

ON INTERVAL VALUED (α, β) -FUZZY IDEALS OF HEMIRINGS

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ABSTRACT. In this paper we define interval valued $(\in, \in \vee q)$ -fuzzy h-quasi-ideals, interval valued $(\in, \in \vee q)$ -fuzzy h-bi-ideals, interval valued $(\bar{\in}, \bar{\in} \vee \bar{q})$ -fuzzy h-ideals, interval valued $(\bar{\in}, \bar{\in} \vee \bar{q})$ -fuzzy h-quasi-ideals, interval valued $(\bar{\in}, \bar{\in} \vee \bar{q})$ -fuzzy h-bi-ideals and characterize different classes of hemirings by the properties of these ideals.

1. Introduction

Semirings, which are considered as the generalization of associative rings, introduced by H. S. Vandiver in 1934 [12]. Semirings are very useful for solving problems in different areas of applied mathematics and information sciences, like as, optimization theory, graph theory, theory of discrete event dynamical systems, matrices, determinants, generalized fuzzy computation, automata theory, formal language theory, coding theory, analysis of computer programs, and so on (see [4]). Hemirings, which are semirings with commutative addition and zero element appears in a natural manner in some applications to the theory of automata and formal languages (see [4]).

Ideals play an important role in the study of hemirings and are useful for many purposes, but they do not coincide with ring ideals. Thus many results of ring theory have no analogues in semirings using only ideals. In order to overcome this problem, in [5], Henriksen defined a class of ideals in semirings, called k -ideals. These ideals have the property that if the semiring R is a ring then a subset of R is a k -ideal if and only if it is a ring ideal. A more restricted class of ideals in hemirings, called h-ideals, was introduced by Iizuka [6].

In 1965, Zadeh [14] introduced the concept of fuzzy set, which proved a very useful tool to describe situation in which the data are imprecise or vague. Fuzzy sets handle such situations by attributing a degree to which a certain object belongs to a set. J. Ahsan initiated the study of fuzzy semirings ([1]). Fuzzy k -ideals in semirings are studied in [3] by Ghosh and fuzzy h -ideals are studied in [7, 13, 15]. By using the concepts of " \in " and " q ", in [2] (α, β) -fuzzy ideals of hemirings are defined, where $\alpha, \beta \in \{\in, q, \in \vee q, \in \wedge q\}$. In

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[8] X. Ma and J. Zhan introduced the concept of interval valued $(\in, \in \vee q)$ -fuzzy h-ideals in hemirings and developed some basic results. In [11] Sun et al characterized h-hemiregular and h-intra-hemiregular hemirings by the properties of their interval valued fuzzy left and right h-ideals. In this paper we extend these ideas and define interval valued $(\in, \in \vee q)$ -fuzzy h-quasi-ideals, interval valued $(\in, \in \vee q)$ -fuzzy h-bi-ideals, interval valued $(\overline{\in}, \overline{\in} \vee \overline{q})$ -fuzzy h-ideals, interval valued $(\overline{\in}, \overline{\in} \vee \overline{q})$ -fuzzy h-quasi-ideals, interval valued $(\overline{\in}, \overline{\in} \vee \overline{q})$ -fuzzy h-bi-ideals and characterize different classes of hemirings by the properties of these ideals.

2. Preliminaries

For basic definitions one can see [4]. A left (right) ideal A of a hemiring R is called a left (right) h-ideal if for all $x, z \in R$ and for any $a, b \in A$ from $x + a + z = b + z$, it follows $x \in A$. [13]. A bi-ideal B of a hemiring R is called an h-bi-ideal of R if for all $x, y \in R$ and $a, b \in B$ from $x + a + y = b + y$, it follows $x \in B$ (cf. [13]).

The h-closure \overline{A} of a non-empty subset A of a hemiring R is defined as

$$\overline{A} = \{x \in R \mid x + a + y = b + y \text{ for some } a, b \in A, y \in R\}.$$

A quasi-ideal Q of a hemiring R is called an h-quasi-ideal of R if $\overline{RQ} \cap \overline{QR} \subseteq Q$ and $x + a + y = b + y$ implies $x \in Q$, for all $x, y \in R$ and $a, b \in Q$ (cf. [13]). Every left (right) h-ideal of a hemiring R is an h-quasi-ideal of R and every h-quasi-ideal is an h-bi-ideal of R . However, the converse is not true in general (cf. [13]).

Definition 2.1. ([13]) A hemiring R is said to be h-hemiregular if for each $x \in R$, there exist $a, b, t \in R$ such that $x + xax + t = xbx + t$.

Definition 2.2. ([13]) A hemiring R is said to be h-intra-hemiregular if for each $x \in R$, there exist $a_i, a'_i, b_j, b'_j, z \in R$ such that $x + \sum_{i=1}^m a_i x^2 a'_i + z = \sum_{j=1}^n b_j x^2 b'_j + z$.

Lemma 2.3. ([13]) A hemiring R is h-hemiregular if and only if for any right h-ideal I and any left h-ideal L of R we have $\overline{IL} = I \cap L$.

Lemma 2.4. ([13]) Let R be a hemiring. Then the following conditions are equivalent.

- (i) R is h-hemiregular.
- (ii) $B = \overline{BRB}$ for every h-bi-ideal B of R .
- (iii) $Q = \overline{QRQ}$ for every h-quasi-ideal Q of R .

Lemma 2.5. ([13]) A hemiring R is h-intra-hemiregular if and only if for any right h-ideal I and any left h-ideal L of R we have $I \cap L \subseteq \overline{LI}$.

Lemma 2.6. ([13]) The following conditions are equivalent for a hemiring R .

- (i) R is both h-hemiregular and h-intra-hemiregular.

- (ii) $B = \overline{B}^2$ for every h -bi-ideal B of R .
- (iii) $Q = \overline{Q}^2$ for every h -quasi-ideal Q of R .

3. Interval valued fuzzy sets

A fuzzy subset λ of a universe X is a function from X into the unit closed interval $[0, 1]$, that is $\lambda : X \rightarrow [0, 1]$. Now let \mathcal{L} be the family of all closed subintervals of $[0, 1]$ with minimal element $\hat{0} = [0, 0]$ and maximal element $\hat{1} = [1, 1]$ according to the partial order $[\alpha, \alpha'] \leq [\beta, \beta']$ if and only if $\alpha \leq \beta, \alpha' \leq \beta'$ defined on \mathcal{L} for all $[\alpha, \alpha'], [\beta, \beta'] \in \mathcal{L}$.

By an interval number \hat{a} we mean an interval $[a^-, a^+] \in \mathcal{L}$, where $0 \leq a^- \leq a^+ \leq 1$. The interval $[a, a]$ can be identified by the number $a \in [0, 1]$.

An interval valued fuzzy subset $\hat{\lambda}$ of a hemiring R is a function $\hat{\lambda} : R \rightarrow \mathcal{L}$. We write $\hat{\lambda}(x) = [\lambda^-(x), \lambda^+(x)] \subseteq [0, 1]$, for all $x \in R$, where $\lambda^-, \lambda^+ : R \rightarrow [0, 1]$ are fuzzy subsets of R such that for each $x \in R$, $0 \leq \lambda^-(x) \leq \lambda^+(x) \leq 1$. For simplicity we write $\hat{\lambda} = [\lambda^-, \lambda^+]$. Let $\hat{\lambda}$ be an interval valued fuzzy subset of R and $[\alpha, \beta] \in \mathcal{L}$ then the level subset $U(\hat{\lambda}, [\alpha, \beta])$ of R is defined as $U(\hat{\lambda}, [\alpha, \beta]) = \{x \in R : \hat{\lambda}(x) \geq [\alpha, \beta]\}$.

Let A be a subset of a hemiring R . Then the interval valued characteristic function $\hat{\chi}_A$ of A is defined to be a function $\hat{\chi}_A : R \rightarrow \mathcal{L}$ such that for all $x \in R$

$$\hat{\chi}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

Clearly the interval valued characteristic function of any subset of R is also an interval valued fuzzy subset of R . Note that $\hat{\chi}_R(x) = 1$ for all $x \in R$.

An interval valued fuzzy subset $\hat{\lambda}$ of R of the form

$$\hat{\lambda}(y) = \begin{cases} \hat{t} (\neq 0) & \text{if } y = x \\ 0 & \text{if } y \neq x. \end{cases}$$

is said to be fuzzy interval value with support x and value \hat{t} and is denoted by $x_{\hat{t}}$. A fuzzy interval value $x_{\hat{t}}$ is said to belong to (resp. be quasi-coincident with) interval valued fuzzy subset $\hat{\lambda}$, written as $x_{\hat{t}} \in \hat{\lambda}$ (resp. $x_{\hat{t}} q \hat{\lambda}$) if $\hat{\lambda}(x) \geq \hat{t}$ (resp. $\hat{\lambda}(x) + \hat{t} > 1$). $x_{\hat{t}} \in \vee q \hat{\lambda}$ means $x_{\hat{t}} \in \hat{\lambda}$ or $x_{\hat{t}} q \hat{\lambda}$ and $x_{\hat{t}} \in \wedge q \hat{\lambda}$ means $x_{\hat{t}} \in \hat{\lambda}$ and $x_{\hat{t}} q \hat{\lambda}$. $x_{\hat{t}} \bar{\alpha} \hat{\lambda}$ means that $x_{\hat{t}} \alpha \hat{\lambda}$ does not hold for $\alpha \in \{\in, q, \in \vee q, \in \wedge q\}$ [cf. [8]].

For any family $\{\hat{\lambda}_i : i \in I\}$ of interval valued fuzzy subsets of R ,

$$(\vee_i \hat{\lambda}_i)(x) = [\vee_i \lambda_i^-(x), \vee_i \lambda_i^+(x)] \quad \text{and} \quad (\wedge_i \hat{\lambda}_i)(x) = [\wedge_i \lambda_i^-(x), \wedge_i \lambda_i^+(x)].$$

For any two interval valued fuzzy subsets $\hat{\lambda}$ and $\hat{\mu}$ of a hemiring R , $\hat{\lambda} \leq \hat{\mu}$ if and only if $\hat{\lambda}(x) \leq \hat{\mu}(x)$, that is $\lambda^-(x) \leq \mu^-(x)$ and $\lambda^+(x) \leq \mu^+(x)$, for all $x \in R$.

Definition 3.1. ([11]) Let $\hat{\lambda}$ and $\hat{\mu}$ be two interval valued fuzzy subsets of a hemiring R . Then the h-intrinsic product of $\hat{\lambda}$ and $\hat{\mu}$ is defined by

$$(\hat{\lambda} \odot \hat{\mu})(x) = \sup \left[\left\{ \left(\bigwedge_{i=1}^m (\hat{\lambda}(a_i) \wedge \hat{\mu}(b_i)) \right) \wedge \left(\bigwedge_{j=1}^n (\hat{\lambda}(a'_j) \wedge \hat{\mu}(b'_j)) \right) \right\} \right]$$

for all $x \in R$, if x can be expressed as $x + \sum_{i=1}^m a_i b_i + z = \sum_{j=1}^n a'_j b'_j + z$, and 0 if x cannot be expressed as $x + \sum_{i=1}^m a_i b_i + z = \sum_{j=1}^n a'_j b'_j + z$.

Lemma 3.2. ([11]) Let R be a hemiring and $A, B \subseteq R$. Then we have

(i) $A \subseteq B$ if and only if $\hat{\chi}_A \leq \hat{\chi}_B$. Where $\hat{\chi}_A$ is the interval valued characteristic function of A .

(ii) $\hat{\chi}_A \wedge \hat{\chi}_B = \hat{\chi}_{A \cap B}$.

(iii) $\hat{\chi}_A \odot \hat{\chi}_B = \hat{\chi}_{\overline{AB}}$.

4. Interval valued $(\in, \in \vee q)$ -fuzzy ideals

Definition 4.1. ([8]) An interval valued fuzzy subset $\hat{\lambda}$ of a hemiring R is said to be an interval valued $(\in, \in \vee q)$ -fuzzy left (resp. right) h-ideal of R if for all $x, y, z, a, b \in R$ and $t, r \in (0, 1]$,

(1a) $x_{\hat{t}}, y_{\hat{r}} \in \hat{\lambda} \rightarrow (x + y)_{\min\{\hat{t}, \hat{r}\}} \in \vee q \hat{\lambda}$

(2a) $y_{\hat{r}} \in \hat{\lambda}$ and $x \in R \rightarrow (xy)_{\hat{r}} \in \vee q \hat{\lambda}$ (resp. (3a) $x_{\hat{t}} \in \hat{\lambda}$ and $y \in R \rightarrow (xy)_{\hat{t}} \in \vee q \hat{\lambda}$)

(4a) $x + a + y = b + y$ and $a_{\hat{t}}, b_{\hat{r}} \in \hat{\lambda} \rightarrow x_{\min\{\hat{t}, \hat{r}\}} \in \vee q \hat{\lambda}$.

An interval valued fuzzy subset $\hat{\lambda} : R \rightarrow \mathcal{L}$ is called an interval valued $(\in, \in \vee q)$ -fuzzy h-ideal of R if it is both, interval valued $(\in, \in \vee q)$ -fuzzy left and right h-ideal of R .

Definition 4.2. An interval valued fuzzy subset $\hat{\lambda}$ of a hemiring R is said to be an interval valued $(\in, \in \vee q)$ -fuzzy h-bi-ideal of R if it satisfies (1a), (4a) and for all $x, y, z \in R$, $t, r \in (0, 1]$

(5a) $x_{\hat{t}}, y_{\hat{r}} \in \hat{\lambda} \rightarrow (xy)_{\min\{\hat{t}, \hat{r}\}} \in \vee q \hat{\lambda}$

(6a) $x_{\hat{t}}, z_{\hat{r}} \in \hat{\lambda}$ implies $(xyz)_{\min\{\hat{t}, \hat{r}\}} \in \vee q \hat{\lambda}$.

Definition 4.3. An interval valued fuzzy subset $\hat{\lambda}$ of a hemiring R is said to be an interval valued $(\in, \in \vee q)$ -fuzzy h-quasi-ideal of R if it satisfies (1a), (4a) and for all $x \in R$, $t \in (0, 1]$

(7a) $x_{\hat{t}} \in (\hat{\lambda} \odot \hat{\mathcal{R}}) \wedge (\hat{\mathcal{R}} \odot \hat{\lambda}) \rightarrow x_{\hat{t}} \in \hat{\lambda}$.

Theorem 4.4. ([8]) An interval valued fuzzy subset $\hat{\lambda}$ of a hemiring R is an interval valued $(\in, \in \vee q)$ -fuzzy left (resp. right) h-ideal of R if and only if it satisfies

(1b) $\hat{\lambda}(x + y) \geq \min\{\hat{\lambda}(x), \hat{\lambda}(y), 0.5\}$

(2b) $\hat{\lambda}(xy) \geq \min\{\hat{\lambda}(y), 0.5\}$ (resp. (3b) $\hat{\lambda}(xy) \geq \min\{\hat{\lambda}(x), 0.5\}$)

(4b) $x + a + y = b + y \Rightarrow \hat{\lambda}(x) \geq \min\{\hat{\lambda}(a), \hat{\lambda}(b), 0.5\}$
 for all $a, b, x, y \in R$.

Theorem 4.5. An interval valued fuzzy subset $\hat{\lambda}$ of a hemiring R is an interval valued $(\in, \in \vee q)$ -fuzzy h -bi-ideal of R if and only if it satisfies (1b), (4b) and

$$(5b) \hat{\lambda}(xy) \geq \min\{\hat{\lambda}(x), \hat{\lambda}(y), 0.5\}$$

$$(6b) \hat{\lambda}(xyz) \geq \min\{\hat{\lambda}(x), \hat{\lambda}(z), 0.5\}$$

for all $x, y, z \in R$.

Theorem 4.6. An interval valued fuzzy subset $\hat{\lambda}$ of a hemiring R is an interval valued $(\in, \in \vee q)$ -fuzzy h -quasi-ideal of R if and only if it satisfies (1b), (4b) and

$$(7b) \hat{\lambda}(x) \geq \min\{(\hat{\lambda} \odot \hat{\mathcal{R}})(x), (\hat{\mathcal{R}} \odot \hat{\lambda})(x), 0.5\}$$

for all $x \in R$.

Theorem 4.7. ([8]) An interval valued fuzzy subset $\hat{\lambda}$ of a hemiring R is an interval valued $(\in, \in \vee q)$ -fuzzy left (resp., right) h -ideal of R if and only if each non-empty level subset $U(\hat{\lambda}, [\alpha, \beta])$ of R is a fuzzy left (resp., right) h -ideal of R .

Theorem 4.8. An interval valued fuzzy subset $\hat{\lambda}$ of a hemiring R is an interval valued $(\in, \in \vee q)$ -fuzzy h -quasi (resp., h -bi) ideal of R if and only if each non-empty level subset $U(\hat{\lambda}, [\alpha, \beta])$ of R is a fuzzy h -quasi (resp., h -bi) ideal of R .

Proof. Proof is straightforward hence omitted. □

Theorem 4.9. Let $\hat{\lambda}$ be an interval valued $(\in, \in \vee q)$ -fuzzy h -bi-ideal of a hemiring R , then $\hat{\lambda} \wedge 0.5$ is an interval valued $(\in, \in \vee q)$ -fuzzy h -bi-ideal of R where $(\hat{\lambda} \wedge 0.5)(x) = \hat{\lambda}(x) \wedge 0.5$ for all $x \in R$.

Proof. Suppose $\hat{\lambda}$ is an interval valued $(\in, \in \vee q)$ -fuzzy h -bi-ideal of a hemiring R and let $a, b, x, y \in R$. Then

$$\begin{aligned} (\hat{\lambda} \wedge 0.5)(x + y) &= \hat{\lambda}(x + y) \wedge 0.5 \geq (\min\{\hat{\lambda}(x), \hat{\lambda}(y), 0.5\}) \wedge 0.5 \\ &= \min\{\hat{\lambda}(x) \wedge 0.5, \hat{\lambda}(y) \wedge 0.5, 0.5\} \\ &= \min\{(\hat{\lambda} \wedge 0.5)(x), (\hat{\lambda} \wedge 0.5)(y), 0.5\}. \end{aligned}$$

Similarly we can show that

$$(\hat{\lambda} \wedge 0.5)(xy) \geq \min\{(\hat{\lambda} \wedge 0.5)(x), (\hat{\lambda} \wedge 0.5)(y), 0.5\}$$

and

$$(\hat{\lambda} \wedge 0.5)(xyz) \geq \min\{(\hat{\lambda} \wedge 0.5)(x), (\hat{\lambda} \wedge 0.5)(z), 0.5\}.$$

Now let $x + a + y = b + y$, then

$$\begin{aligned} (\hat{\lambda} \wedge 0.5)(x) &= \hat{\lambda}(x) \wedge 0.5 \geq (\min\{\hat{\lambda}(a), \hat{\lambda}(b), 0.5\}) \wedge 0.5 \\ &= \min\{\hat{\lambda}(a) \wedge 0.5, \hat{\lambda}(b) \wedge 0.5, 0.5\} \end{aligned}$$

$$= \min\{(\hat{\lambda} \wedge 0.5)(a), (\hat{\lambda} \wedge 0.5)(b), 0.5\}.$$

This shows that $(\hat{\lambda} \wedge 0.5)$ is an interval valued $(\in, \in \vee q)$ -fuzzy h-bi-ideal of R . \square

Similarly we can show that:

Theorem 4.10. *If $\hat{\lambda}$ is an interval valued $(\in, \in \vee q)$ -fuzzy left (right) h-ideal of a hemiring R , then $\hat{\lambda} \wedge 0.5$ is an interval valued $(\in, \in \vee q)$ -fuzzy left (right) h-ideal of R .*

Definition 4.11. Let $\hat{\lambda}, \hat{\mu}$ be interval valued fuzzy subsets of a hemiring R . Then the fuzzy subsets $\hat{\lambda} \wedge_{0.5} \hat{\mu}$ and $\hat{\lambda} \odot_{0.5} \hat{\mu}$ of R are defined as following:

$$\begin{aligned} \hat{\lambda} \wedge_{0.5} \hat{\mu}(x) &= \min\{\hat{\lambda}(x), \hat{\mu}(x), 0.5\} \\ (\hat{\lambda} \odot_{0.5} \hat{\mu})(x) &= (\hat{\lambda} \odot \hat{\mu})(x) \wedge 0.5 \quad \text{for all } x \in R. \end{aligned}$$

Now we define addition of two interval valued fuzzy subsets of a hemiring.

Definition 4.12. Let $\hat{\lambda}, \hat{\mu}$ be interval valued fuzzy subsets of a hemiring R . The fuzzy subset $\hat{\lambda} + \hat{\mu}$ of R is defined by

$$(\hat{\lambda} + \hat{\mu})(x) = \sup\{\hat{\lambda}(a_1) \wedge \hat{\lambda}(a_2) \wedge \hat{\mu}(b_1) \wedge \hat{\mu}(b_2)\}$$

for all $a_1, a_2, b_1, b_2, x, z \in R$ such that $x + (a_1 + b_1) + z = (a_2 + b_2) + z$.

We also define $\hat{\lambda} +_{0.5} \hat{\mu} = (\hat{\lambda} + \hat{\mu}) \wedge 0.5$.

Lemma 4.13. *Let A, B be subsets of R , then*

$$\hat{\chi}_A +_{0.5} \hat{\chi}_B = \hat{\chi}_{\overline{A+B}} \wedge 0.5.$$

Lemma 4.14. *An interval valued fuzzy subset $\hat{\lambda}$ of a hemiring R satisfies conditions (1b) and (4b) if and only if it satisfies condition*

$$(8b) \quad \hat{\lambda} +_{0.5} \hat{\lambda} \leq \hat{\lambda} \wedge 0.5.$$

Proof. Suppose $\hat{\lambda}$ satisfies conditions (1b) and (4b). Let $x \in R$, then for all expressions $x + (a_1 + b_1) + z = (a_2 + b_2) + z$ of x , for $a_1, a_2, b_1, b_2, z \in R$,

$$\begin{aligned} (\hat{\lambda} +_{0.5} \hat{\lambda})(x) &= \left(\sup\{\hat{\lambda}(a_1) \wedge \hat{\lambda}(a_2) \wedge \hat{\lambda}(b_1) \wedge \hat{\lambda}(b_2)\} \right) \wedge 0.5 \\ &= \left(\sup\{((\hat{\lambda}(a_1) \wedge \hat{\lambda}(b_1) \wedge 0.5) \wedge (\hat{\lambda}(a_2) \wedge \hat{\lambda}(b_2) \wedge 0.5))\} \right) \wedge 0.5 \\ &\leq \left(\sup\{(\hat{\lambda}(a_1 + b_1)) \wedge (\hat{\lambda}(a_2 + b_2))\} \right) \wedge 0.5 \quad \text{by condition(1b)} \\ &\leq \hat{\lambda}(x) \wedge 0.5 \quad \text{by condition(4b)}. \end{aligned}$$

Thus $\hat{\lambda} +_{0.5} \hat{\lambda} \leq \hat{\lambda} \wedge 0.5$.

Conversely, assume that $\hat{\lambda} +_{0.5} \hat{\lambda} \leq \hat{\lambda} \wedge 0.5$. Then for each $x \in R$ we have

$$\begin{aligned} \hat{\lambda}(0) &\geq \hat{\lambda}(0) \wedge 0.5 \geq (\hat{\lambda} +_{0.5} \hat{\lambda})(0) \\ &= \left(\sup \left\{ \hat{\lambda}(a_1) \wedge \hat{\lambda}(a_2) \wedge \hat{\lambda}(b_1) \wedge \hat{\lambda}(b_2) \right\} \right) \wedge 0.5 \\ &\geq \hat{\lambda}(x) \wedge 0.5 \quad \text{because } 0 + (x + x) + 0 = (x + x) + 0. \end{aligned}$$

Thus $\hat{\lambda}(0) \geq \hat{\lambda}(x) \wedge 0.5$.

Let $x, y \in R$, then for all expressions $(x + y) + (a_1 + b_1) + z = (a_2 + b_2) + z$

$$\begin{aligned} \hat{\lambda}(x + y) &\geq \hat{\lambda}(x + y) \wedge 0.5 \\ &\geq (\hat{\lambda} +_{0.5} \hat{\lambda})(x + y) \\ &= \left(\sup \left\{ \hat{\lambda}(a_1) \wedge \hat{\lambda}(a_2) \wedge \hat{\lambda}(b_1) \wedge \hat{\lambda}(b_2) \right\} \right) \wedge 0.5 \\ &\geq \left\{ \hat{\lambda}(0) \wedge \hat{\lambda}(x) \wedge \hat{\lambda}(0) \wedge \hat{\lambda}(y) \right\} \wedge 0.5 \\ &\quad \text{because } (x + y) + (0 + 0) + 0 = (x + y) + 0, \\ &= \left\{ \hat{\lambda}(x) \wedge \hat{\lambda}(y) \wedge 0.5 \right\} \quad \text{because } \hat{\lambda}(0) \geq \hat{\lambda}(x) \wedge 0.5. \end{aligned}$$

Thus $\hat{\lambda}$ satisfies condition (1b).

Let $a, b, x, z \in R$ such that $x + a + z = b + z$. Then for all such expressions

$$\begin{aligned} \hat{\lambda}(x) &\geq \hat{\lambda}(x) \wedge 0.5 \\ &\geq (\hat{\lambda} +_{0.5} \hat{\lambda})(x) \\ &= \left(\sup \left\{ \hat{\lambda}(a_1) \wedge \hat{\lambda}(a_2) \wedge \hat{\lambda}(b_1) \wedge \hat{\lambda}(b_2) \right\} \right) \wedge 0.5 \\ &\geq \left\{ \hat{\lambda}(a) \wedge \hat{\lambda}(0) \wedge \hat{\lambda}(b) \wedge 0.5 \right\} \quad \text{because } x + (a + 0) + z = (b + 0) + z, \\ &= \left\{ \hat{\lambda}(a) \wedge \hat{\lambda}(b) \wedge 0.5 \right\} \quad \text{because } \hat{\lambda}(0) \geq \hat{\lambda}(x) \wedge 0.5. \end{aligned}$$

This shows that $\hat{\lambda}$ satisfies condition (4b). \square

By using Lemma 4.14, we can also prove the following:

Theorem 4.15. *An interval valued fuzzy subset $\hat{\lambda}$ of a hemiring R is an interval valued $(\in, \in \vee q)$ -fuzzy left (resp. right) h -ideal of R if and only if $\hat{\lambda}$ satisfies conditions*

$$\begin{aligned} (8b) \quad &\hat{\lambda} +_{0.5} \hat{\lambda} \leq \hat{\lambda} \wedge 0.5 \\ (9b) \quad &\hat{\mathcal{R}} \odot_{0.5} \hat{\lambda} \leq \hat{\lambda} \wedge 0.5 \quad \left(\text{resp. } \hat{\lambda} \odot_{0.5} \hat{\mathcal{R}} \leq \hat{\lambda} \wedge 0.5 \right). \end{aligned}$$

Theorem 4.16. *An interval valued fuzzy subset $\hat{\lambda}$ of a hemiring R is an interval valued $(\in, \in \vee q)$ -fuzzy h -quasi-ideal of R if and only if $\hat{\lambda}$ satisfies conditions (7b) and (8b).*

Proof. Proof is straight forward because conditions (1b) and (4b) are equivalent to condition (8b). \square

Theorem 4.17. *Every interval valued $(\in, \in \vee q)$ -fuzzy left h -ideal of a hemiring R is an interval valued $(\in, \in \vee q)$ -fuzzy h -quasi-ideal of R .*

Lemma 4.18. *Every interval valued $(\in, \in \vee q)$ -fuzzy h -quasi-ideal of R is an interval valued $(\in, \in \vee q)$ -fuzzy h -bi-ideal of R .*

Proofs of the following results are straightforward by using the techniques as in [9].

Lemma 4.19. *If $\hat{\lambda}$ and $\hat{\mu}$ are interval valued $(\in, \in \vee q)$ -fuzzy right and left h -ideal of R respectively, then $\hat{\lambda} \odot_{0.5} \hat{\mu} \leq \hat{\lambda} \wedge_{0.5} \hat{\mu}$.*

Theorem 4.20. *For a hemiring R the following conditions are equivalent.*

- (i) R is h -hemiregular.
- (ii) $\hat{\lambda} \wedge_{0.5} \hat{\mu} = \hat{\lambda} \odot_{0.5} \hat{\mu}$ for every interval valued $(\in, \in \vee q)$ -fuzzy right h -ideal $\hat{\lambda}$ and every interval valued $(\in, \in \vee q)$ -fuzzy left h -ideal $\hat{\mu}$ of R .

Theorem 4.21. *For a hemiring R , the following conditions are equivalent.*

- (i) R is h -hemiregular.
- (ii) $\hat{\lambda} \wedge_{0.5} \leq \hat{\lambda} \odot_{0.5} \hat{\mathcal{R}} \odot_{0.5} \hat{\lambda}$ for every interval valued $(\in, \in \vee q)$ -fuzzy h -bi-ideal $\hat{\lambda}$ of R .
- (iii) $\hat{\lambda} \wedge_{0.5} \leq \hat{\lambda} \odot_{0.5} \hat{\mathcal{R}} \odot_{0.5} \hat{\lambda}$ for every interval valued $(\in, \in \vee q)$ -fuzzy h -quasi-ideal $\hat{\lambda}$ of R .

Theorem 4.22. *For a hemiring R , the following conditions are equivalent.*

- (i) R is h -hemiregular.
- (ii) $\hat{\lambda} \wedge_{0.5} \hat{\mu} \leq \hat{\lambda} \odot_{0.5} \hat{\mu} \odot_{0.5} \hat{\lambda}$ for every interval valued $(\in, \in \vee q)$ -fuzzy h -bi-ideal $\hat{\lambda}$ and every interval valued $(\in, \in \vee q)$ -fuzzy h -ideal $\hat{\mu}$ of R .
- (iii) $\hat{\lambda} \wedge_{0.5} \hat{\mu} \leq \hat{\lambda} \odot_{0.5} \hat{\mu} \odot_{0.5} \hat{\lambda}$ for every interval valued $(\in, \in \vee q)$ -fuzzy h -quasi-ideal $\hat{\lambda}$ and every interval valued $(\in, \in \vee q)$ -fuzzy h -ideal $\hat{\mu}$ of R .

Theorem 4.23. *For a hemiring R , the following conditions are equivalent.*

- (i) R is h -hemiregular.
- (ii) $\hat{\lambda} \wedge_{0.5} \hat{\mu} \leq \hat{\lambda} \odot_{0.5} \hat{\mu}$ for every interval valued $(\in, \in \vee q)$ -fuzzy h -bi-ideal $\hat{\lambda}$ and every interval valued $(\in, \in \vee q)$ -fuzzy left h -ideal $\hat{\mu}$ of R .
- (iii) $\hat{\lambda} \wedge_{0.5} \hat{\mu} \leq \hat{\lambda} \odot_{0.5} \hat{\mu}$ for every interval valued $(\in, \in \vee q)$ -fuzzy h -quasi-ideal $\hat{\lambda}$ and every interval valued $(\in, \in \vee q)$ -fuzzy left h -ideal $\hat{\mu}$ of R .
- (iv) $\hat{\lambda} \wedge_{0.5} \hat{\mu} \leq \hat{\lambda} \odot_{0.5} \hat{\mu}$ for every interval valued $(\in, \in \vee q)$ -fuzzy right h -ideal $\hat{\lambda}$ and every interval valued $(\in, \in \vee q)$ -fuzzy h -bi-ideal $\hat{\mu}$ of R .
- (v) $\hat{\lambda} \wedge_{0.5} \hat{\mu} \leq \hat{\lambda} \odot_{0.5} \hat{\mu}$ for every interval valued $(\in, \in \vee q)$ -fuzzy right h -ideal $\hat{\lambda}$ and every interval valued $(\in, \in \vee q)$ -fuzzy h -quasi-ideal $\hat{\mu}$ of R .
- (vi) $\hat{\lambda} \wedge_{0.5} \hat{\mu} \wedge_{0.5} \hat{\nu} \leq \hat{\lambda} \odot_{0.5} \hat{\mu} \odot_{0.5} \hat{\nu}$ for every interval valued $(\in, \in \vee q)$ -fuzzy right h -ideal $\hat{\lambda}$, every interval valued $(\in, \in \vee q)$ -fuzzy h -bi-ideal $\hat{\mu}$ and every interval valued $(\in, \in \vee q)$ -fuzzy left h -ideal $\hat{\nu}$ of R .

(vii) $\hat{\lambda} \wedge_{0.5} \hat{\mu} \wedge_{0.5} \hat{\nu} \leq \hat{\lambda} \odot_{0.5} \hat{\mu} \odot_{0.5} \hat{\nu}$ for every interval valued $(\in, \in \vee q)$ -fuzzy right h -ideal $\hat{\lambda}$, every interval valued $(\in, \in \vee q)$ -fuzzy h -quasi-ideal $\hat{\mu}$ and every interval valued $(\in, \in \vee q)$ -fuzzy left h -ideal $\hat{\nu}$ of R .

Proof. (i) \Rightarrow (ii) Let $\hat{\lambda}$ be any interval valued $(\in, \in \vee q)$ -fuzzy h -bi-ideal and $\hat{\mu}$ any interval valued $(\in, \in \vee q)$ -fuzzy left h -ideal of R . Since R is h -hemiregular, so for any $a \in R$ there exist $x_1, x_2, z \in R$ such that $a + ax_1a + z = ax_2a + z$. Then for all expressions of "a" of the form $a + \sum_{i=1}^m a_i b_i + z = \sum_{j=1}^n a'_j b'_j + z$ in R we have

$$\begin{aligned} & (\hat{\lambda} \odot_{0.5} \hat{\mu})(a) \\ &= \sup \left(\left(\bigwedge_{i=1}^m (\hat{\lambda}(a_i) \wedge \hat{\mu}(b_i)) \right) \wedge \left(\bigwedge_{j=1}^n (\hat{\lambda}(a'_j) \wedge \hat{\mu}(b'_j)) \right) \right) \wedge 0.5 \\ &\geq \{\hat{\lambda}(a) \wedge \hat{\mu}(x_1a) \wedge \hat{\mu}(x_2a)\} \wedge 0.5 \quad \text{because } a + ax_1a + z = ax_2a + z \\ &\geq \{\hat{\lambda}(a) \wedge \hat{\mu}(a) \wedge 0.5\} \wedge 0.5 \\ &= \{\hat{\lambda}(a) \wedge \hat{\mu}(a) \wedge 0.5\} = (\hat{\lambda} \wedge_{0.5} \hat{\mu})(a). \end{aligned}$$

So $\hat{\lambda} \odot_{0.5} \hat{\mu} \geq \hat{\lambda} \wedge_{0.5} \hat{\mu}$.

(ii) \Rightarrow (iii) Obvious.

(iii) \Rightarrow (i) Let $\hat{\lambda}$ be an interval valued $(\in, \in \vee q)$ -fuzzy right h -ideal and $\hat{\mu}$ be an interval valued $(\in, \in \vee q)$ -fuzzy left h -ideal of R . Since every interval valued $(\in, \in \vee q)$ -fuzzy right h -ideal is an interval valued $(\in, \in \vee q)$ -fuzzy h -quasi-ideal, so by (iii) we have $\hat{\lambda} \odot_{0.5} \hat{\mu} \geq \hat{\lambda} \wedge_{0.5} \hat{\mu}$. But by Lemma 4.19, $\hat{\lambda} \odot_{0.5} \hat{\mu} \leq \hat{\lambda} \wedge_{0.5} \hat{\mu}$. Hence $\hat{\lambda} \odot_{0.5} \hat{\mu} = \hat{\lambda} \wedge_{0.5} \hat{\mu}$ for every interval valued $(\in, \in \vee q)$ -fuzzy right h -ideal $\hat{\lambda}$ of R , and every interval valued $(\in, \in \vee q)$ -fuzzy left h -ideal $\hat{\mu}$ of R . Thus by Theorem 4.20, R is h -hemiregular.

Similarly we can show that (i) \Leftrightarrow (iv) \Leftrightarrow (v).

(i) \Rightarrow (vi) Let $\hat{\lambda}$ be an interval valued $(\in, \in \vee q)$ -fuzzy right h -ideal, $\hat{\mu}$ be an interval valued $(\in, \in \vee q)$ -fuzzy h -bi-ideal and $\hat{\nu}$ be an interval valued $(\in, \in \vee q)$ -fuzzy left h -ideal of R . Since R is h -hemiregular, so for any $a \in R$ there exist $x_1, x_2, z \in R$ such that $a + ax_1a + z = ax_2a + z$. Then for all expressions of "a" of the form $a + \sum_{i=1}^m a_i b_i + z = \sum_{j=1}^n a'_j b'_j + z$ in R we have

$$\begin{aligned} & (\hat{\lambda} \odot_{0.5} \hat{\mu} \odot_{0.5} \hat{\nu})(a) \\ &= \sup \left(\left(\bigwedge_{i=1}^m ((\hat{\lambda} \odot_{0.5} \hat{\mu})(a_i) \wedge \hat{\nu}(b_i)) \right) \wedge \left(\bigwedge_{j=1}^n ((\hat{\lambda} \odot_{0.5} \hat{\mu})(a'_j) \wedge \hat{\nu}(b'_j)) \right) \right) \\ &\quad \wedge 0.5 \\ &\geq \{(\hat{\lambda} \odot_{0.5} \hat{\mu})(a) \wedge \hat{\nu}(x_1a) \wedge \hat{\nu}(x_2a) \wedge 0.5\} \end{aligned}$$

$$\begin{aligned}
&\geq \sup \left(\left(\bigwedge_{i=1}^m (\hat{\lambda}(a_i) \wedge \hat{\mu}(b_i)) \right) \wedge \left(\bigwedge_{j=1}^n (\hat{\lambda}(a'_j) \wedge \hat{\mu}(b'_j)) \right) \right) \wedge 0.5 \wedge \hat{\nu}(a) \\
&\geq \{\hat{\lambda}(ax_1) \wedge \hat{\lambda}(ax_2) \wedge \hat{\mu}(a) \wedge \hat{\nu}(a) \wedge 0.5\} \\
&\geq \{\hat{\lambda}(a) \wedge \hat{\mu}(a) \wedge \hat{\nu}(a) \wedge 0.5\} \\
&= \left(\hat{\lambda} \wedge_{0.5} \hat{\mu} \wedge_{0.5} \hat{\nu} \right) (a).
\end{aligned}$$

Thus $\hat{\lambda} \wedge_{0.5} \hat{\mu} \wedge_{0.5} \hat{\nu} \leq \hat{\lambda} \odot_{0.5} \hat{\mu} \odot_{0.5} \hat{\nu}$.

(vi) \Rightarrow (vii) Straightforward..

(vii) \Rightarrow (i) Let $\hat{\lambda}$ be an interval valued $(\in, \in \vee q)$ -fuzzy right h -ideal, and $\hat{\nu}$ be an interval valued $(\in, \in \vee q)$ -fuzzy left h -ideal of R . Then

$$\hat{\lambda} \wedge_{0.5} \hat{\nu} = \hat{\lambda} \wedge_{0.5} \hat{\mathcal{R}} \wedge_{0.5} \hat{\nu} \leq \hat{\lambda} \odot_{0.5} \hat{\mathcal{R}} \odot_{0.5} \hat{\nu} \leq \hat{\lambda} \odot_{0.5} \hat{\nu}.$$

But $\hat{\lambda} \odot_{0.5} \hat{\nu} \leq \hat{\lambda} \wedge_{0.5} \hat{\nu}$. Hence $\hat{\lambda} \odot_{0.5} \hat{\nu} = \hat{\lambda} \wedge_{0.5} \hat{\nu}$ for every interval valued $(\in, \in \vee q)$ -fuzzy right h -ideal $\hat{\lambda}$ and for every interval valued $(\in, \in \vee q)$ -fuzzy left h -ideal $\hat{\nu}$ of R . Thus by Theorem 4.20, R is h -hemiregular. \square

Lemma 4.24. *A hemiring R is h -intra-hemiregular if and only if $\hat{\lambda} \wedge_{0.5} \hat{\mu} \leq \hat{\lambda} \odot_{0.5} \hat{\mu}$ for every interval valued $(\in, \in \vee q)$ -fuzzy left h -ideal $\hat{\lambda}$ and for every interval valued $(\in, \in \vee q)$ -fuzzy right h -ideal $\hat{\mu}$ of R .*

Proof. Proof is straightforward by using the techniques as are in [9]. \square

Theorem 4.25. *The following conditions are equivalent for a hemiring R :*

- (i) R is both h -hemiregular and h -intra-hemiregular.
- (ii) $\hat{\lambda} \wedge 0.5 = \hat{\lambda} \odot_{0.5} \hat{\lambda}$ for every interval valued $(\in, \in \vee q)$ -fuzzy h -bi-ideal $\hat{\lambda}$ of R .
- (iii) $\hat{\lambda} \wedge 0.5 = \hat{\lambda} \odot_{0.5} \hat{\lambda}$ for every interval valued $(\in, \in \vee q)$ -fuzzy h -quasi-ideal $\hat{\lambda}$ of R .

Proof. (i) \Rightarrow (ii) Let $\hat{\lambda}$ be an interval valued $(\in, \in \vee q)$ -fuzzy h -bi-ideal of R and $x \in R$. Since R is both h -hemiregular and h -intra-hemiregular, there exist elements $a_1, a_2, p_i, p'_i, q_j, q'_j, z \in R$ such that

$$\begin{aligned}
&x + \sum_{j=1}^n (xa_2q_jx)(xq'_ja_1x) + \sum_{j=1}^n (xa_1q_jx)(xq'_ja_2x) + \sum_{i=1}^m (xa_1p_ix)(xp'_ia_1x) \\
&+ \sum_{i=1}^m (xa_2p_ix)(xp'_ia_2x) + z = \sum_{i=1}^m (xa_2p_ix)(xp'_ia_1x) + \sum_{i=1}^m (xa_1p_ix)(xp'_ia_2x) \\
&+ \sum_{j=1}^n (xa_1q_jx)(xq'_ja_1x) + \sum_{j=1}^n (xa_2q_jx)(xq'_ja_2x) + z
\end{aligned}$$

(As given in Lemma 5.6 [13]). Then for all such expressions

$$\begin{aligned}
&(\hat{\lambda} \odot_{0.5} \hat{\lambda})(x) \\
&= \sup \left(\left(\bigwedge_{i=1}^m (\hat{\lambda}(a_i) \wedge \hat{\lambda}(b_i)) \right) \wedge \left(\bigwedge_{j=1}^n (\hat{\lambda}(a'_j) \wedge \hat{\lambda}(b'_j)) \right) \right) \wedge 0.5 \\
&\geq \left(\bigwedge_{j=1}^n (\hat{\lambda}(xa_2q_jx) \wedge \hat{\lambda}(xq'_ja_1x) \wedge \hat{\lambda}(xa_1q_jx) \wedge \hat{\lambda}(xq'_ja_2x)) \right) \wedge
\end{aligned}$$

$$\begin{aligned} & \left(\bigwedge_{i=1}^m \left(\left(\hat{\lambda}(xa_1p_i x) \wedge \hat{\lambda}(xp'_i a_1 x) \right) \wedge \hat{\lambda}(xa_2p_i x) \wedge \hat{\lambda}(xp'_i a_2 x) \right) \right) \wedge 0.5 \\ & \geq \min\{\hat{\lambda}(x), 0.5\} \wedge 0.5 = \hat{\lambda}(x) \wedge 0.5. \end{aligned}$$

This implies that $\hat{\lambda} \odot_{0.5} \hat{\lambda} \geq \hat{\lambda} \wedge 0.5$. Further if $x + \sum_{i=1}^m a_i b_i + z = \sum_{j=1}^n a'_j b'_j + z$, we have

$$\begin{aligned} \hat{\lambda}(x) \wedge 0.5 & \geq \min\{\hat{\lambda}(\sum_{i=1}^m a_i b_i), \hat{\lambda}(\sum_{j=1}^n a'_j b'_j), 0.5\} \wedge 0.5 \\ & \geq \min \left\{ \left(\bigwedge_{i=1}^m \hat{\lambda}(a_i b_i) \right), \left(\bigwedge_{j=1}^n \hat{\lambda}(a'_j b'_j) \right), 0.5 \right\} \\ & \geq \min \left\{ \left(\bigwedge_{i=1}^m \hat{\lambda}(a_i) \wedge \hat{\lambda}(b_i) \right), \left(\bigwedge_{j=1}^n \hat{\lambda}(a'_j) \wedge \hat{\lambda}(b'_j) \right), 0.5 \right\} \end{aligned}$$

because $\hat{\lambda}$ is an interval valued $(\in, \in \vee q)$ -fuzzy h -bi-ideal of R . Then for all expressions $x + \sum_{i=1}^m a_i b_i + z = \sum_{j=1}^n a'_j b'_j + z$ of x in R

$$\begin{aligned} & (\hat{\lambda} \odot_{0.5} \hat{\lambda})(x) \\ & = \sup \left(\left(\bigwedge_{i=1}^m \left(\hat{\lambda}(a_i) \wedge \hat{\lambda}(b_i) \right) \right) \wedge \left(\bigwedge_{j=1}^n \left(\hat{\lambda}(a'_j) \wedge \hat{\lambda}(b'_j) \right) \right) \right) \wedge 0.5 \\ & \leq \hat{\lambda}(x) \wedge 0.5. \end{aligned}$$

Consequently $\hat{\lambda} \wedge 0.5 = \hat{\lambda} \odot_{0.5} \hat{\lambda}$.

(ii) \Rightarrow (iii) This is straightforward because every interval valued $(\in, \in \vee q)$ -fuzzy h -quasi-ideal of R is an interval valued $(\in, \in \vee q)$ -fuzzy h -bi-ideal of R .

(iii) \Rightarrow (i) Let Q be an h -quasi-ideal of R . Then $\hat{\chi}_Q$ is an interval valued $(\in, \in \vee q)$ -fuzzy h -quasi-ideal of R . Thus by hypothesis

$$\hat{\chi}_Q \wedge 0.5 = \hat{\chi}_Q \odot_{0.5} \hat{\chi}_Q = \hat{\chi}_Q \odot \hat{\chi}_Q \wedge 0.5 = \hat{\chi}_{Q^2} \wedge 0.5.$$

Then it follows $Q = \overline{Q^2}$. Hence by Lemma 2.6, R is both h -hemiregular and h -intra-hemiregular. \square

Theorem 4.26. *The following conditions are equivalent for a hemiring R :*

- (i) R is both h -hemiregular and h -intra-hemiregular.
- (ii) $\hat{\lambda} \wedge_{0.5} \hat{\mu} \leq \hat{\lambda} \odot_{0.5} \hat{\mu}$ for all interval valued $(\in, \in \vee q)$ -fuzzy h -bi-ideals $\hat{\lambda}$ and $\hat{\mu}$ of R .
- (iii) $\hat{\lambda} \wedge_{0.5} \hat{\mu} \leq \hat{\lambda} \odot_{0.5} \hat{\mu}$ for every interval valued $(\in, \in \vee q)$ -fuzzy h -bi-ideal $\hat{\lambda}$ and every interval valued $(\in, \in \vee q)$ -fuzzy h -quasi-ideals $\hat{\mu}$ of R .
- (iv) $\hat{\lambda} \wedge_{0.5} \hat{\mu} \leq \hat{\lambda} \odot_{0.5} \hat{\mu}$ for every interval valued $(\in, \in \vee q)$ -fuzzy h -quasi-ideal $\hat{\lambda}$ and every interval valued $(\in, \in \vee q)$ -fuzzy h -bi-ideals $\hat{\mu}$ of R .
- (v) $\hat{\lambda} \wedge_{0.5} \hat{\mu} \leq \hat{\lambda} \odot_{0.5} \hat{\mu}$ for all interval valued $(\in, \in \vee q)$ -fuzzy h -quasi-ideals $\hat{\lambda}$ and $\hat{\mu}$ of R .

Proof. (i) \Rightarrow (ii) Similar as in Theorem 4.25.

(ii) \Rightarrow (iii) \Rightarrow (iv) and (ii) \Rightarrow (iv) \Rightarrow (v) are clear.

(v) \Rightarrow (i) Let $\hat{\lambda}$ be an interval valued $(\in, \in \vee q)$ -fuzzy left h -ideals of R and $\hat{\mu}$ be an interval valued $(\in, \in \vee q)$ -fuzzy right h -ideal of R . Then $\hat{\lambda}$ and $\hat{\mu}$ are interval valued $(\in, \in \vee q)$ -fuzzy h -bi-ideals of R . So by hypothesis $\hat{\lambda} \wedge_{0.5} \hat{\mu} \leq \hat{\lambda} \odot_{0.5} \hat{\mu}$ but $\hat{\lambda} \wedge_{0.5} \hat{\mu} \geq \hat{\lambda} \odot_{0.5} \hat{\mu}$ by Lemma 4.19. Thus $\hat{\lambda} \wedge_{0.5} \hat{\mu} = \hat{\lambda} \odot_{0.5} \hat{\mu}$. Hence by Theorem 4.20, R is h -hemiregular. On the other hand by hypothesis we also have $\hat{\lambda} \wedge_{0.5} \hat{\mu} \leq \hat{\lambda} \odot_{0.5} \hat{\mu}$. By Lemma 4.24, R is h -intra-hemiregular. \square

5. Interval valued $(\bar{\in}, \bar{\in} \vee \bar{q})$ -fuzzy h -ideal

In this section we define interval valued $(\bar{\in}, \bar{\in} \vee \bar{q})$ -fuzzy h -ideals, interval valued $(\bar{\in}, \bar{\in} \vee \bar{q})$ -fuzzy h -quasi-ideals, interval valued $(\bar{\in}, \bar{\in} \vee \bar{q})$ -fuzzy h -bi-ideals and characterize h -hemiregular and h -intra-hemiregular hemirings by the properties of these ideals.

Definition 5.1. An interval valued fuzzy subset $\hat{\lambda}$ of a hemiring R is said to be an interval valued $(\bar{\in}, \bar{\in} \vee \bar{q})$ -fuzzy left (resp. right) h -ideal of R if for all $x, y, z, a, b \in R$ and $t, r \in (0, 1]$,

$$(1'a) (x + y)_{\min\{t, r\}} \bar{\in} \hat{\lambda} \rightarrow x_t \bar{\in} \vee \bar{q} \hat{\lambda} \text{ or } y_r \bar{\in} \vee \bar{q} \hat{\lambda}.$$

$$(2'a) (xy)_t \bar{\in} \hat{\lambda} \rightarrow y_t \bar{\in} \vee \bar{q} \hat{\lambda} \text{ (resp. } (3'a) (xy)_t \bar{\in} \hat{\lambda} \rightarrow x_t \bar{\in} \vee \bar{q} \hat{\lambda})$$

$$(4'a) (x)_{\min\{t, r\}} \bar{\in} \hat{\lambda} \rightarrow a_t \bar{\in} \vee \bar{q} \hat{\lambda} \text{ or } b_r \bar{\in} \vee \bar{q} \hat{\lambda}, \text{ for all } a, b, x, z \in R \text{ with } x + a + z = b + z.$$

An interval valued fuzzy subset $\hat{\lambda} : R \rightarrow \mathcal{L}$ is called an interval valued $(\bar{\in}, \bar{\in} \vee \bar{q})$ -fuzzy h -ideal of R if it is both, interval valued $(\bar{\in}, \bar{\in} \vee \bar{q})$ -fuzzy left and right h -ideal of R .

Definition 5.2. An interval valued fuzzy subset $\hat{\lambda}$ of a hemiring R is said to be an interval valued $(\bar{\in}, \bar{\in} \vee \bar{q})$ -fuzzy h -bi-ideal of R if it satisfies (1'a), (4'a) and for all $x, y, z \in R$, $t, r \in (0, 1]$

$$(5'a) (xy)_{\min\{t, r\}} \bar{\in} \hat{\lambda} \rightarrow x_t \bar{\in} \vee \bar{q} \hat{\lambda} \text{ or } y_r \bar{\in} \vee \bar{q} \hat{\lambda}.$$

$$(6'a) (xyz)_{\min\{t, r\}} \bar{\in} \hat{\lambda} \rightarrow x_t \bar{\in} \vee \bar{q} \hat{\lambda} \text{ or } z_r \bar{\in} \vee \bar{q} \hat{\lambda}.$$

Definition 5.3. An interval valued fuzzy subset $\hat{\lambda}$ of a hemiring R is said to be an interval valued $(\bar{\in}, \bar{\in} \vee \bar{q})$ -fuzzy h -quasi-ideal of R if it satisfies (1'a), (4'a) and for all $x \in R$, $t \in (0, 1]$

$$(7'a) x_t \bar{\in} \hat{\lambda} \rightarrow x_t \bar{\in} \vee \bar{q}(\hat{\lambda} \odot \hat{\mathcal{R}} \wedge \hat{\mathcal{R}} \odot \hat{\lambda}).$$

Theorem 5.4. An interval valued fuzzy subset $\hat{\lambda}$ of a hemiring R is an interval valued $(\bar{\in}, \bar{\in} \vee \bar{q})$ -fuzzy left (resp. right) h -ideal of R if and only if it satisfies

$$(1'b) \max\{\hat{\lambda}(x + y), 0.5\} \geq \min\{\hat{\lambda}(x), \hat{\lambda}(y)\}$$

$$(2'b) \max\{\hat{\lambda}(xy), 0.5\} \geq \hat{\lambda}(y) \text{ (resp. } (3'b) \max\{\hat{\lambda}(xy), 0.5\} \geq \hat{\lambda}(x))$$

$$(4'b) x + a + y = b + y \Rightarrow \max\{\hat{\lambda}(x), 0.5\} \geq \min\{\hat{\lambda}(a), \hat{\lambda}(b)\}$$

for all $a, b, x, y \in R$.

Proof. (1'a) \Rightarrow (1'b), suppose (1'b) does not hold, then there exists $x, y \in R$ such that $\max\{\hat{\lambda}(x+y), 0.5\} < \min\{\hat{\lambda}(x), \hat{\lambda}(y)\}$. Then for some \hat{t} , $\max\{\hat{\lambda}(x+y), 0.5\} < \hat{t} < \min\{\hat{\lambda}(x), \hat{\lambda}(y)\}$. Which implies $(x+y)_{\hat{t}} \in \hat{\lambda}$, $0.5 < \hat{t} \leq 1$, $\hat{\lambda}(x) > \hat{t}$ and $\hat{\lambda}(y) > \hat{t}$. Then $x_{\hat{t}} \in \wedge q \hat{\lambda}$ and $y_{\hat{t}} \in \wedge q \hat{\lambda}$. Which contradicts (1'a).

(1'b) \Rightarrow (1'a), Let $(x+y)_{\min\{\hat{t}, \hat{r}\}} \in \hat{\lambda}$, then $\hat{\lambda}(x+y) < \min\{\hat{t}, \hat{r}\}$.

Case I: If $\hat{\lambda}(x+y) \geq \min\{\hat{\lambda}(x), \hat{\lambda}(y)\}$, then $\min\{\hat{\lambda}(x), \hat{\lambda}(y)\} < \min\{\hat{t}, \hat{r}\}$, which implies $\hat{\lambda}(x) < \hat{t}$ or $\hat{\lambda}(y) < \hat{r}$. Then $x_{\hat{t}} \in \vee \bar{q} \hat{\lambda}$ or $y_{\hat{r}} \in \vee \bar{q} \hat{\lambda}$.

Case II: If $\hat{\lambda}(x+y) < \min\{\hat{\lambda}(x), \hat{\lambda}(y)\}$, then $0.5 \geq \min\{\hat{\lambda}(x), \hat{\lambda}(y)\}$. Let for some $\hat{t}, \hat{r}, x_{\hat{t}} \in \hat{\lambda}, y_{\hat{r}} \in \hat{\lambda}$. Then $\hat{t} \leq \hat{\lambda}(x) \leq 0.5$ or $\hat{r} \leq \hat{\lambda}(y) \leq 0.5$, consequently $x_{\hat{t}} \in \vee \bar{q} \hat{\lambda}$ or $y_{\hat{r}} \in \vee \bar{q} \hat{\lambda}$.

Similarly it can be proved (2'a), (3'a) and (4'a) are equivalent to (2'b), (3'b) and (4'b) respectively. \square

Theorem 5.5. An interval valued fuzzy subset $\hat{\lambda}$ of a hemiring R is an interval valued $(\bar{\in}, \bar{\in} \vee \bar{q})$ -fuzzy h -bi-ideal of R if and only if it satisfies (1b), (4b) and

$$(5'b) \max\{\hat{\lambda}(xy), 0.5\} \geq \min\{\hat{\lambda}(x), \hat{\lambda}(y)\}$$

$$(6'b) \max\{\hat{\lambda}(xyz), 0.5\} \geq \min\{\hat{\lambda}(x), \hat{\lambda}(z)\}$$

for all $x, y, z \in R$.

Theorem 5.6. An interval valued fuzzy subset $\hat{\lambda}$ of a hemiring R is an interval valued $(\bar{\in}, \bar{\in} \vee \bar{q})$ -fuzzy h -quasi-ideal of R if and only if it satisfies (1b), (4b) and

$$(7'b) \max\{\hat{\lambda}(x), 0.5\} \geq \min\{(\hat{\lambda} \odot \hat{\mathcal{R}})(x), (\hat{\mathcal{R}} \odot \hat{\lambda})(x)\}$$

for all $x \in R$.

Theorem 5.7. ([8]) An interval valued fuzzy subset $\hat{\lambda}$ of a hemiring R is an interval valued $(\bar{\in}, \bar{\in} \vee \bar{q})$ -fuzzy left (resp., right) h -ideal of R if and only if each non-empty level subset $U(\hat{\lambda}, [\alpha, \beta])$ of R is a fuzzy left (resp., right) h -ideal of R .

Theorem 5.8. An interval valued fuzzy subset $\hat{\lambda}$ of a hemiring R is an interval valued $(\bar{\in}, \bar{\in} \vee \bar{q})$ -fuzzy h -quasi (resp., h -bi) ideal of R if and only if each non-empty level subset $U(\hat{\lambda}, [\alpha, \beta])$ of R is a fuzzy h -quasi (resp., h -bi) ideal of R .

Theorem 5.9. A non-empty subset A of R is a left h -ideal (resp. right h -ideal) of R if and only if $\hat{\chi}_A$ is an interval valued $(\bar{\in}, \bar{\in} \vee \bar{q})$ -fuzzy left h -ideal (resp. right h -ideal) of R .

Theorem 5.10. A non-empty subset A of R is an h -quasi-ideal (resp. h -bi-ideal) of R if and only if $\hat{\chi}_A$ is an interval valued $(\bar{\in}, \bar{\in} \vee \bar{q})$ -fuzzy h -quasi-ideal (resp. h -bi-ideal) of R .

Definition 5.11. Let $\hat{\lambda}$ and $\hat{\mu}$ be interval valued fuzzy subsets of a hemiring R then the fuzzy subsets $\hat{\lambda} \vee 0.5$, $\hat{\lambda} \wedge^{0.5} \hat{\mu}$, $\hat{\lambda} \odot^{0.5} \hat{\mu}$ and $\hat{\lambda} +^{0.5} \hat{\mu}$ of R are defined as

$$\begin{aligned}(\hat{\lambda} \vee 0.5)(x) &= \hat{\lambda}(x) \vee 0.5 \\(\hat{\lambda} \wedge^{0.5} \hat{\mu})(x) &= (\hat{\lambda} \wedge \hat{\mu})(x) \vee 0.5 \\(\hat{\lambda} \odot^{0.5} \hat{\mu})(x) &= (\hat{\lambda} \odot \hat{\mu})(x) \vee 0.5 \\(\hat{\lambda} +^{0.5} \hat{\mu})(x) &= (\hat{\lambda} + \hat{\mu})(x) \vee 0.5\end{aligned}$$

for all $x \in R$.

Theorem 5.12. *Let $\hat{\lambda}$ be an interval valued $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy h -bi-ideal of a hemiring R , then $\hat{\lambda} \vee 0.5$ is an interval valued $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy h -bi-ideal of R .*

Proof. Suppose $\hat{\lambda}$ is an interval valued $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy h -bi-ideal of a hemiring R and $a, b, x, y \in R$. Then

$$\begin{aligned}((\hat{\lambda} \vee 0.5)(x+y) \vee 0.5) &= (\hat{\lambda}(x+y) \vee 0.5) \vee 0.5 \geq (\min\{\hat{\lambda}(x), \hat{\lambda}(y)\}) \vee 0.5 \\ &= \min\{\hat{\lambda}(x) \vee 0.5, \hat{\lambda}(y) \vee 0.5\} = \min\{(\hat{\lambda} \vee 0.5)(x), (\hat{\lambda} \vee 0.5)(y)\}.\end{aligned}$$

This shows that $\max\{(\hat{\lambda} \vee 0.5)(x+y), 0.5\} \geq \min\{(\hat{\lambda} \vee 0.5)(x), (\hat{\lambda} \vee 0.5)(y)\}$.

Similarly we can show

$$\max\{(\hat{\lambda} \vee 0.5)(xy), 0.5\} \geq \min\{(\hat{\lambda} \vee 0.5)(x), (\hat{\lambda} \vee 0.5)(y)\}$$

and

$$\max\{(\hat{\lambda} \vee 0.5)(xyz), 0.5\} \geq \min\{(\hat{\lambda} \vee 0.5)(x), (\hat{\lambda} \vee 0.5)(z)\}.$$

Now let $x + a + y = b + y$, then

$$\begin{aligned}\max\{(\hat{\lambda} \vee 0.5)(x), 0.5\} &= \max\{\hat{\lambda}(x) \vee 0.5, 0.5\} \\ &\geq \{(\min\{\hat{\lambda}(a), \hat{\lambda}(b)\}) \vee 0.5\} \\ &= \min\{(\hat{\lambda} \vee 0.5)(a), (\hat{\lambda} \vee 0.5)(b)\}.\end{aligned}$$

This shows that $(\hat{\lambda} \vee 0.5)$ is an interval valued $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy h -bi-ideal of R . \square

Similarly we can show:

Theorem 5.13. *If $\hat{\lambda}$ is an interval valued $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy left (right) h -ideal of a hemiring R , then $(\hat{\lambda} \vee 0.5)$ is an interval valued $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy left (right) h -ideal of R .*

Lemma 5.14. *Let A, B be subsets of R . Then*

$$\hat{\chi}_A +^{0.5} \hat{\chi}_B = \hat{\chi}_{A+B} \vee 0.5.$$

Lemma 5.15. *An interval valued fuzzy subset $\hat{\lambda}$ of a hemiring R satisfies conditions (1'b) and (4'b) if and only if it satisfies condition*

$$(8'b) \quad \hat{\lambda} +^{0.5} \hat{\lambda} \leq \hat{\lambda} \vee 0.5.$$

Proof. Suppose $\hat{\lambda}$ satisfies conditions (1'b) and (4'b). Let $x \in R$, then for all expressions $x + (a_1 + b_1) + z = (a_2 + b_2) + z$ in R ,

$$\begin{aligned} & (\hat{\lambda} +^{0.5} \hat{\lambda})(x) \\ &= \sup \left\{ \hat{\lambda}(a_1) \wedge \hat{\lambda}(a_2) \wedge \hat{\lambda}(b_1) \wedge \hat{\lambda}(b_2) \right\} \vee 0.5 \\ &= \sup \left\{ (\hat{\lambda}(a_1) \wedge \hat{\lambda}(b_1)) \wedge (\hat{\lambda}(a_2) \wedge \hat{\lambda}(b_2)) \right\} \vee 0.5 \\ &\leq \sup \left\{ \max(\hat{\lambda}(a_1 + b_1), 0.5) \wedge \max(\hat{\lambda}(a_2 + b_2), 0.5) \right\} \vee 0.5 \text{ by condition(1'b)} \\ &\leq \sup \left\{ (\hat{\lambda}(a_1 + b_1) \wedge \hat{\lambda}(a_2 + b_2)) \vee 0.5 \right\} \vee 0.5 \\ &\leq \hat{\lambda}(x) \vee 0.5 \quad \text{by condition(4'b)}. \end{aligned}$$

Thus $\hat{\lambda} +^{0.5} \hat{\lambda} \leq \hat{\lambda} \vee 0.5$.

Conversely, assume that $\hat{\lambda} +^{0.5} \hat{\lambda} \leq \hat{\lambda} \vee 0.5$. Then for each $x \in R$ we have

$$\begin{aligned} \hat{\lambda}(0) \vee 0.5 &\geq (\hat{\lambda} +^{0.5} \hat{\lambda})(0) \\ &= \sup \left\{ \hat{\lambda}(a_1) \wedge \hat{\lambda}(a_2) \wedge \hat{\lambda}(b_1) \wedge \hat{\lambda}(b_2) \right\} \vee 0.5 \\ &\geq \hat{\lambda}(x) \vee 0.5 \quad \text{because } 0 + (x + x) + 0 = (x + x) + 0. \end{aligned}$$

Thus $\hat{\lambda}(0) \vee 0.5 \geq \hat{\lambda}(x) \vee 0.5$ for all $x \in R$.

Let $x, y \in R$, then for all expressions $(x + y) + (a_1 + b_1) + z = (a_2 + b_2) + z$ in R ,

$$\begin{aligned} \hat{\lambda}(x + y) \vee 0.5 &= (\hat{\lambda} +^{0.5} \hat{\lambda})(x + y) \\ &= \sup \left\{ \hat{\lambda}(a_1) \wedge \hat{\lambda}(a_2) \wedge \hat{\lambda}(b_1) \wedge \hat{\lambda}(b_2) \right\} \vee 0.5 \\ &\geq \left\{ \hat{\lambda}(0) \wedge \hat{\lambda}(x) \wedge \hat{\lambda}(0) \wedge \hat{\lambda}(y) \right\} \vee 0.5 \\ &\quad \text{because } (x + y) + (0 + 0) + 0 = (x + y) + 0, \\ &\geq \{(\hat{\lambda}(x) \wedge \hat{\lambda}(y))\} \quad \text{because } \hat{\lambda}(0) \vee 0.5 \geq \hat{\lambda}(x) \vee 0.5. \end{aligned}$$

Thus $\hat{\lambda}$ satisfies condition (1'b).

Let $a, b, x, z \in R$ such that $x + a + z = b + z$. Then

$$\begin{aligned} \hat{\lambda}(x) \vee 0.5 &\geq (\hat{\lambda} +^{0.5} \hat{\lambda})(x) \\ &= \sup \left\{ \hat{\lambda}(a_1) \wedge \hat{\lambda}(a_2) \wedge \hat{\lambda}(b_1) \wedge \hat{\lambda}(b_2) \right\} \vee 0.5 \\ &\geq \{ \hat{\lambda}(a) \wedge \hat{\lambda}(0) \wedge \hat{\lambda}(b) \} \vee 0.5 \\ &\quad \text{because } x + (a + 0) + z = (b + 0) + z, \\ &\geq \{ \hat{\lambda}(a) \wedge \hat{\lambda}(b) \} \quad \text{because } \hat{\lambda}(0) \vee 0.5 \geq \hat{\lambda}(x) \vee 0.5. \end{aligned}$$

This shows that $\hat{\lambda}$ satisfies condition (4'b). □

Similarly we can prove:

Theorem 5.16. *An interval valued fuzzy subset $\hat{\lambda}$ of a hemiring R is an interval valued $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy left (resp. right) h -ideal of R if and only if $\hat{\lambda}$ satisfies conditions*

$$(8'b) \quad \hat{\lambda} +^{0.5} \hat{\lambda} \leq \hat{\lambda} \vee 0.5$$

$$(9'b) \quad \mathcal{R} \odot^{0.5} \hat{\lambda} \leq \hat{\lambda} \vee 0.5 \quad \left(\text{resp. } \hat{\lambda} \odot^{0.5} \mathcal{R} \leq \hat{\lambda} \vee 0.5 \right).$$

Theorem 5.17. *An interval valued fuzzy subset $\hat{\lambda}$ of a hemiring R is an interval valued $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy h -quasi-ideal of R if and only if $\hat{\lambda}$ satisfies conditions (7'b) and (8'b).*

Proof. Proof is straight forward because conditions (1'b) and (4'b) are equivalent to condition (8'b). \square

Lemma 5.18. *Every interval valued $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy left h -ideal of a hemiring R is an interval valued $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy h -quasi-ideal of R .*

Lemma 5.19. *Every interval valued $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy h -quasi-ideal of R is an interval valued $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy h -bi-ideal of R .*

Lemma 5.20. *If $\hat{\lambda}$ and $\hat{\mu}$ are interval valued $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy right and left h -ideal of R , respectively, then $\hat{\lambda} \odot^{0.5} \hat{\mu} \leq \hat{\lambda} \wedge^{0.5} \hat{\mu}$.*

Proof. Let $\hat{\lambda}$ and $\hat{\mu}$ be an interval valued $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy right and left h -ideal of R , respectively. Then for all expressions $x + \sum_{i=1}^m a_i b_i + z = \sum_{j=1}^n c_j d_j + z$ in R

$$\begin{aligned} & \left(\hat{\lambda} \odot^{0.5} \hat{\mu} \right) (x) \\ &= \sup \left\{ \left(\bigwedge_{i=1}^m \left(\hat{\lambda}(a_i) \wedge \hat{\mu}(b_i) \right) \right) \wedge \left(\bigwedge_{j=1}^n \left(\hat{\lambda}(c_j) \wedge \hat{\mu}(d_j) \right) \right) \right\} \vee 0.5 \\ &\leq \sup \left\{ \left(\bigwedge_{i=1}^m \left(\left(\hat{\lambda}(a_i b_i) \vee 0.5 \right) \wedge \left(\hat{\mu}(a_i b_i) \vee 0.5 \right) \right) \right) \wedge \left(\bigwedge_{j=1}^n \left(\left(\hat{\lambda}(c_j d_j) \vee 0.5 \right) \wedge \left(\hat{\mu}(c_j d_j) \vee 0.5 \right) \right) \right) \right\} \vee 0.5 \\ &\leq \sup \left(\left(\left(\bigwedge_{i=1}^m \left(\hat{\lambda}(a_i b_i) \wedge \hat{\mu}(a_i b_i) \right) \right) \wedge \left(\bigwedge_{j=1}^n \left(\hat{\lambda}(c_j d_j) \wedge \hat{\mu}(c_j d_j) \right) \right) \right) \vee 0.5 \right) \vee 0.5 \\ &= \sup \left(\left(\left(\bigwedge_{i=1}^m \left(\hat{\lambda}(a_i b_i) \wedge \hat{\mu}(a_i b_i) \right) \right) \wedge \left(\bigwedge_{j=1}^n \left(\hat{\lambda}(c_j d_j) \wedge \hat{\mu}(c_j d_j) \right) \right) \right) \vee 0.5 \right) \\ &\leq \left(\sup \left(\left(\left(\sum_{i=1}^m \hat{\lambda}(a_i b_i) \vee 0.5 \right) \wedge \left(\sum_{i=1}^m \hat{\mu}(a_i b_i) \vee 0.5 \right) \right) \wedge \left(\left(\sum_{j=1}^n \hat{\lambda}(c_j d_j) \vee 0.5 \right) \wedge \left(\sum_{j=1}^n \hat{\mu}(c_j d_j) \vee 0.5 \right) \right) \right) \right) \vee 0.5 \\ &= \left(\sup \left(\left(\left(\sum_{i=1}^m \hat{\lambda}(a_i b_i) \right) \wedge \left(\sum_{i=1}^m \hat{\mu}(a_i b_i) \right) \right) \wedge \left(\left(\sum_{j=1}^n \hat{\lambda}(c_j d_j) \right) \wedge \left(\sum_{j=1}^n \hat{\mu}(c_j d_j) \right) \right) \right) \right) \vee 0.5 \end{aligned}$$

$$\begin{aligned}
 &= \left(\sup \left(\frac{(\sum_{i=1}^m \hat{\lambda}(a_i b_i)) \wedge (\sum_{j=1}^n \hat{\lambda}(c_j d_j)) \wedge}{(\sum_{i=1}^m \hat{\mu}(a_i b_i)) \leq \wedge (\sum_{j=1}^n \hat{\mu}(c_j d_j))} \right) \right) \vee 0.5 \\
 &\leq \left(\left(\sup \left(\frac{(\sum_{i=1}^m \hat{\lambda}(a_i b_i)) \wedge}{(\sum_{j=1}^n \hat{\lambda}(c_j d_j)) \wedge} \right) \right) \wedge \left(\sup \left(\frac{(\sum_{i=1}^m \hat{\mu}(a_i b_i)) \wedge}{(\sum_{j=1}^n \hat{\mu}(c_j d_j)) \wedge} \right) \right) \right) \vee 0.5 \\
 &\leq \left((\hat{\lambda}(x) \vee 0.5) \wedge (\hat{\mu}(x) \vee 0.5) \right) \vee 0.5 \\
 &= (\hat{\lambda}(x) \wedge \hat{\mu}(x)) \vee 0.5 = (\hat{\lambda} \wedge^{0.5} \hat{\mu})(x). \\
 &\text{Thus } \hat{\lambda} \odot^{0.5} \hat{\mu} \leq \hat{\lambda} \wedge^{0.5} \hat{\mu}. \quad \square
 \end{aligned}$$

Theorem 5.21. For a hemiring R the following conditions are equivalent.

- (i) R is h -hemiregular.
- (ii) $\hat{\lambda} \wedge^{0.5} \hat{\mu} = \hat{\lambda} \odot^{0.5} \hat{\mu}$ for every interval valued $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy right h -ideal R and every interval valued $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy left h -ideal $\hat{\mu}$ of R .

Proof. (i) \Rightarrow (ii): Let $\hat{\lambda}$ be an interval valued $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy right h -ideal and $\hat{\mu}$ be an interval valued $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy left h -ideal, then by Lemma 5.20, $\hat{\lambda} \odot^{0.5} \hat{\mu} \leq \hat{\lambda} \wedge^{0.5} \hat{\mu}$. Let $a \in R$, then there exist $x_1, x_2, z \in R$ such that $a + ax_1a + z = ax_2a + z$. Then for all expressions $a + \sum_{i=1}^m a_i b_i + z = \sum_{j=1}^n c_j d_j + z$ in R we have

$$\begin{aligned}
 &(\hat{\lambda} \odot^{0.5} \hat{\mu})(a) \\
 &= \sup \left(\left(\bigwedge_{i=1}^m (\hat{\lambda}(a_i) \wedge \hat{\mu}(b_i)) \right) \wedge \left(\bigwedge_{j=1}^n (\hat{\lambda}(c_j) \wedge \hat{\mu}(d_j)) \right) \right) \vee 0.5 \\
 &\geq \{(\hat{\lambda}(a) \wedge \hat{\mu}(x_1a) \wedge \hat{\mu}(x_2a)) \vee 0.5\} \\
 &\geq \{(\hat{\lambda}(a) \wedge (\hat{\mu}(x_1a) \vee 0.5) \wedge (\hat{\mu}(x_2a) \vee 0.5)) \vee 0.5\} \\
 &\geq \{\hat{\lambda}(a) \wedge \hat{\mu}(a) \vee 0.5\} \\
 &= (\hat{\lambda} \wedge^{0.5} \hat{\mu})(a).
 \end{aligned}$$

So $(\hat{\lambda} \odot^{0.5} \hat{\mu}) \geq (\hat{\lambda} \wedge^{0.5} \hat{\mu})$. Hence $(\hat{\lambda} \odot^{0.5} \hat{\mu}) = (\hat{\lambda} \wedge^{0.5} \hat{\mu})$.

(ii) \Rightarrow (i): Let I and L be right and left h -ideals of R , respectively. Then $\hat{\chi}_I$ and $\hat{\chi}_L$ are interval valued $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy right h -ideal and interval valued $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy left h -ideal of R . Hence by hypothesis $\hat{\chi}_I \odot^{0.5} \hat{\chi}_L = \hat{\chi}_I \wedge^{0.5} \hat{\chi}_L$. Thus $(\hat{\chi}_I \odot \hat{\chi}_L) \vee 0.5 = (\hat{\chi}_I \wedge \hat{\chi}_L) \vee 0.5$. This implies $\hat{\chi}_{\bar{I}\bar{L}} \vee 0.5 = \hat{\chi}_I \vee 0.5$. Hence $\bar{I}\bar{L} = I \cap L$, so by Lemma 2.3 R is h -hemiregular. \square

Theorem 5.22. For a hemiring R , the following conditions are equivalent.

- (i) R is h -hemiregular.
- (ii) $\hat{\lambda} \vee 0.5 \leq (\hat{\lambda} \odot^{0.5} \hat{\mathcal{R}} \odot^{0.5} \hat{\lambda})$ for every interval valued $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy h -bi-ideal $\hat{\lambda}$ of R .

(iii) $\hat{\lambda} \vee 0.5 \leq (\hat{\lambda} \odot^{0.5} \hat{\mathcal{R}} \odot^{0.5} \hat{\lambda})$ for every interval valued $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy h -quasi-ideal $\hat{\lambda}$ of R .

Proof. (i) \Rightarrow (ii): Let R be an h -hemiregular semiring and $\hat{\lambda}$ be an interval valued $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy h -bi-ideal of R . Let $a \in R$, then there exist $x, x', z \in R$ such that $a + axa + z = ax'a + z$. Then for all expressions $a + \sum_{i=1}^m a_i b_i + z = \sum_{j=1}^n c_j d_j + z$ we have

$$\begin{aligned} & (\hat{\lambda} \odot^{0.5} \hat{\mathcal{R}} \odot^{0.5} \hat{\lambda})(a) \\ &= \sup \left(\left(\bigwedge_{i=1}^m (\hat{\lambda} \odot^{0.5} \hat{\mathcal{R}})(a_i) \wedge \hat{\lambda}(b_i) \right) \wedge \left(\bigwedge_{j=1}^n (\hat{\lambda} \odot^{0.5} \hat{\mathcal{R}})(c_j) \wedge \hat{\lambda}(d_j) \right) \right) \\ & \quad \vee 0.5 \\ & \geq \{ (\hat{\lambda} \odot^{0.5} \hat{\mathcal{R}})(ax) \wedge \hat{\lambda}(a) \wedge (\hat{\lambda} \odot^{0.5} \hat{\mathcal{R}})(ax') \vee 0.5 \} \\ &= \left(\left\{ \sup \left(\left(\bigwedge_{i=1}^m \hat{\lambda}(a'_i) \right) \wedge \left(\bigwedge_{j=1}^n \hat{\lambda}(c'_j) \right) \right) \vee 0.5 \right\} \wedge \hat{\lambda}(a) \wedge \right. \\ & \quad \left. \left\{ \left(\sup \left(\left(\bigwedge_{i=1}^m \hat{\lambda}(a''_i) \right) \wedge \left(\bigwedge_{j=1}^n \hat{\lambda}(c''_j) \right) \right) \right) \vee 0.5 \right\} \right) \vee 0.5 \end{aligned}$$

(for all expressions $ax + \sum_{i=1}^m a'_i b'_i + z = \sum_{j=1}^n c'_j d'_j + z$ and $ax' + \sum_{i=1}^m a''_i b''_i + z = \sum_{j=1}^n c''_j d''_j + z$) $\geq \{ (\hat{\lambda}(axa) \wedge \hat{\lambda}(a) \wedge \hat{\lambda}(ax'a) \vee 0.5 \}$ (because $ax + axax + z = ax'ax + z$ and $ax' + axax' + zx' = ax'ax' + zx'$) $\geq \hat{\lambda}(a) \vee 0.5$.

(ii) \Rightarrow (iii): This is straight forward because every interval valued $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy h -quasi-ideal is interval valued $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy h -bi-ideal.

(iii) \Rightarrow (i): Let Q be an h -quasi-ideal of R . By Proposition 5.10, $\hat{\chi}_Q$ is an interval valued $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy h -quasi-ideal of R . Thus by hypothesis $\hat{\chi}_Q \vee 0.5 \leq (\hat{\chi}_Q \odot^{0.5} \hat{\mathcal{R}} \odot^{0.5} \hat{\chi}_Q) = C_{\overline{QRQ}} \vee 0.5$. Hence $Q \subseteq \overline{QRQ}$ but $\overline{QRQ} \subseteq Q$. Therefore $Q = \overline{QRQ}$. Thus by Lemma 2.4, R is h -hemiregular. \square

Theorem 5.23. For a hemiring R , the following conditions are equivalent.

- (i) R is h -hemiregular.
- (ii) $\hat{\lambda} \wedge^{0.5} \hat{\mu} \leq \hat{\lambda} \odot^{0.5} \hat{\mu} \odot^{0.5} \hat{\lambda}$ for every interval valued $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy h -bi-ideal $\hat{\lambda}$ and every interval valued $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy h -ideal $\hat{\mu}$ of R .
- (iii) $\hat{\lambda} \wedge^{0.5} \hat{\mu} \leq \hat{\lambda} \odot^{0.5} \hat{\mu} \odot^{0.5} \hat{\lambda}$ for every interval valued $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy h -quasi-ideal and every interval valued $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy h -ideal $\hat{\mu}$ of R .

Proof. (i) \Rightarrow (ii): Let R be an h -hemiregular semiring and $\hat{\lambda}$ be an interval valued $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy h -bi-ideal, $\hat{\mu}$ be an interval valued $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy h -ideal of R . Let $a \in R$, then there exist $x, x', z \in R$ such that $a + axa + z = ax'a + z$.

Then for all expressions $a + \sum_{i=1}^m a_i b_i + z = \sum_{j=1}^n c_j d_j + z$ in R we have

$$\begin{aligned} & (\hat{\lambda} \odot^{0.5} \hat{\mu} \odot^{0.5} \hat{\lambda})(a) \\ &= \sup \left(\left(\bigwedge_{i=1}^m (\hat{\lambda} \odot^{0.5} \hat{\mu})(a_i) \wedge \hat{\lambda}(b_i) \right) \wedge \left(\bigwedge_{j=1}^n (\hat{\lambda} \odot^{0.5} \hat{\mu})(c_j) \wedge \hat{\lambda}(d_j) \right) \right) \\ & \quad \vee 0.5 \\ & \geq \{ (\hat{\lambda} \odot^{0.5} \hat{\mu})(ax) \wedge \hat{\lambda}(a) \wedge (\hat{\lambda} \odot^{0.5} \hat{\mu})(ax') \vee 0.5 \} \\ &= \left(\left\{ \left(\sup \left(\left(\bigwedge_{i=1}^m (\hat{\lambda}(a'_i) \wedge \hat{\mu}(b'_i)) \right) \wedge \left(\bigwedge_{j=1}^n (\hat{\lambda}(c'_j) \wedge \hat{\mu}(d'_j)) \right) \right) \right) \vee 0.5 \right\} \wedge \hat{\lambda}(a) \right) \\ & \quad \wedge \left(\left\{ \left(\sup \left(\left(\bigwedge_{i=1}^m (\hat{\lambda}(a''_i) \wedge \hat{\mu}(b''_i)) \right) \wedge \left(\bigwedge_{j=1}^n (\hat{\lambda}(c''_j) \wedge \hat{\mu}(d''_j)) \right) \right) \right) \vee 0.5 \right\} \right) \\ & \quad \vee 0.5 \end{aligned}$$

(for all expressions $ax + \sum_{i=1}^m a'_i b'_i + z = \sum_{j=1}^n c'_j d'_j + z$ and $ax' + \sum_{i=1}^m a''_i b''_i + z = \sum_{j=1}^n c''_j d''_j + z$) $\geq \{ (\hat{\lambda}(a) \wedge \hat{\mu}(xax) \wedge \hat{\mu}(xax') \wedge \hat{\mu}(x'ax')) \vee 0.5 \}$ (because $ax + axax + zx = ax'ax + zx$ and $ax' + axax' + zx' = ax'ax' + zx'$) $\geq (\hat{\lambda}(a) \wedge \hat{\mu}(a)) \vee 0.5 = (\hat{\lambda} \wedge^{0.5} \hat{\mu})(a)$.

(ii) \Rightarrow (iii): This is straight forward because every interval valued $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q})$ -fuzzy h -quasi-ideal is an interval valued $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q})$ -fuzzy h -bi-ideal.

(iii) \Rightarrow (i): Let $\hat{\lambda}$ be an interval valued $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q})$ -fuzzy h -quasi-ideal of R and $\hat{\mathcal{R}}$ an interval valued $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q})$ -fuzzy h -ideal of R . By hypothesis $(\hat{\lambda} \wedge^{0.5} \hat{\mathcal{R}}) \leq (\hat{\lambda} \odot^{0.5} \hat{\mathcal{R}} \odot^{0.5} \hat{\lambda}) \Rightarrow \hat{\lambda} \vee 0.5 \leq (\hat{\lambda} \odot^{0.5} \hat{\mathcal{R}} \odot^{0.5} \hat{\lambda})$. Thus by Theorem 5.22, R is h -hemiregular. \square

Theorem 5.24. For a hemiring R , the following conditions are equivalent.

- (i) R is h -hemiregular.
- (ii) $\hat{\lambda} \wedge^{0.5} \hat{\mu} \leq \hat{\lambda} \odot^{0.5} \hat{\mu}$ for every interval valued $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q})$ -fuzzy h -bi-ideal $\hat{\lambda}$ and every interval valued $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q})$ -fuzzy left h -ideal $\hat{\mu}$ of R .
- (iii) $\hat{\lambda} \wedge^{0.5} \hat{\mu} \leq \hat{\lambda} \odot^{0.5} \hat{\mu}$ for every interval valued $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q})$ -fuzzy h -quasi-ideal $\hat{\lambda}$ and every interval valued $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q})$ -fuzzy left h -ideal $\hat{\mu}$ of R .
- (iv) $\hat{\lambda} \wedge^{0.5} \hat{\mu} \leq \hat{\lambda} \odot^{0.5} \hat{\mu}$ for every interval valued $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q})$ -fuzzy right h -ideal $\hat{\lambda}$ and every interval valued $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q})$ -fuzzy h -bi-ideal $\hat{\mu}$ of R .
- (v) $\hat{\lambda} \wedge^{0.5} \hat{\mu} \leq \hat{\lambda} \odot^{0.5} \hat{\mu}$ for every interval valued $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q})$ -fuzzy right h -ideal $\hat{\lambda}$ and every interval valued $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q})$ -fuzzy h -quasi-ideal $\hat{\mu}$ of R .
- (vi) $\hat{\lambda} \wedge^{0.5} \hat{\mu} \wedge^{0.5} \hat{\nu} \leq \hat{\lambda} \odot^{0.5} \hat{\mu} \odot^{0.5} \hat{\nu}$ for every interval valued $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q})$ -fuzzy right h -ideal $\hat{\lambda}$, every interval valued $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q})$ -fuzzy h -bi-ideal $\hat{\mu}$ and every interval valued $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q})$ -fuzzy right h -ideal $\hat{\nu}$ of R .

(vii) $\hat{\lambda} \wedge^{0.5} \hat{\mu} \wedge^{0.5} \hat{\nu} \leq \hat{\lambda} \odot^{0.5} \hat{\mu} \odot^{0.5} \hat{\nu}$ for every interval valued $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy right h -ideal $\hat{\lambda}$, every interval valued $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy h -quasi-ideal $\hat{\mu}$ and every interval valued $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy right h -ideal $\hat{\nu}$ of R .

Proof. (i) \Rightarrow (ii) Let $\hat{\lambda}$ be an interval valued $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy h -bi-ideal and $\hat{\mu}$ an interval valued $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy left h -ideal of R . Since R is h -hemiregular, so for $a \in R$ there exist $x_1, x_2, z \in R$ such that $a + ax_1a + z = ax_2a + z$. Then for all expressions $a + \sum_{i=1}^m a_i b_i + z = \sum_{j=1}^n a'_j b'_j + z$ we have

$$\begin{aligned} & \left(\hat{\lambda} \odot^{0.5} \hat{\mu} \right) (a) \\ &= \sup \left\{ \left(\bigwedge_{i=1}^m \left(\hat{\lambda} (a_i) \wedge \hat{\mu} (b_i) \right) \right) \wedge \left(\bigwedge_{j=1}^n \left(\hat{\lambda} (a'_j) \wedge \hat{\mu} (b'_j) \right) \right) \right\} \vee 0.5 \\ &\geq \{ \hat{\lambda} (a) \wedge \hat{\mu} (x_1 a) \wedge \hat{\mu} (x_2 a) \} \vee 0.5 \quad \text{because } a + ax_1a + z = ax_2a + z \\ &\geq \{ \hat{\lambda} (a) \wedge \hat{\mu} (a) \wedge 0.5 \} \vee 0.5 \\ &= \{ \hat{\lambda} (a) \wedge \hat{\mu} (a) \vee 0.5 \} \\ &= \left(\hat{\lambda} \wedge^{0.5} \hat{\mu} \right) (a). \end{aligned}$$

So $\left(\hat{\lambda} \odot^{0.5} \hat{\mu} \right) \geq \left(\hat{\lambda} \wedge^{0.5} \hat{\mu} \right)$.

(ii) \Rightarrow (iii) This is obvious because every interval valued $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy h -quasi-ideal is an interval valued $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy h -bi-ideal.

(iii) \Rightarrow (i) Let $\hat{\lambda}$ be an interval valued $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy right h -ideal and $\hat{\mu}$ be an interval valued $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy left h -ideal of R . Since every interval valued $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy right h -ideal is an interval valued $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy h -quasi-ideal, so by (iii) we have $\hat{\lambda} \odot^{0.5} \hat{\mu} \geq \hat{\lambda} \wedge^{0.5} \hat{\mu}$. But by Lemma 5.20, $\hat{\lambda} \odot^{0.5} \hat{\mu} \leq \hat{\lambda} \wedge^{0.5} \hat{\mu}$. Hence $\hat{\lambda} \odot^{0.5} \hat{\mu} = \hat{\lambda} \wedge^{0.5} \hat{\mu}$ for every interval valued $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy right h -ideal $\hat{\lambda}$ of R , and every interval valued $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy left h -ideal $\hat{\mu}$ of R . Thus by Theorem 5.21, R is h -hemiregular.

Similarly we can show that (i) \Leftrightarrow (iv) \Leftrightarrow (v).

(i) \Rightarrow (vi) Let $\hat{\lambda}$ be an interval valued $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy right h -ideal, $\hat{\mu}$ be an interval valued $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy h -bi-ideal and $\hat{\nu}$ be an interval valued $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy left h -ideal of R . Since R is h -hemiregular, so for any $a \in R$ there exist $x_1, x_2, z \in R$ such that $a + ax_1a + z = ax_2a + z$. Then for all expressions $a + \sum_{i=1}^m a_i b_i + z = \sum_{j=1}^n a'_j b'_j + z$ in R we have

$$\begin{aligned} & \left(\hat{\lambda} \odot^{0.5} \hat{\mu} \odot^{0.5} \hat{\nu} \right) (a) \\ &= \sup \left\{ \left(\bigwedge_{i=1}^m \left(\left(\hat{\lambda} \odot^{0.5} \hat{\mu} \right) (a_i) \wedge \hat{\nu} (b_i) \right) \right) \wedge \left(\bigwedge_{j=1}^n \left(\left(\hat{\lambda} \odot^{0.5} \hat{\mu} \right) (a'_j) \wedge \hat{\nu} (b'_j) \right) \right) \right\} \\ & \quad \vee 0.5 \end{aligned}$$

$$\begin{aligned}
 &\geq \{(\hat{\lambda} \odot_{0.5} \hat{\mu})(a) \wedge \hat{\nu}(x_1a) \wedge \hat{\nu}(x_2a) \vee 0.5\} \\
 &\geq \sup \left\{ \left(\bigwedge_{i=1}^m (\hat{\lambda}(a_i) \wedge \hat{\mu}(b_i)) \right) \wedge \left(\bigwedge_{j=1}^n (\hat{\lambda}(a'_j) \wedge \hat{\mu}(b'_j)) \right) \right\} \vee 0.5 \wedge \hat{\nu}(x_1a) \\
 &\quad \wedge \hat{\nu}(x_2a) \\
 &\geq \{\hat{\lambda}(ax_1) \wedge \hat{\lambda}(ax_2) \wedge \hat{\mu}(a) \wedge \hat{\nu}(x_1a) \wedge \hat{\nu}(x_2a) \vee 0.5\} \\
 &\geq \{\hat{\lambda}(a) \wedge \hat{\mu}(a) \wedge \hat{\nu}(a) \vee 0.5\} \\
 &= (\hat{\lambda} \wedge^{0.5} \hat{\mu} \wedge^{0.5} \hat{\nu})(a).
 \end{aligned}$$

Thus $\hat{\lambda} \wedge^{0.5} \hat{\mu} \wedge^{0.5} \hat{\nu} \leq \hat{\lambda} \odot_{0.5} \hat{\mu} \odot_{0.5} \hat{\nu}$.

(vi) \Rightarrow (vii) Straightforward.

(vii) \Rightarrow (i) Let $\hat{\lambda}$ be an interval valued $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy right h -ideal, and $\hat{\nu}$ be an interval valued $(\in, \in \vee q)$ -fuzzy left h -ideal of R . Then

$$\hat{\lambda} \wedge^{0.5} \hat{\nu} = \hat{\lambda} \wedge^{0.5} \mathcal{R} \wedge^{0.5} \hat{\nu} \leq \hat{\lambda} \odot_{0.5} \mathcal{R} \odot_{0.5} \hat{\nu} \leq \hat{\lambda} \odot_{0.5} \hat{\nu}.$$

But $\hat{\lambda} \odot_{0.5} h \leq \hat{\lambda} \wedge^{0.5} \hat{\nu}$. Hence $\hat{\lambda} \odot_{0.5} \hat{\nu} = \hat{\lambda} \wedge^{0.5} \hat{\nu}$ for every interval valued $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy right h -ideal $\hat{\lambda}$ and for every interval valued $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy left h -ideal h of R . Thus by Theorem 5.21, R is h -hemiregular. \square

Lemma 5.25. *A hemiring R is h -intra-hemiregular if and only if $\hat{\lambda} \wedge^{0.5} \hat{\mu} \leq \hat{\lambda} \odot_{0.5} \hat{\mu}$ for every interval valued $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy left h -ideal $\hat{\lambda}$ and for every interval valued $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy right h -ideal $\hat{\mu}$ of R .*

Proof. Let $\hat{\lambda}$ and $\hat{\mu}$ be interval valued $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy left h -ideal and interval valued $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy right h -ideal of R , respectively. Let $a \in R$, then there exist $x_i, x'_i, y_j, y'_j, z \in R$ such that $a + \sum_{i=1}^m x_i a^2 x'_i + z = \sum_{j=1}^n y_j a^2 y'_j + z$. Then for all such expressions we have

$$\begin{aligned}
 &(\hat{\lambda} \odot_{0.5} \hat{\mu})(a) \\
 &= \sup \left\{ \left(\bigwedge_{i=1}^m (\hat{\lambda}(a_i) \wedge \hat{\mu}(b_i)) \right) \wedge \left(\bigwedge_{j=1}^n (\hat{\lambda}(a'_j) \wedge \hat{\mu}(b'_j)) \right) \right\} \vee 0.5 \\
 &\geq \left\{ \left(\bigwedge_{i=1}^m (\hat{\lambda}(x_i a) \wedge \hat{\mu}(a x'_i)) \right) \wedge \left(\bigwedge_{j=1}^n (\hat{\lambda}(y_j a) \wedge \hat{\mu}(a y'_j)) \right) \right\} \vee 0.5 \\
 &\geq \{\hat{\lambda}(a) \wedge \hat{\mu}(a)\} \vee 0.5 = \{\hat{\lambda}(a) \wedge \hat{\mu}(a) \vee 0.5\} = (\hat{\lambda} \wedge^{0.5} \hat{\mu})(a).
 \end{aligned}$$

So $\hat{\lambda} \odot_{0.5} \hat{\mu} \geq \hat{\lambda} \wedge^{0.5} \hat{\mu}$.

Conversely, let L be left h -ideal and I be right h -ideal of R , then $\hat{\chi}_I$ and $\hat{\chi}_L$ are respectively interval valued $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy left h -ideal and interval valued $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy right h -ideal of R . Then by hypothesis $\hat{\chi}_L \wedge^{0.5} \hat{\chi}_I \leq \hat{\chi}_L \odot_{0.5} \hat{\chi}_I$ implies $L \cap I \subseteq \overline{LI}$. Then by Lemma 2.5 R is h -intra-hemiregular. \square

Theorem 5.26. *The following conditions are equivalent for a hemiring R :*

- (i) R is both h -hemiregular and h -intra-hemiregular.
- (ii) $\hat{\lambda} \vee 0.5 = \hat{\lambda} \odot^{0.5} \hat{\lambda}$ for every interval valued $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy h -bi-ideal $\hat{\lambda}$ of R .
- (iii) $\hat{\lambda} \vee 0.5 = \hat{\lambda} \odot^{0.5} \hat{\lambda}$ for every interval valued $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy h -quasi-ideal $\hat{\lambda}$ of R .

Proof. (i) \Rightarrow (ii) Let $\hat{\lambda}$ be an interval valued $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy h -bi-ideal of R . Then for all expressions $x + \sum_{i=1}^m a_i b_i + z = \sum_{j=1}^n a'_j b'_j + z$ in R

$$\begin{aligned} & (\hat{\lambda} \odot^{0.5} \hat{\lambda})(x) \\ &= \sup \left\{ \left(\bigwedge_{i=1}^m (\hat{\lambda}(a_i) \wedge \hat{\lambda}(b_i)) \right) \wedge \left(\bigwedge_{j=1}^n (\hat{\lambda}(a'_j) \wedge \hat{\lambda}(b'_j)) \right) \right\} \vee 0.5 \\ &\geq \left(\bigwedge_{j=1}^n (\hat{\lambda}(x a_2 q_j x) \wedge \hat{\lambda}(x q'_j a_1 x) \wedge \hat{\lambda}(x a_1 q_j x) \wedge \hat{\lambda}(x q'_j a_2 x)) \right) \\ &\quad \vee \left(\bigwedge_{i=1}^m (\hat{\lambda}(x a_1 p_i x) \wedge \hat{\lambda}(x p'_i a_1 x) \wedge \hat{\lambda}(x a_2 p_i x) \wedge \hat{\lambda}(x p'_i a_2 x)) \right) \vee 0.5 \\ &\hspace{15em} \text{(by using Theorem 4.25)} \\ &\geq \min\{\hat{\lambda}(x), 0.5\} \vee 0.5 \\ &= \hat{\lambda}(x) \vee 0.5. \end{aligned}$$

This implies that $\hat{\lambda} \odot^{0.5} \hat{\lambda} \geq \hat{\lambda} \vee 0.5$. Also if $x + \sum_{i=1}^m a_i b_i + z = \sum_{j=1}^n a'_j b'_j + z$, we have

$$\begin{aligned} \hat{\lambda}(x) \vee 0.5 &\geq \min\{\hat{\lambda}(\sum_{i=1}^m a_i b_i), \hat{\lambda}(\sum_{j=1}^n a'_j b'_j), 0.5\} \wedge 0.5 \\ &\geq \min \left\{ \left(\bigwedge_{i=1}^m \hat{\lambda}(a_i b_i) \right), \left(\bigwedge_{j=1}^n \hat{\lambda}(a'_j b'_j) \right), 0.5 \right\} \\ &\geq \min \left\{ \left(\bigwedge_{i=1}^m \hat{\lambda}(a_i) \wedge \hat{\lambda}(b_i) \right), \left(\bigwedge_{j=1}^n \hat{\lambda}(a'_j) \wedge \hat{\lambda}(b'_j) \right), 0.5 \right\} \end{aligned}$$

because $\hat{\lambda}$ is an interval valued $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy h -bi-ideal of R . Thus

$$\begin{aligned} & (\hat{\lambda} \odot^{0.5} \hat{\lambda})(x) \\ &= \sup \left\{ \left(\bigwedge_{i=1}^m (\hat{\lambda}(a_i) \wedge \hat{\lambda}(b_i)) \right) \wedge \left(\bigwedge_{j=1}^n (\hat{\lambda}(a'_j) \wedge \hat{\lambda}(b'_j)) \right) \right\} \vee 0.5 \\ &\leq \hat{\lambda}(x) \vee 0.5. \end{aligned}$$

Consequently $\hat{\lambda} \vee 0.5 = \hat{\lambda} \odot^{0.5} \hat{\lambda}$.

(ii) \Rightarrow (iii) This is straightforward because every interval valued $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q})$ -fuzzy h -quasi-ideal of R is an interval valued $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q})$ -fuzzy h -bi-ideal of R .

(iii) \Rightarrow (i) Let Q be an h -quasi-ideal of R . Then $\hat{\chi}_Q$ is an interval valued $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q})$ -fuzzy h -quasi-ideal of R . Thus by hypothesis

$$\hat{\chi}_Q \vee 0.5 = \hat{\chi}_Q \odot^{0.5} \hat{\chi}_Q = \hat{\chi}_Q \odot \hat{\chi}_Q \vee 0.5 = C_{\overline{Q^2}} \vee 0.5.$$

Then it follows $Q = \overline{Q^2}$. Hence by Lemma 2.6, R is both h -hemiregular and h -intra-hemiregular. \square

Theorem 5.27. *The following conditions are equivalent for a hemiring R :*

- (i) R is both h -hemiregular and h -intra-hemiregular.
- (ii) $\hat{\lambda} \wedge^{0.5} \hat{\mu} \leq \hat{\lambda} \odot^{0.5} \hat{\mu}$ for all interval valued $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q})$ -fuzzy h -bi-ideals $\hat{\lambda}$ and $\hat{\mu}$ of R .
- (iii) $\hat{\lambda} \wedge^{0.5} \hat{\mu} \leq \hat{\lambda} \odot^{0.5} \hat{\mu}$ for every interval valued $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q})$ -fuzzy h -bi-ideal $\hat{\lambda}$ and every interval valued $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q})$ -fuzzy h -quasi-ideals $\hat{\lambda}$ of R .
- (iv) $\hat{\lambda} \wedge^{0.5} \hat{\mu} \leq \hat{\lambda} \odot^{0.5} \hat{\mu}$ for every interval valued $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q})$ -fuzzy h -quasi-ideal $\hat{\lambda}$ and every interval valued $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q})$ -fuzzy h -bi-ideals $\hat{\lambda}$ of R .
- (v) $\hat{\lambda} \wedge^{0.5} \hat{\mu} \leq \hat{\lambda} \odot^{0.5} \hat{\mu}$ for all interval valued $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q})$ -fuzzy h -quasi-ideals $\hat{\lambda}$ and $\hat{\mu}$ of R .

Proof. (i) \Rightarrow (ii) Similar as in Theorem 5.26.

(ii) \Rightarrow (iii) \Rightarrow (v) and (ii) \Rightarrow (iv) \Rightarrow (v) are clear.

(v) \Rightarrow (i) Let $\hat{\lambda}$ be an interval valued $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q})$ -fuzzy left h -ideals of R and $\hat{\mu}$ be an interval valued $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q})$ -fuzzy right h -ideal of R . Then $\hat{\lambda}$ and $\hat{\mu}$ are interval valued $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q})$ -fuzzy h -bi-ideals of R . So by hypothesis $\hat{\lambda} \wedge^{0.5} \hat{\mu} \leq \hat{\lambda} \odot^{0.5} \hat{\mu}$ but $\hat{\lambda} \wedge^{0.5} \hat{\mu} \geq \hat{\lambda} \odot^{0.5} \hat{\mu}$ by Lemma 5.20, Thus $\hat{\lambda} \wedge^{0.5} \hat{\mu} = \hat{\lambda} \odot^{0.5} \hat{\mu}$. Hence by Theorem 5.21, R is h -hemiregular. On the other hand by hypothesis we also have $\hat{\lambda} \wedge^{0.5} \hat{\mu} \leq \hat{\mu} \odot^{0.5} \hat{\lambda}$. By Lemma 5.25, R is h -intra-hemiregular. \square

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