.1) with the kernel defined by (4.2) can be written as

(4.7) $f(x_1,...,x_m)$

$$= \prod_{i=1}^{m}\left[\frac{|r_i|}{\mu_i^{l_i} l_i^{n_i}}(-\tau_i)^{(\mu_i l_i+n_i(l_i+q_i)+\lambda_i+\alpha_i)/r_i}\Gamma\left(\frac{-\beta_i}{\tau_i} - c_i n_i\right) x_i^{-\mu_i l_i - \lambda_i}\right]$$

$$\int_0^\infty ... \int_0^\infty y_1^{-1}...y_m^{-1}(y_1^{\mu_1+1}Dy_1)^{l_1}...(y_m^{\mu_m+1}Dy_m)^{l_m}\left\{y_1^{\lambda_1}...y_m^{\lambda_m} g(y_1,...,y_m)\right\}$$

$$H^{0,0;1,1;...;1,1}_{0,0;3,3;...;3,3}\left[\frac{x_1}{y_1(-\tau_1)^{1/r_1}},...,\frac{x_m}{y_m(-\tau_m)^{1/r_m}}\,\middle|\, \begin{matrix} \text{———} : \left(1+n_1+\frac{\mu_1 l_1+\lambda_1}{l_1}, 1/l_1\right), (0,-1/\mu_1), \left(\frac{\mu_1 l_1+\lambda_1+n_1(l_1+q_1)+\alpha_1}{r_1}, 1/r_1\right); ...; \left(1+n_m+\frac{\mu_m l_m+\lambda_m}{l_m}, 1/l_m\right), (0,-1/\mu_m), \left(\frac{\mu_m l_m+\lambda_m+n_m(l_m+q_m)+\alpha_m}{r_m}, 1/r_m\right) \\ \text{———} : (-l_1,-1/\mu_1), \left(1+\frac{\mu_1 l_1+\lambda_1}{l_1}, 1/l_1\right), \left(1+\frac{\beta_1}{\tau_1}+c_1 n_1+\frac{\mu_1 l_1+\lambda_1+n_1(l_1+q_1)+\alpha_1}{r_1}, 1/r_1\right); ...; (-l_m,-1/\mu_m), \left(1+\frac{\mu_m l_m+\lambda_m}{l_m}, 1/l_m\right), \left(1+\frac{\beta_m}{\tau_m}+c_m n_m+\frac{\mu_m l_m+\lambda_m+n_m(l_m+q_m)+\alpha_m}{r_m}, 1/r_m\right) \end{matrix}\right] dy_1...dy_m,$$

for $r_i > 0$, $\mu_i < 0$, $l_i > 0$ $(i = 1, 2, .., m)$.

Also, on setting $A_i = 1$, $B_i = 0$, $k_i = 0$ as $\tau_i \to 0$, we get

$$S_{n_i}^{(\alpha_i+l_in_i),\beta_i,0}[x_i; r_i, -l_i, 1, 0, 0, l_i] = n_i! G_{n_i}^{(\alpha_i)}(x_i, r_i, \beta_i, l_i), \tag{4.8}$$

where $G_n^\alpha(x, r, \beta, l)$ is the class of polynomials studied by Srivastava and Singhal [14], so we have the following Corollary.

Corollary 4: *Under the hypothesis of Theorem, the multidimensional integral equation*

$$\int_0^\infty \cdots \int_0^\infty y_1^{-1} \dots y_m^{-1}\ u_4\left(\frac{x_1}{y_1}, \dots, \frac{x_m}{y_m}\right) f(y_1, \dots, y_m)\ dy_1 \dots dy_m = g(x_1, \dots, x_m), \tag{4.9}$$

where $x_i > 0, \forall\ i = 1, \dots, m$, *and*

$$\begin{aligned} u_4(x_1, \dots, x_m) &= \prod_{i=1}^m n_i! x_i^{\alpha_i+l_in_i}\ exp\ (-\beta_i x_i^{r_i}) G_{n_i}^{(\alpha_i)}(x_i, r_i, \beta_i, l_i) \\ &= \prod_{i=1}^m (x_i^{l_i+1} D_{x_i})^{n_i} \left\{x_i^{\alpha_i} exp\ (-\beta_i x_i^{r_i})\right\}, \end{aligned} \tag{4.10}$$

has its solution given by

$$\begin{aligned} f(x_1, \dots, x_m) = x_1^{-\mu_1 l_1 - \lambda_1} \dots x_m^{-\mu_m l_m - \lambda_m} \int_0^\infty \cdots \int_0^\infty y_1^{-1} \dots y_m^{-1}\ \bar{u}_4\left(\frac{x_1}{y_1}, \dots, \frac{x_m}{y_m}\right) \\ (y_1^{\mu_1+1} D y_1)^{l_1} \dots (y_m^{\mu_m+1} D y_m)^{l_m} \left\{y_1^{\lambda_1} \dots y_m^{\lambda_m} g(y_1, \dots, y_m)\right\}\ dy_1 \dots dy_m, \end{aligned} \tag{4.11}$$

provided the integral exist, and $\bar{u}_4(x_1, \dots, x_m)$ *is the the multidimensional Mellin inverse transform of*

$$\bar{U}_4(s_1, \dots, s_m) = \prod_{i=1}^m \left[\frac{|r_i|}{\mu_i^{l_i} l_i^{n_i}} (\beta_i)^{(s_i+\mu_i l_i+n_i l_i+\lambda_i+\alpha_i)/r_i} \frac{\Gamma\left(-l_i - \frac{s_i}{\mu_i}\right) \Gamma\left(-n_i - \frac{s_i+\mu_i l_i+\lambda_i}{l_i}\right)}{\Gamma\left(\frac{s_i+\mu_i l_i+\lambda_i+n_i l_i+\alpha_i}{r_i}\right) \Gamma\left(-\frac{s_i}{\mu_i}\right) \Gamma\left(-\frac{s_i+\mu_i l_i+\lambda_i}{l_i}\right)}\right], \tag{4.12}$$

provided that $Re(s_i + \mu_i l_i + \lambda_i + n_i l_i + \alpha_i) > 0$, *when* $r_i > 0$; $Re(s_i + \mu_i l_i + \lambda_i + n_i l_i + \alpha_i) < 0$, *when* $r_i < 0$, *for* $(i = 1, \dots, m), m \in N_0$.

Now, applying the Mellin inversion formula (3.1) in (4.12) and replacing s_i by

-s_i w get

$$\bar{U}_4(s_1, ..., s_m) \tag{4.13}$$

$$= \prod_{i=1}^{m} \left[\frac{|r_i|}{\mu_i^{l_i} l_i^{n_i}} (\beta i)^{(\mu_i l_i + n_i l_i + \lambda_i + \alpha_i)/r_i} \frac{1}{(2\pi i)^m} \int_{\gamma_1 - i\infty}^{\gamma_1 + i\infty} ... \int_{\gamma_m - i\infty}^{\gamma_m + i\infty} \right.$$

$$\left. \varphi_1(s_1)...\varphi_m(s_m) \left(\frac{x_1}{(\beta_1)^{1/r_1}} \right)^{s_1} ... \left(\frac{x_m}{(\beta_m)^{1/r_m}} \right)^{s_m} ds_1...ds_m \right],$$

where

$$\varphi_i(s_i) = \frac{\Gamma\left(-l_i + \frac{s_i}{\mu_i}\right) \Gamma\left(-n_i - \frac{-s_i + \mu_i l_i + \lambda_i}{l_i}\right)}{\Gamma\left(\frac{s_i}{\mu_i}\right) \Gamma\left(\frac{s_i - \mu_i l_i + \lambda_i}{l_i}\right) \Gamma\left(\frac{-s_i + \mu_i l_i + \lambda_i + n_i l_i + \alpha_i}{r_i}\right)}. \tag{4.14}$$

In particular, if $\lambda_i > 0$, $\mu_i > 0$, $r_i > 0$, then the solution (4.11) can be written as

$$f(x_1, ..., x_m) \tag{4.15}$$

$$= \prod_{i=1}^{m} \left(\frac{|r_i|}{\mu_i^{l_i} l_i^{n_i}} (\beta_i)^{(\mu_i l_i + n_i l_i + \lambda_i + \alpha_i)/r_i} x_i^{-\mu_i l_i - \lambda_i} \right)$$

$$\int_0^\infty ... \int_0^\infty H^{0,0;0,2;...;0,2}_{0,0;3,2;...;3,2} \left[\frac{x_1}{y_1(\beta_1)^{1/r_1}}, ..., \frac{x_m}{y_m(\beta_m)^{1/r_m}} \middle| \begin{matrix} \text{---} : \left(1+n_1+\frac{\mu_1 l_1+\lambda_1}{l_1}, 1/l_1\right), (1+l_1, 1/\mu_1), \\ \text{---} : (1, 1/\mu_1), \left(1+\frac{\mu_1 l_1+\lambda_1}{l_1}, 1/l_1\right) \end{matrix} \right.$$

$$\left. \begin{matrix} \left(\frac{\mu_1 l_1+\lambda_1+n_1 l_1+\alpha_1}{r_1}, 1/r_1\right); ...; \left(1+n_m+\frac{\mu_m l_m+\lambda_m}{l_m}, 1/l_m\right), (1+l_m, 1/\mu_m), \left(\frac{\mu_m l_m+\lambda_m+n_m l_m+\alpha_m}{r_m}, 1/r_m\right) \\ ; ...; (1, 1/\mu_m), \left(1+\frac{\mu_m l_m+\lambda_m}{l_m}, 1/l_m\right) \end{matrix} \right]$$

$dy_1...dy_m$

If we set $m = 1$ in (4.15), we get the result obtained by Srivastava [15] and by setting $A_1 = 0$, $B_1 = 0$, $k_1 = q_1 = 0$, $l_1 = \mu_1 = -1$, $m = 1$, and $\tau_1 \to 0$ in (1.6), we get the solution of the integral equation considered by Lala and Shrivastava [6,7].

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References

[1] Agrawal B. D. and Chaubey J. P., Operational derivation of generating relations for generalized polynomials, Indian J. Pure Appl. Math., **11**(1980), 1155-1157; ibid. **11**(1981), 357-379.

[2] Chatterjea S. K., *Quelques fonctions génératrics des Poylnomes d'Hermite du point de vue de algebre de Lie,* C. R. Acad. Sci. Paris, Ser., **A-B268**(1969), 600-602.

[3] Erdélyi A., Magnus W., Oberhettinger F. and Tricomi F. G., Tables of integral transforms Vol. I, McGraw-Hill, New York, 1954.

[4] Gould H. W. and Hopper A. T., *Operational formulas connected with two generalizations of Hermite polynomials,* Duke Math. J., **29**(1962), 51-63.

[5] Krall H. L. and Frink O. A., *A new class of orthogonal polynomials; Bessel polynomials,* Trans. Am. Math. Soc., **65**(1949), 100-115.

[6] Lala A.and Shrivastava P. N., *Inversion of an integral involving a generalized Hermite polynomial,* Indian J. Pure Appl. Math., **21**(1990), 163-166.

[7] Lala A. and Shrivastava P. N., *Inversion of an integral involving a generalized function,* Bull. Calcutta Math. Soc., **82**(1990), 115-118.

[8] Raizad S. K., *A study of unified representation of special functions of mathematical physics and their use in statistical and boundary value problems*, Ph. D. thesis (Bundelkhand Univ., Jhansi, India), 1991.

[9] Saigo M., Goyal S. P. and Sexena S. A., *A theorem relating a generalized Weyl fractional integral, Laplace and Varma transforms with applications,* J. Fract. calc., **13**(1998), 43-56.

[10] Srivastava H. M., Gupta K. C. and Goyal S. P., The H-Functions of One and Two Variables with Applications, South Asian Publ., New Delhi-Madras, 1982.

[11] Srivastava H. M. and Monocha H. L., A Treatise on Generating Functions, Wiley/Halsted, New York, 1984.

[12] Srivastava H. M. and Panda R., *Certain multidimensional integral transformation I and II,* Nederl. Akad. Wetensch. Indag Math., **40**(1978), 118-131, 132-144.

[13] Srivastava H. M. and Panda R., *On the unified presentation of certain classical polynomials*, Boll. Un. Mat. Ital., **12**(4)(1975), 306-314.

[14] Srivastava H. M. and Singhal J. P., *A class of polynomials defined by generalized Rodrigues formula,* Ann. Math. Pure Appl., **90**(1971), 75-85.

[15] Srivastava R., *The inversion of an integral equation involving a general class of polynomials,* J. Math. Anal. Appl., **186**(1994), 11-20.