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TRANSFORMATION FORMULAS FOR THE GENERATING FUNCTIONS FOR CRANKS

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Abstract. B. C. Berndt [6] has evaluated the transformation formula for a large class of functions that includes and generalizes the classical Dedekind eta-function. In this paper, we consider a twisted version of his formula. Using this transformation formula, we derive modular transformation formulas for the generating functions for cranks which were central to deduce K. Mahlburg's results in [11].

1. Introduction

In 1944, Freeman Dyson [7] defined the rank of a partition to be the largest part minus the number of parts. He conjectured that using this, one could give combinatorial interpretations and proofs of two of the Ramanujan's congruences for the partition function $p(n)$, namely, $p(5n+4) \equiv 0 \pmod 5$ and $p(7n+5) \equiv 0 \pmod 7$. His conjectures were proved by Atkin and Swinnerton-Dyer [3]. However, the rank function did not give any combinatorial interpretation for Ramanujan's third congruence $p(11n+6) \equiv 0 \pmod{11}$. Dyson conjectured the existence of another partition statistic, a "crank", that would interpret the congruence modulo 11 combinatorially. Forty years later, Andrews and Garvan [2] found such a statistic that explains all three Ramanujan congruences combinatorially. For a partition λ of a positive integer n, let $\rho(\lambda)$ be the largest part of λ, let $\mu(\lambda)$ be the number of ones in λ, and let $\nu(\lambda)$ be the number of parts of λ strictly larger than $\mu(\lambda)$. The crank $c(\lambda)$ is defined by

$$c(\lambda) = \begin{cases} \rho(\lambda), & \text{if } \mu(\lambda) = 0, \\ \nu(\lambda) - \mu(\lambda), & \text{if } \mu(\lambda) > 0. \end{cases}$$

Let $M(m,n)$, $n > 1$, denote the number of partitions of n with crank m. For $n \le 1$, we define [8]

$$M(m,n) = \begin{cases} -1, & \text{if } (m,n) = (0,1), \\ 1, & \text{if } (m,n) = (0,0), (1,1), (-1,1), \\ 0, & \text{otherwise.} \end{cases}$$

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The generating function for $M(m,n)$ is given by [2]

$$\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty} M(m,n)a^m q^n = \frac{(q;q)_\infty}{(aq;q)_\infty (q/a;q)_\infty},$$

where

$$(a;q)_n := \prod_{k=0}^{n-1}\left(1-aq^k\right),$$
$$(a;q)_\infty := \lim_{n\to\infty}(a;q)_n,\ |q|<1.$$

Let $M(m,N,n)$ denote the number of partitions λ of n with crank congruent to m modulo N. Then

$$M(m,N,n) = \sum_{k=-\infty}^{\infty} M(m+kN,n),$$

and the generating function for $M(m,N,n)$ is given by

$$\sum_{n=0}^{\infty} M(m,N,n)q^n = \frac{1}{N}\sum_{g=0}^{N-1}\frac{(q;q)_\infty \zeta^{-mg}}{(\zeta^{-g}q;q)_\infty(\zeta^g q;q)_\infty},$$

where $\zeta = e^{2\pi i/N}$. K. Mahlburg [11] proved that for any positive integers k and j, and for a prime $\ell \geq 5$, there are infinitely many non-nested arithmetic progressions $An+B$ such that

$$M(m,\ell^j,An+B) \equiv 0 \pmod{\ell^k},$$

simultaneously for every $0 \leq m \leq \ell^j - 1$. Let ℓ be a prime and N be a positive power of ℓ. Let

$$K_m(z) := \frac{1}{2\pi i}\sum_{g=1}^{N-1}\zeta^{-mg}q^{-\frac{1}{24}}\eta^\ell(\ell z)F(\zeta^g,z) + \frac{\eta^\ell(\ell z)}{\eta(z)}, \tag{1.1}$$

where $\eta(z)$ is the Dedekind eta function, $q = e^{2\pi i z}$ for $z \in \mathbb{C}$ with Im $z > 0$, and

$$\begin{aligned} F(\zeta^g,z) &:= \frac{(q;q)_\infty}{(\zeta^{-g}q;q)_\infty(\zeta^g q;q)_\infty} \\ &= \prod_{k=0}^{\infty}(1-q^{k+1})(1-\zeta^{-g}q^{k+1})^{-1}(1-\zeta^g q^{k+1})^{-1}. \end{aligned}$$

Let $G_m(z)$ and $P(z)$, respectively, denote the two summands in (1.1). In his famous work [11], K. Mahlburg, using modular properties of related functions, showed that for $\ell \geq 5$, $G_m(z)$ is a nearly holomorphic modular form of weight $(\ell+1)/2$ for the modular group $\Gamma(2N^2)$ and $P(z)$ is a modular form of weight $(\ell-1)/2$ for the modular group $\Gamma_0(\ell)$ with character $\left(\frac{\cdot}{\ell}\right)$. These modular properties of $G_m(z)$ and $P(z)$ play an important role in his work, although he did not give modular transformation formulas for these functions. He claimed that he had been really appealing to the work of Kubert and Lang [10] at Number

Theory Fest 2007 which was held in UIUC. In this paper, using a twisted version of the transformation formula given by B. Berndt [6], we establish modular transformation formulas for $G_m(z)$ and $P(z)$.

2. Notations and preliminaries

We follow notations in [6] except for some twisted functions. If w is a complex, we choose the branch of the argument defined by $-\pi \leq \arg w < \pi$. Let $e(w) = e^{2\pi i w}$. For a positive integer N, let λ_N denote the characteristic function of the integers modulo N, i.e.,

$$\lambda_N(m) = \begin{cases} 1, & \text{if } m \equiv 0 \pmod{N}, \\ 0, & \text{otherwise.} \end{cases}$$

In the sequel, $V\tau = V(\tau) = \frac{a\tau+b}{c\tau+d}$ always denotes a modular transformation with $c > 0$ and $c \equiv 0 \pmod{N}$ for every complex τ. We say that V corresponds to a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Let $r = (r_1, r_2)$ and $h = (h_1, h_2)$ denote real vectors, and define the associated vectors R and H by

$$R = (R_1, R_2) = (ar_1 + cr_2, br_1 + dr_2)$$

and

$$H = (H_1, H_2) = (dh_1 - bh_2, -ch_1 + ah_2).$$

For real x, α and $\mathrm{Re}(s) > 1$, let

$$\psi(x, \alpha, s) := \sum_{n+\alpha>0} \frac{e(nx)}{(n+\alpha)^s}.$$

In fact, $\psi(x, \alpha, s)$ can be analytically continued to the entire s-plane [4]. For a real number x, $[x]$ denotes the greatest integer less than or equal to x and $\{x\} := x - [x]$. Let $\mathbb{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$, the upper half-plane. For $\tau \in \mathbb{H}$ and an arbitrary complex number s, define

$$A_N(\tau, s; r, h) := \sum_{Nm+r_1>0} \sum_{n-h_2>0} \frac{e\left(Nmh_1 + ((Nm+r_1)\tau + r_2)(n-h_2)\right)}{(n-h_2)^{1-s}}. \tag{2.1}$$

Let

$$H_N(\tau, s; r, h) := A_N(\tau, s; r, h) + e\left(\frac{s}{2}\right) A_N(\tau, s; -r, -h).$$

Now we state a twisted version of Theorem 2.1 in [6].

Theorem 2.1. ([6]) *Let* $Q = \left\{\tau \in \mathbb{C} \mid \mathrm{Re}(\tau) > -\frac{d}{c}\right\}$, $\varrho_N = c\{R_2\} - Nd\{R_1/N\}$ *and let* $c = c^{'}N$. *Then for* $\tau \in Q$ *and all* $s \in \mathbb{C}$,

$$\begin{aligned}
&(c\tau+d)^{-s} H_N(V\tau, s; r, h)\\
&= H_N(\tau, s; R, H) - \lambda_N(r_1) e(-r_1h_1)(c\tau+d)^{-s}\Gamma(s)(-2\pi i)^{-s}\Bigg(\psi(h_2, r_2, s)\\
&\quad + e\left(\frac{s}{2}\right)\psi(-h_2, -r_2, s)\Bigg) + \lambda_N(R_1)e(-R_1H_1)\Gamma(s)(-2\pi i)^{-s}\Bigg(\psi(H_2, R_2, s)\\
&\quad + e\left(-\frac{s}{2}\right)\psi(-H_2, -R_2, s)\Bigg) + (2\pi i)^{-s} L_N(\tau, s; R, H),
\end{aligned}$$

where

$$\begin{aligned}
L_N(\tau, s; R, H) := \sum_{j=1}^{c'} e(-H_1(Nj + N[R_1/N] - c) - H_2([R_2] + 1 + [(Njd + \varrho_N)/c] - d))\\
\cdot \int_C u^{s-1} \frac{e^{-(c\tau+d)(Nj - N\{R_1/N\})u/c}}{e^{-(c\tau+d)u} - e(cH_1 + dH_2)} \frac{e^{\{(Njd+\varrho_N)/c\}u}}{e^u - e(-H_2)} du,
\end{aligned}$$

where C *is a loop beginning at* $+\infty$, *proceeding in the upper half-plane, encircling the origin in the positive direction so that* $u = 0$ *is the only zero of*

$$\left(e^{-(c\tau+d)u} - e(cH_1 + dH_2)\right)(e^u - e(-H_2))$$

lying "inside" the loop, and then returning to $+\infty$ *in the lower half plane. Here, we choose the branch of* u^s *with* $0 < \arg u < 2\pi$.

Here, for brevity, we set

$$D_1(h_2, r_2, s) := -(c\tau+d)^{-s}\Gamma(s)(-2\pi i)^{-s}\left(\psi(h_2, r_2, s) + e\left(\frac{s}{2}\right)\psi(-h_2, -r_2, s)\right),$$

and

$$D_2(H_2, R_2, s) := \Gamma(s)(-2\pi i)^{-s}\left(\psi(H_2, R_2, s) + e\left(-\frac{s}{2}\right)\psi(-H_2, -R_2, s)\right)$$

To compute $L_N(\tau, s; R, H)$ for any integer s, we shall use the generating function

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \ (|t| < 2\pi)$$

for Bernoulli polynomials $B_n(x)$, $n \geq 0$. The n-th Bernoulli number B_n, $n \geq 0$, is defined by $B_n = B_n(0)$. Put $\bar{B}_n(x) = B_n(\{x\})$, $n \geq 0$. Recall that $B_{2n+1} = 0$, $n \geq 1$, and that $B_{2n+1}\left(\frac{1}{2}\right) = 0$, $n \geq 0$. We often use the following formulas [1];

$$B_n(1-x) = (-1)^n B_n(x), \tag{2.2}$$

$$\sum_{j=0}^{c-1} B_n\left(\frac{j}{c} + x\right) = c^{1-n}B_n(cx), \tag{2.3}$$

Let $\zeta(s,x)$ be the Hurwitz zeta-function. For $D_1(h_2,r_2,s)$ and $D_2(H_2,R_2,s)$ when $s=-2n$, we have two lemmas as follows.

Lemma 2.2. *Let n be an arbitrary integer and assume that r_2, R_2 are not integers. If $n<0$, then*

$$\begin{aligned}\lim_{s\to -2n} D_1(0,r_2,s) &= -(2\pi i)^{2n}\Gamma(-2n)(c\tau+d)^{2n}(\zeta(-2n,\{r_2\})\\ &\qquad + \zeta(-2n,1-\{r_2\})),\\ \lim_{s\to -2n} D_2(0,R_2,s) &= (2\pi i)^{2n}\Gamma(-2n)(\zeta(-2n,\{R_2\})+\zeta(-2n,1-\{R_2\})).\end{aligned}$$

If $n=0$, then

$$\begin{aligned}&\lim_{s\to 0} D_1(0,r_2,s) = \log\left(1-e^{-2\pi i r_2}\right),\\ &\lim_{s\to 0} D_2(0,R_2,s) = -\log\left(1-e^{-2\pi i R_2}\right) - 2\pi i \bar{B}_1(R_2).\end{aligned}$$

If $n>0$, then

$$\begin{aligned}&\lim_{s\to -2n} D_1(0,r_2,s) = -(c\tau+d)^{2n}\psi(-r_2,0,1+2n),\\ &\lim_{s\to -2n} D_2(0,R_2,s) = \psi(R_2,0,1+2n).\end{aligned}$$

Lemma 2.3. *Let n be an arbitrary integer and assume that r_2, R_2 are integers. If $n=0$, then*

$$\lim_{s\to 0}(D_1(0,r_2,s)+D_2(0,R_2,s)) = \pi i - \log(c\tau+d).$$

If $n\neq 0$, then

$$\begin{aligned}&\lim_{s\to -2n} D_1(0,r_2,s) = -\zeta(1+2n)(c\tau+d)^{2n},\\ &\lim_{s\to -2n} D_2(0,R_2,s) = \zeta(1+2n).\end{aligned}$$

These lemmas are immediate consequences of facts in [5], pp. 501–502 and an elementary calculus.

3. Transformation formulas for the generating functions for cranks

In this section, let $H(\tau,s;r,h) := H_1(\tau,s;r,h)$, $L(\tau,s;r,h) := L_1(\tau,s;r,h)$, $\varrho := \varrho_1$ and $\lambda := \lambda_1$. We start with computation for the transformation formula of the generating function for $M(m,N,n)$. At first, we have

$$\begin{aligned}&\log F(\zeta^g,z)\zeta^{-mg}\\ &= \log\zeta^{-mg} + \sum_{k=0}^{\infty}\log(1-q^{k+1}) - \sum_{k=0}^{\infty}\log(1-\zeta^{-g}q^{k+1}) - \sum_{k=0}^{\infty}\log(1-\zeta^{g}q^{k+1})\\ &= -\frac{2\pi i m g}{N} - \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{e(knz)}{n} + \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{e(-\frac{gn}{N}+knz)}{n} + \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{e(\frac{gn}{N}+knz)}{n}\\ &= -\frac{2\pi i m g}{N} - \frac{1}{2}H(z,0;0,0) + H(z,0;r,0),\end{aligned}$$

where $r = (0, g/N)$. Let V be a modular transformation corresponding to the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}_2(\mathbb{Z}).$$

We have

$$\log F(\zeta^g, Vz)\zeta^{-mg} = -\frac{2\pi img}{N} - \frac{1}{2}H(Vz,0;0,0) + H(Vz,0;r,0).$$

By Theorem 2.1, Lemma 2.3, (2.2) and (2.3),

$$\begin{aligned} &H(Vz,0;0,0) \\ &= H(z,0;0,0) + \lim_{s\to 0}\left(D_1(0,0,s) + D_2(0,0,s)\right) + L(z,0;0,0) \\ &= H(z,0;0,0) - \log(cz+d) - \frac{\pi i}{6c}\left((cz+d) + (cz+d)^{-1}\right) \\ &\quad + 2\pi i\ s(d,c) + \frac{\pi i}{2}, \end{aligned} \tag{3.1}$$

where

$$s(d,c) := \sum_{j=1}^{c-1} \left(\left(\frac{j}{c}\right)\right)\left(\left(\frac{dj}{c}\right)\right)$$

and

$$((x)) := \begin{cases} \{x\} - \frac{1}{2}, & \text{if } x \notin \mathbb{Z}, \\ 0, & \text{otherwise.} \end{cases}$$

Next, since $R = (R_1, R_2) = (cg/N, dg/N)$, we have

$$\begin{aligned} H(Vz,0;r,0) &= H(z,0;R,0) + L(z,0;R,0) \\ &\quad + \lim_{s\to 0}\left(D_1\left(0,\frac{g}{N},s\right) + \lambda\left(\frac{cg}{N}\right)D_2\left(0,\frac{dg}{N},s\right)\right). \end{aligned} \tag{3.2}$$

If $g = 0$, then $H(Vz,0;r,0) = H(Vz,0;0,0)$. Thus, by (3.1), we find that

$$\begin{aligned} &\log\left(F(\zeta^g, Vz)\zeta^{-mg}\right) = \log F(1,Vz) = \frac{1}{2}H(Vz,0;0,0) \\ &= \log F(1,z) - \frac{1}{2}\log(cz+d) - \frac{\pi i}{12c}\left((cz+d)+(cz+d)^{-1}\right) \\ &\quad + \pi i s(d,c) + \frac{\pi i}{4}. \end{aligned}$$

We now assume that $g \neq 0$ and $N \mid c$. It is easy to see that

$$\begin{aligned} H(Vz,0;R,0) &= \sum_{k+cg/N>0}\sum_{n=1}^{\infty}\frac{e(((k+cg/N)z + dg/N)n)}{n} \\ &\quad + \sum_{k-cg/N>0}\sum_{n=1}^{\infty}\frac{e(((k-cg/N)z - dg/N)n)}{n} \\ &= \log F\left(\zeta^{dg}, z\right). \end{aligned} \tag{3.3}$$

Employing Lemma 2.2, we find that

$$\begin{aligned}&\lim_{s\to 0}\left(D_1\left(0,\frac{g}{N},s\right)+\lambda\left(\frac{cg}{N}\right)D_2\left(0,\frac{dg}{N},s\right)\right)\\&=\log\left(1-e^{-2\pi i\frac{g}{N}}\right)-\log\left(1-e^{-2\pi i\frac{dg}{N}}\right)-2\pi i\bar{B}_1\left(\frac{dg}{N}\right).\end{aligned}\tag{3.4}$$

Since $\varrho=c\{R_2\}-d\{R_1\}=c\{dg/N\}-d\{cg/N\}$ and $N\mid c$, ϱ is an integer. Thus by (2.2) and (2.3), we obtain

$$\begin{aligned}&L(z,0;R,0)\\&=2\pi i\sum_{j=1}^{c}\sum_{k=1}^{2}\frac{B_k((j-\{cg/N\})/c)B_{2-k}(\{(jd+\varrho)/c\})}{k!(2-k)!}(-1)^{k-1}(cz+d)^{k-1}\\&=-\pi i\sum_{j=1}^{c}\bar{B}_2\left(\frac{jd+\varrho}{c}\right)(cz+d)^{-1}-\pi i\sum_{j=1}^{c}B_2\left(\frac{j-\{cg/N\}}{c}\right)(cz+d)\\&\quad+2\pi i\sum_{j=1}^{c}B_1\left(\frac{j-\{cg/N\}}{c}\right)\bar{B}_1\left(\frac{jd+\varrho}{c}\right)\\&=-\frac{\pi i}{c}\left(\bar{B}_2(\varrho)(cz+d)^{-1}+\bar{B}_2\left(\frac{cg}{N}\right)(cz+d)\right)\\&\qquad+2\pi i\sum_{j=1}^{c}B_1\left(\frac{j-\{cg/N\}}{c}\right)\bar{B}_1\left(\frac{jd+\varrho}{c}\right)\\&=-\frac{\pi i}{6c}\left((cz+d)^{-1}+(cz+d)\right)+2\pi i\sum_{j=1}^{c}B_1\left(\frac{j}{c}\right)\bar{B}_1\left(\frac{jd}{c}+\left\{\frac{dg}{N}\right\}\right).\end{aligned}\tag{3.5}$$

We see that

$$\begin{aligned}&\sum_{j=1}^{c}B_1\left(\frac{j}{c}\right)\bar{B}_1\left(\frac{jd}{c}+\left\{\frac{dg}{N}\right\}\right)\\&=s\left(d,c;\frac{dg}{N},-\frac{cg}{N}\right)+\frac{1}{2}\bar{B}_1\left(\frac{dg}{N}\right)+\frac{1}{2}B_1\left(\frac{g}{N}\right),\end{aligned}\tag{3.6}$$

where, for any real numbers x, y,

$$s(d,c;x,y):=\sum_{j=0}^{c-1}\left(\left(d\frac{j+y}{c}+x\right)\right)\left(\left(\frac{j+y}{c}\right)\right).$$

Hence, substituting (3.3), (3.4), (3.5), (3.6) into (3.2), we conclude that

$$\begin{aligned}\log\left(F(\zeta^g,Vz)\zeta^{-mg}\right)\;=\;&\log\left(F\left(\zeta^{dg},z\right)\zeta^{-mg}\right)+\frac{1}{2}\log(cz+d)-\pi is(d,c)\\&-\frac{\pi i}{12c}\left((cz+d)^{-1}+(cz+d)\right)+2\pi is\left(d,c;\frac{dg}{N},-\frac{cg}{N}\right)\\&-\pi i\bar{B}_1\left(\frac{dg}{N}\right)+\pi iB_1\left(\frac{g}{N}\right)+\log\left(1-e^{-\frac{2\pi ig}{N}}\right)\end{aligned}\tag{3.7}$$

$$-\log\left(1-e^{-\frac{2\pi idg}{N}}\right)-\frac{\pi i}{4}.$$

We now derive transformation formulas for $P(z)$ and $G_m(z)$. Let V be a modular transformation corresponding to the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix},\ c=c'\ell.$$

From (1.1),

$$K_m(Vz)=\frac{1}{2\pi i}\sum_{g=1}^{N-1}\zeta^{-mg}e(Vz)^{-\frac{1}{24}}\eta^{\ell}(\ell Vz)F(\zeta^g,Vz)+\frac{\eta^{\ell}(\ell Vz)}{\eta(Vz)}.$$

Since

$$\log\eta(z)=\frac{\pi i}{12}z-\frac{1}{2}H(z,0;0,0),$$
$$\log\eta(\ell z)=\frac{\pi i}{12}\ell z-\frac{1}{2}H(\ell z,0;0,0).$$

In [5], we see that

$$\log\eta(Vz)=\log\eta(z)+\frac{1}{2}\log(cz+d)-\pi is(d,c)-\frac{\pi i}{4}+\pi i\frac{a+d}{12c}. \tag{3.8}$$

By Theorem 2.1, we have

$$\begin{aligned}H(\ell Vz,0;0,0)&=H_\ell(Vz,0;0,0)\\&=H_\ell(z,0;0,0)+\lim_{s\to0}(D_1(0,0,s)+D_2(0,0,s))\\&\quad+L_\ell(z,0;0,0).\end{aligned} \tag{3.9}$$

Lemma 2.3 gives us that

$$\lim_{s\to0}(D_1(0,0,s)+D_2(0,0,s))=\pi i-\log(cz+d). \tag{3.10}$$

It follows from (2.2) and (2.3) that

$$\begin{aligned}L_\ell(z,0;0,0)&=2\pi i\sum_{j=1}^{c'}\sum_{k=0}^{2}\frac{B_k(j/c')B_{2-k}(\{jd/c'\})}{k!(2-k)!}(-1)^{k-1}(cz+d)^{k-1}\\&=-\frac{\pi i}{6c'}((cz+d)+(cz+d)^{-1})+2\pi is(d,c')-\frac{\pi i}{2}.\end{aligned} \tag{3.11}$$

Thus we obtain from (3.9), (3.10) and (3.11) that

$$\begin{aligned}\log\eta(\ell Vz)&=\frac{\pi i}{12}\ell Vz-\frac{1}{2}H_\ell(Vz,0;0,0)\\&=\frac{\pi i}{12}\ell Vz-\frac{1}{2}H_\ell(z,0;0,0)-\frac{\pi i}{6c'}((cz+d)+(cz+d)^{-1})\\&\quad+\frac{1}{2}\log(cz+d)+2\pi is(d,c')\\&=\log\eta(\ell z)+\frac{1}{2}\log(cz+d)-\pi is(d,c')-\frac{\pi i}{4}\\&\quad+\frac{\pi i}{12}\frac{a+d}{c'}.\end{aligned} \tag{3.12}$$

Theorem 3.1. *For $V \in \Gamma_0(\ell)$,*

$$\log P(Vz) = \log P(z) + \frac{\ell-1}{2}\log(cz+d) + \pi i(s(d,c) - \ell s(d, c/\ell)) \\ -\frac{\pi i}{4}(\ell-1) + \frac{\pi i}{12}\frac{a+d}{c}(\ell^2-1).$$

Proof. We see that

$$\log P(Vz) = \log\left(\frac{\eta^\ell(\ell Vz)}{\eta(Vz)}\right) = \ell\log\eta(\ell Vz) - \log\eta(Vz).$$

Then the desired results follow from (3.8), (3.12). □

For $g \neq 0$, let

$$H_{m,g}(z) := \zeta^{-mg} e\left(-\frac{z}{24}\right)\eta^\ell(\ell z)F(\zeta^g, z).$$

Then

$$G_m(z) = \frac{1}{2\pi i}\sum_{g=1}^{N-1} H_{m,g}(z),$$

and

$$G_m(Vz) = \frac{1}{2\pi i}\sum_{g=1}^{N-1} H_{m,g}(Vz).$$

Thus it is sufficient to compute the transformation formulas of $H_{m,g}(z)$ to obtain the transformation formulas of $G_m(z)$. Using (3.7) and (3.12), we obtain that

$$\begin{aligned} &\log H_{m,g}(Vz) \\ = \;& -\frac{2\pi i mg}{N} - \frac{2\pi i Vz}{24} + \ell\log\eta(\ell Vz) + \log F(\zeta^g, Vz) \\ = \;& \log H_{m,g}(z) + \log F(\zeta^{dg}, z) - \log F(\zeta^g, z) + \frac{\ell+1}{2}\log(cz+d) \\ & -\pi i(s(d,c) + \ell s(d,c')) + 2\pi i\ s\left(d, c; \frac{dg}{N}, -\frac{cg}{N}\right) - \pi i \bar{B}_1\left(\frac{dg}{N}\right) \\ & +\pi i B_1\left(\frac{g}{N}\right) + \log\left(1 - e\left(-\frac{g}{N}\right)\right) - \log\left(1 - e\left(-\frac{dg}{N}\right)\right) \\ & +\frac{\pi i}{12}\cdot\frac{a+d}{c}(\ell^2-1) - \frac{\pi i}{4}(\ell+1). \end{aligned} \tag{3.13}$$

Theorem 3.2. *For $V \in \Gamma_1(2N^2)$,*

$$\log H_{m,g}(Vz) = \log H_{m,g}(z) + \frac{\ell+1}{2}\log(cz+d) - \pi i(s(d,c) + \ell s(d, c/\ell)) \\ +2\pi i\ s\left(d, c; \frac{dg}{N}, -\frac{cg}{N}\right) + \frac{\pi i}{12}\cdot\frac{a+d}{c}(\ell^2-1) - \frac{\pi i}{4}(\ell+1).$$

Proof. If $V \in \Gamma_1(2N^2)$, then

$$a \equiv d \equiv 1 \pmod{2N^2} \text{ and } c \equiv 0 \pmod{2N^2}.$$

Thus we have $d \equiv 1 \pmod N$. Then the results follow from (3.13). □

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