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ON THE CONVERGENCE OF INEXACT TWO-STEP NEWTON-TYPE METHODS USING RECURRENT FUNCTIONS

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Abstract. We approximate a locally unique solution of a nonlinear equation in a Banach space setting using an inexact two–step Newton–type method. It turn out that under our new idea of recurrent functions, our semilocal analysis provides tighter error bounds than before, and in many interesting cases, weaker sufficient convergence conditions. Applications including the solution of nonlinear Chandrasekhar–type integral equations appearing in radiative transfer and two point boundary value problems are also provided in this study.

1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution $x^\star$ of equation

$$F(x) = 0, \tag{1.1}$$

where F is a Fréchet–differentiable operator defined on a convex subset $\mathcal{D}$ of a Banach space $\mathcal{X}$ with values in a Banach space $\mathcal{Y}$.

The field of computational sciences has seen a considerable development in mathematics, engineering sciences, and economic equilibrium theory. For example, dynamic systems are mathematically modeled by difference or differential equations, and their solutions usually represent the states of the systems. For the sake of simplicity, assume that a time–invariant system is driven by the equation $\dot{x} = T(x)$, for some suitable operator T, where x is the state. Then the equilibrium states are determined by solving equation (1.1). Similar equations are used in the case of discrete systems. The unknowns of engineering equations can be functions (difference, differential, and integral equations), vectors (systems of linear or nonlinear algebraic equations), or real or complex numbers (single algebraic equations with single unknowns). Except in special

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cases, the most commonly used solution methods are iterative–when starting from one or several initial approximations a sequence is constructed that converges to a solution of the equation. Iteration methods are also applied for solving optimization problems. In such cases, the iteration sequences converge to an optimal solution of the problem at hand. Since all of these methods have the same recursive structure, they can be introduced and discussed in a general framework. We note that in computational sciences, the practice of numerical analysis for finding such solutions is essentially connected to variants of Newton's method.

In [2], [3], [6], we introduced the inexact two–step Newton–type method (ITSNTM):

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}\, F(x_n), \\ x_{n+1} &= y_n - z_n, \quad (n \geq 0), \quad (x_0 \in \mathcal{D}) \end{aligned} \tag{1.2}$$

to generate a sequence $\{x_n\}$ approximating $x^\star$. Here, $F'(x) \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ ($x \in \mathcal{D}$) the space of bounded linear operators from $\mathcal{X}$ into $\mathcal{Y}$, and $\{z_n\}$ is a null predetermined sequence in $\mathcal{X}$ depending on $\{x_n\}$, and earlier to $\{x_n\}$ iterates. If $z_n = 0$, we obtain Newton's method whereas if $z_n = F'(y_n)^{-1}\, F(y_n)$, we obtain the two–step Newton method. Many other choices of $\{z_n\}$ were given in [2], [3], [6]. Several authors have also examined the convergence for (ITSNTM) but for special choices of sequences $\{z_n\}$ [1]–[33].

Using a Kantorovich–type approach, we provided a semilocal convergence analysis for (ITSNTM) under general conditions on the operators involved [2]–[4], [6], [10]–[12]. Relevant work can be found [1], [5], [7]–[9], [13]–[33].

In this study we shall expand the applicability of (ITSNTM). The main hypothesis in all studies involving inexact Newton methods (INM) is the Lipschitz condition

$$\| F'(x_0)^{-1}\, (F'(x) - F'(y)) \| \leq L \; \| x - y \| \quad \text{for all} \quad x, y \in \mathcal{D}, \tag{1.3}$$

where $L > 0$, and $F'(x_0)^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ ($x_0 \in \mathcal{D}$). Let $x_0 \in U(x_0, 1/L)$ the open ball with center x_0 and of radius $1/L$. Then, by the Banach Lemma on invertible operators [6] (see also (2.36)), we obtain the estimate

$$\| F'(x)^{-1}\, F'(x_0) \| \leq \frac{1}{1 - L \; \| x - x_0 \|} \tag{1.4}$$

for all $x \in U(x_0, 1/L)$.

Estimate (1.4) is used by the Kantorovich approach to construct the majorizing sequence for (INM). However, the upper bound on the norm $\| F'(x)^{-1}$ $F'(x_0) \|$ can be improved. Indeed, in view of (1.3) there exists $L_0 > 0$ such that

$$\| F'(x_0)^{-1}\, (F'(x) - F'(x_0)) \| \leq L_0 \; \| x - x_0 \| \quad \text{for all} \;\; x \in U(x_0, 1/L_0), \tag{1.5}$$

leading to the corresponding to (1.4) estimate

$$\| F'(x)^{-1} F'(x_0) \| \leq \frac{1}{1 - L_0 \| x - x_0 \|} \tag{1.6}$$

for all $x \in U(x_0, 1/L_0)$.

Note that in general

$$L_0 \leq L \tag{1.7}$$

holds, and $\frac{L}{L_0}$ can be arbitrarily large (see, Section 3). In the case $L_0 < L$, the upper bound of $\| F'(x)^{-1} F'(x_0) \|$ in (1.6) is tighter than in (1.4). In our approach, we use estimate (1.6) instead (1.4) to construct a more precise majorizing sequence for (INM) than in the earlier works (using (1.4)). This is our new idea. Then utilizing our new concept of recurrent functions instead of the less flexible Kantorovich analysis (which cannot use L_0 instead of L), we provide a new semilocal convergence analysis for (ITSNTM) with the following advantages over earlier works for $z_n = 0$ or not:

> Tighter than before error bounds on the distances $\| x_{n+1} - x_n \|$ $(n \geq 0)$, and at least as tight on $\| x_n - x^\star \|$ (under the same or weaker sufficient convergence conditions).

Simply replace L by L_0 at the denominator of the majorizing sequences appearing in all works using (1.3) [2]–[4], [13]–[33]. Moreover, we can show that using the recurrent functions approach instead of the Kantorovich's analysis, the sufficient convergence conditions can also be weakened, and under the same computational cost, since in practice the computation of constant L requires that of L_0. In particular for the special case of Newton's method, our sufficient convergence conditions provide tighter error bounds under weaker hypotheses (see Remark 3.6) than the celebrated Kantorovich theorem for solving equations using Newton's method [26].

The results obtained here can be extended to hold for (ITSNTM) involving outer or generalized inverses along the works of Nashed, Chen [17], and ours [12].

The paper is organized as follows: Section 2 contains the semilocal convergence analysis of (ITSNTM), and comparison with earlier results. Section 3 contains special cases, and numerical example involving a nonlinear integral equation of Chandrasekhar–type appearing in radiative transfer [1], [16], and two point boundary value problems involving integral equations with a Green's kernel.

2. Semilocal convergence analysis of (ITSNTM)

The semilocal convergence analysis of (ITSNTM) under weak conditions is provided in this section. First, we need the following result on majorizing sequences for (ITSNTM).

Lemma 2.1. *Let $a \geq 0$, $b \geq 0$, $c > 0$, $L_0 > 0$, $L > 0$, and $\eta \geq 0$ be given constants.*

Define constants α, β, γ, and δ by

$$\alpha = \frac{2\,L\,(1+a^2\,\eta^{2b})}{L\,(1+a^2\,\eta^{2b}) + \sqrt{(L\,(1+a^2\,\eta^{2b}))^2 + 8\,L_0\,L\,(1+a)\,(1+a^2\,\eta^{2b})}}, \tag{2.1}$$

$$\begin{aligned} \beta &= 2\,L_0\,(1+a\,\eta^b)\,\eta\,\alpha^2 + \Big((L+2\,L_0)\,\eta + L\,a^2\,\eta^{1+2\,b} + 2\,L_0\,a\,\eta^{1+b}\Big)\,\alpha \\ &\quad +2\,a\,c\,\eta^b, \end{aligned} \tag{2.2}$$

$$\gamma = L\,a^2\,\eta^{1+2b} + (L+2\,\alpha\,L_0)\,\eta + 2\,a\,c\,\eta^b + 2\,\alpha\,L_0\,a\,\eta^{1+b}, \tag{2.3}$$

$$\delta = \max\{\beta,\,\gamma\}, \tag{2.4}$$

and scalar sequences $\{s_n\}$, $\{t_n\}$ by

$$\begin{aligned} & t_0 = 0,\ s_0 = \eta,\ t_{n+1} = s_n + a\,(s_n - t_n)^{1+b}, \\ & s_{n+1} = t_{n+1} + \frac{L\,(t_{n+1}-s_n)^2 + L\,(s_n-t_n)^2 + 2\,c\,(t_{n+1}-s_n)}{2\,(1-L_0\,t_{n+1})}. \end{aligned} \tag{2.5}$$

Assume:

$$\delta \leq 2\,\alpha. \tag{2.6}$$

Then, scalar sequence $\{s_n\}$ ($n \geq 0$) is increasing, bounded from above by

$$s^{\star\star} = \left(\frac{1}{1-\alpha} + \frac{a\,\eta^b}{1-\alpha^{1+b}} + \alpha\right)\eta, \tag{2.7}$$

and converges to its unique least upper bound $s^\star$ satisfying $s^\star \in [0, s^{\star\star}]$.

Moreover, the following estimates hold for all $n \geq 0$:

$$0 \leq s_{n+1} - t_{n+1} \leq \alpha\,(s_n - t_n). \tag{2.8}$$

Proof. It follows from (2.1) that $\alpha \in (0,1)$.

We shall show using induction on the integer k:

$$0 \leq \frac{L\,a^2\,(s_k-t_k)^{1+2b} + L\,(s_k-t_k) + 2\,a\,c(s_k-t_k)^b}{1-L_0\,t_{k+1}} \leq 2\,\alpha. \tag{2.9}$$

Estimate (2.8) will then follow from (2.5), and (2.9). Using the definition of γ, (2.4), and (2.6), we conclude that (2.8) and (2.9) hold for $k = 0$.

Let us assume (2.8), and (2.9) hold for all $n \leq k$. We have in turn:

$$\begin{aligned}
t_{k+1} &= s_k + a\,(s_k - t_k)^{1+b} \\
&\leq t_k + \alpha^k\,\eta + a\,(s_k - t_k)^{1+b} \\
&\leq s_{k-1} + a\,(s_{k-1} - t_{k-1})^{1+b} + \alpha^k\,\eta + a\,(s_k - t_k)^{1+b} \\
&\leq \alpha^{k-1}\,\eta + s_{k-2} + a\,(s_{k-2} - t_{k-2})^{1+b} + a\,(s_{k-1} - t_{k-1})^{1+b} + \alpha^k\,\eta \\
&\quad + a\,(s_k - t_k)^{1+b} \\
&\leq s_1 + (\alpha^2 + \alpha^3 + \cdots + \alpha^k)\,\eta + a\,((s_1 - t_1)^{1+b} + \cdots + (s_k - t_k)^{1+b}) \\
&\leq t_1 + \alpha\,\eta + (\alpha^2 + \alpha^3 + \cdots + \alpha^k)\,\eta + a\,((s_1 - t_1)^{1+b} + \cdots \\
&\quad + (s_k - t_k)^{1+b}) \\
&\leq s_0 + a\,(s_0 - t_0)^{1+b} + (\alpha + \alpha^2 + \cdots + \alpha^k)\,\eta + a\,((s_1 - t_1)^{1+b} \\
&\quad + \cdots + (s_k - t_k)^{1+b}) \\
&\leq \eta + (\alpha + \alpha^2 + \cdots + \alpha^k)\,\eta + a\,((s_0 - t_0)^{1+b} + (s_1 - t_1)^{1+b} \\
&\quad + \cdots + (s_k - t_k)^{1+b}) \\
&= \frac{1 - \alpha^{k+1}}{1 - \alpha}\,\eta + a\,(\eta^{1+b} + (\alpha\,\eta)^{1+b} + \cdots + (\alpha^k\,\eta)^{1+b}) \\
&= \frac{1 - \alpha^{k+1}}{1 - \alpha}\,\eta + a\,(1 + \alpha^{1+b} + (\alpha^{1+b})^2 + \cdots + (\alpha^{1+b})^k)\,\eta^{1+b} \\
&= \frac{1 - \alpha^{k+1}}{1 - \alpha}\,\eta + a\,\frac{1 - \alpha^{(1+b)\,(k+1)}}{1 - \alpha^{1+b}}\,\eta^{1+b} \\
&< \frac{\eta}{1 - \alpha} + \frac{a}{1 - \alpha^{1+b}}\,\eta^{1+b} < s^{\star\star},
\end{aligned} \tag{2.10}$$

and

$$s_{k+1} \leq t_{k+1} + \alpha\,(s_k - t_k) \leq \frac{\eta}{1 - \alpha} + \frac{a\,\eta^{1+b}}{1 - \alpha^{1+b}} + \alpha^{k+1}\,\eta \leq s^{\star\star}. \tag{2.11}$$

In view of the induction hypotheses, (2.10), and (2.11), estimate (2.9) shall be true if

$$\begin{aligned}
&L\,a^2(\alpha^k\,\eta)^{1+b} + L\,\alpha^k\,\eta + 2\,a\,c\,\alpha^k\,\eta^b + \\
&2\,\alpha\,L_0\left(\frac{1 - \alpha^{k+1}}{1 - \alpha}\,\eta + a\,\frac{1 - \alpha^{(1+b)\,(k+1)}}{1 - \alpha^{1+b}}\,\eta^{1+b}\right) - 2\,\alpha \leq 0.
\end{aligned} \tag{2.12}$$

Estimate (2.12) motivates us to introduce functions f_k on $[0, +\infty)$ $(k \geq 1)$ for $t = \alpha$ by:

$$\begin{aligned}
f_k(t) &= L\,a^2\,\eta^{1+2\,b}\,t^k + L\,\eta\,t^k + 2\,a\,c\,\eta^b + \\
&\quad 2\,L_0\,((1 + t + \cdots + t^k)\,\eta + a\,(1 + t + \cdots + t^k)\,\eta^{1+b})\,t - 2\,t.
\end{aligned} \tag{2.13}$$

We need a relationship between two consecutive polynomials f_k:

$$\begin{aligned}
f_{k+1}(t) &= L\,a^2\,\eta^{1+2\,b}\,t^{k+1} + L\,\eta\,t^{k+1} + 2\,a\,c\,\eta^b + \\
&\quad 2\,L_0\,((1 + t + \cdots + t^{k+1})\,\eta + a\,(1 + t + \cdots + t^{k+1})\,\eta^{1+b})\,t - 2\,t - \\
&\quad L\,a^2\,\eta^{1+2\,b}\,t^k - L\,\eta\,t^k - 2\,a\,c\,\eta^b - \\
&\quad 2\,L_0\,((1 + t + \cdots + t^k)\,\eta + a\,(1 + t + \cdots + t^k)\,\eta^{1+b})\,t + 2\,t + f_k(t) \\
&= f_k(t) + g(t)\,t^k\,\eta,
\end{aligned} \tag{2.14}$$

where,

$$g(t) = 2\,L_0\,(1+a)\,t^2 + L\,(1+a^2\,\eta^{2b})\,t - L\,(1+a^2\,\eta^{2\,b}). \tag{2.15}$$

Note that α given by (2.1) is the unique positive zero of function g.

In view of (2.14), we have

$$f_k(\alpha) = f_1(\alpha) \qquad (k \geq 1). \tag{2.16}$$

Consequently, estimate (2.12) holds if

$$f_k(\alpha) \leq 0 \qquad (k \geq 1),$$

or (by (2.16))

$$f_1(\alpha) \leq 0. \tag{2.17}$$

But (2.17) holds by the choice of β, and (2.6).

Define

$$f_\infty(\alpha) = \lim_{k \to \infty} f_k(\alpha).$$

Then, we get by (2.17)

$$f_\infty(\alpha) = \lim_{k \to \infty} f_k(\alpha) \leq 0. \tag{2.18}$$

That completes the induction.

It follows that sequence $\{s_n\}$ is non–decreasing, bounded from above by $s^{\star\star}$, and as such it converges to $s^\star$.

That completes the proof of Lemma 2.1. □

We shall show the main semilocal convergence result for (ITSNTM).

Theorem 2.2. *Let $F : \mathcal{D} \subseteq \mathcal{X} \longrightarrow \mathcal{Y}$ be a Fréchet–differentiable operator.*
Assume:
there exist $x_0 \in \mathcal{D}$, a sequence $\{z_n\} \subseteq \mathcal{X}$, and constants $a \geq 0$, $b \geq 0$, $L_0 > 0$, $L > 0$, $\eta \geq 0$, $s_0 \geq \eta$, such that for all $x, y \in \mathcal{D}$:

$$F'(x_0)^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X}), \tag{2.19}$$

$$\parallel F'(x_0)^{-1}\ F(x_0) \parallel \leq \eta, \tag{2.20}$$

$$\parallel F'(x_0)^{-1}\ (F'(x) - F'(x_0)) \parallel \leq L_0\ \parallel x - x_0 \parallel, \tag{2.21}$$

$$\parallel F'(x_0)^{-1}\ (F'(x) - F'(y)) \parallel \leq L\ \parallel x - y \parallel, \tag{2.22}$$

$$\parallel x_{n+1} - y_n \parallel = \parallel z_n \parallel \leq a\ \parallel F'(x_n)^{-1}\,F(x_n) \parallel^{1+b}, \tag{2.23}$$

$$\overline{U}(x_0, s^\star) = \{x \in \mathcal{X} : \parallel x - x_0 \parallel \leq s^\star\} \subseteq \mathcal{D}, \tag{2.24}$$

and hypothesis (2.6) of Lemma 2.1 holds, where, $\{s_n\}$, δ, α, $s^\star$, $s^{\star\star}$ are given in Lemma 2.1, with

$$c = 1 + L_0\, s^{\star\star}. \tag{2.25}$$

Then, sequence $\{y_n\}$ *(*$n \geq 0$*) generated by (ITSNTM) is well defined, remains in* $\overline{U}(x_0, s^\star)$ *for all* $n \geq 0$*, and converges to a solution* $x^\star \in \overline{U}(x_0, s^\star)$ *of equation* $F(x) = 0$*.*

Moreover, the following estimates hold:

$$\| y_n - x_n \| \leq s_n - t_n, \tag{2.26}$$

$$\| x_{n+1} - y_n \| \leq t_{n+1} - s_n, \tag{2.27}$$

$$\| x_{n+1} - x_n \| \leq t_{n+1} - t_n, \tag{2.28}$$

$$\| y_{n+1} - y_n \| \leq s_{n+1} - s_n, \tag{2.29}$$

$$\| y_n - x^\star \| \leq s^\star - s_n, \tag{2.30}$$

and

$$\| x_n - x^\star \| \leq s^\star - t_n. \tag{2.31}$$

Furthemore, if there exists $R \geq s^\star$ *such that*

$$\overline{U}(x_0, R) \subseteq \mathcal{D}, \tag{2.32}$$

and

$$L_0\,(s^\star + R) < 2, \tag{2.33}$$

then $x^\star$ *is the unique solution of equation (1.1) in* $\overline{U}(x_0, R)$*.*

Proof. We shall use mathematical induction to show (2.26)–(2.31) hold for all n. Estimate (2.26) holds for $n = 0$ by (2.5), and (2.20). We have also that $y_0 \in \overline{U}(x_0, s^\star)$, since $s^\star \geq \eta$. It follows from (2.5), (2.10), and (2.11) that

$$t_0 \leq s_0 \leq t_1 \leq s_1 \leq s^\star.$$

We get in turn

$$\| x_1 - y_0 \| = \| z_0 \| \leq a \ \| y_0 - x_0 \|^{1+b} \leq a\,(s_0 - t_0)^{1+b} = t_1 - s_0,$$

and

$$\| x_1 - x_0 \| \leq \| x_1 - y_0 \| + \| y_0 - x_0 \| \leq t_1 - s_0 + s_0 - t_0 = t_1 - t_0 \leq s^\star. \tag{2.34}$$

That is $x_1 \in \overline{U}(x_0, s^\star)$, and (2.26), (2.27) hold for $n = 0$.

We suppose that (2.26), (2.27), and $x_{k+1} \in \overline{U}(x_0, s^\star)$ hold for all $k \leq n$.

Using (2.21) for $x = x_{k+1}$, we get:

$$\begin{aligned} \| F'(x_0)^{-1}\,(F'(x_{k+1}) - F'(x_0)) \| &\leq L_0 \ \| x_{k+1} - x_0 \| \\ &\leq L_0\, t_{k+1} \leq L_0\, s^\star < 1 \quad \text{(by (2.18))}. \end{aligned} \tag{2.35}$$

It follows from (2.35), and the Banach Lemma on invertible operators [6], [26] that $F'(x_{k+1})^{-1}$ exists, so that

$$\| F'(x_{k+1})^{-1}\, F'(x_0) \| \leq \frac{1}{1 - L_0 \ \| x_{k+1} - x_0 \|} \leq \frac{1}{1 - L_0 \ t_{k+1}}\,. \tag{2.36}$$

In view of (ITSNTM), we obtain the approximation:

$$\begin{aligned}F(x_{k+1}) &= (F(x_{k+1})-F(y_k)-F'(y_k)\,(x_{k+1}-y_k))\\ &\quad +(F(y_k)+F'(y_k)\,(x_{k+1}-y_k))\\ &= \int_0^1 (F'(y_k+\theta\;(x_{k+1}-y_k))-F'(y_k))\;(x_{k+1}-y_k)\;d\theta\\ &\quad +(F(y_k)+F'(y_k)\,(x_{k+1}-y_k))\\ &= \int_0^1 (F'(y_k+\theta\;(x_{k+1}-y_k))-F'(y_k))\;(x_{k+1}-y_k)\;d\theta\\ &\quad +(F(y_k)-F(x_k)-F'(x_k)\,(y_k-x_k))\\ &\quad +F'(y_k)\,(x_{k+1}-y_k).\end{aligned} \tag{2.37}$$

Using (2.21), we get

$$\begin{aligned}&\parallel F'(x_0)^{-1}\,F'(y_k)\parallel\\ =\;&\parallel F'(x_0)^{-1}\,(F'(y_k)-F'(x_0))+F'(x_0)^{-1}\,F'(x_0)\parallel\\ \leq\;&\parallel F'(x_0)^{-1}\,F'(x_0)\parallel+\parallel F'(x_0)^{-1}\,(F'(y_k)-F'(x_0))\parallel\\ \leq\;& L_0\;\parallel y_k-x_0\parallel+1\\ \leq\;& 1+L_0\,s_k\\ \leq\;& 1+L_0\,s^{\star\star}\\ =\;& c.\end{aligned} \tag{2.38}$$

Moreover, by (2.22), (2.37), and (2.38), we have in turn:

$$\begin{aligned}&\parallel F'(x_0)^{-1}\,F(x_{k+1})\parallel\\ \leq\;&\parallel \int_0^1 F'(x_0)^{-1}\,(F'(y_k+\theta\;(x_{k+1}-y_k))-F'(y_k))\;(x_{k+1}-y_k)\parallel\;d\theta\\ &+\parallel \int_0^1 F'(x_0)^{-1}\,(F'(x_k+\theta\;(y_k-x_k))-F'(x_k))\;(y_k-x_k)\parallel\;d\theta\\ &+\parallel F'(x_0)^{-1}\,F'(y_k)\,(x_{k+1}-y_k)\parallel\\ \leq\;&\frac{L}{2}\parallel x_{k+1}-y_k\parallel^2+\frac{L}{2}\parallel y_k-x_k\parallel^2+\parallel F'(x_0)^{-1}\,F'(y_k)\parallel\parallel x_{k+1}-y_k\parallel\\ \leq\;&\frac{L}{2}\,(t_{k+1}-s_k)^2+\frac{L}{2}\,(s_k-t_k)^2+(1+L_0\,s_k)\,(t_{k+1}-s_k).\end{aligned} \tag{2.39}$$

Furthemore, by (ITSNTM), (2.36), and (2.39), we get:

$$\begin{aligned}\parallel y_{k+1}-x_{k+1}\parallel &= \parallel (F'(x_{k+1})^{-1}\,F'(x_0))\,(F'(x_0)^{-1}\,F(x_{k+1}))\parallel\\ &\leq \parallel F'(x_{k+1})^{-1}\,F'(x_0)\parallel\parallel F'(x_0)^{-1}\,F(x_{k+1})\parallel\\ &\leq \frac{L\,(t_{k+1}-s_k)^2+L\,(s_k-t_k)^2+2\,(1+L_0\,s_k)\,(t_{k+1}-s_k)}{2\,(1-L_0\,t_{k+1})}\\ &= \frac{L\,(t_{k+1}-s_k)^2+L\,(s_k-t_k)^2+2\,c\,(t_{k+1}-s_k)}{2\,(1-L_0\,t_{k+1})}\\ &= s_{k+1}-t_{k+1},\end{aligned} \tag{2.40}$$

which shows (2.26) for all n.

We also have:

$$\begin{aligned} \| x_{k+1} - y_k \| = \| z_k \| &\leq a \| y_k - x_k \|^{1+b} \\ &\leq a\,(s_k - t_k)^{1+b} = t_{k+1} - s_k, \end{aligned} \tag{2.41}$$

so,

$$\begin{aligned} \| x_{k+1} - x_k \| &\leq \| x_{k+1} - y_k \| + \| y_k - x_k \| \\ &\leq t_{k+1} - s_k + s_k - t_k = t_{k+1} - t_k. \end{aligned} \tag{2.42}$$

$$\begin{aligned} \| y_{k+1} - y_k \| &\leq \| y_{k+1} - x_{k+1} \| + \| x_{k+1} - y_k \| \\ &\leq s_{k+1} - t_{k+1} + t_{k+1} - s_k = s_{k+1} - s_k, \end{aligned} \tag{2.43}$$

$$\| x_{k+1} - x_0 \| \leq \sum_{i=1}^{k+1} \| x_i - x_{i-1} \| \leq \sum_{i=1}^{k+1} (t_i - t_{i-1}) = t_{k+1} \leq s^\star,$$

and

$$\begin{aligned} \| y_{k+1} - x_0 \| &\leq \| y_{k+1} - x_{k+1} \| + \| x_{k+1} - x_0 \| \\ &\leq s_{k+1} - t_{k+1} + t_{k+1} - t_0 = s_{k+1} \leq s^\star \end{aligned}$$

which complete the induction.

In view of Lemma 2.1, sequence $\{s_n\}$ is Cauchy. It then follows from (2.26)–(2.29) that $\{y_n\}$ $(n \geq 0)$ is a Cauchy sequence too in a Banach space $\mathcal{X}$, and as such it converges to some $x^\star \in \overline{U}(x_0, s^\star)$ (since $\overline{U}(x_0, s^\star)$ is a closed set).

By letting $k \longrightarrow \infty$ in (2.39), and noticing that $s_k \leq s^{\star\star}$, we obtain $F(x^\star) = 0$. Estimates (2.30), and (2.31) follow from (2.26)–(2.29) by using standard majorization techniques [6], [10].

Finally, to show the uniqueness part, let $y^\star \in \overline{U}(x_0, R)$ be a solution of $F(x) = 0$, and set

$$\mathcal{M} = \int_0^1 F'(y^\star + \theta\,(x^\star - y^\star))\, d\theta. \tag{2.44}$$

Using (2.21), (2.32), and (2.33), we obtain in turns in (2.35):

$$\begin{aligned} \| F'(x_0)^{-1}\,(\mathcal{M} - F'(x_0)) \| &\leq L_0 \int_0^1 \| y^\star + \theta\,(x^\star - y^\star) - x_0 \| \; d\theta \\ &\leq L_0 \int_0^1 (\theta \; \| x^\star - x_0 \| + (1-\theta) \; \| y^\star - x_0 \|)\, d\theta \\ &\leq \frac{L_0}{2}\,(s^\star + R) < 1. \end{aligned} \tag{2.45}$$

It follows from (2.45), and the Banach Lemma on invertible operators that $\mathcal{M}^{-1}$ exists. By (2.44), and the identity

$$0 = F(x^\star) - F(y^\star) = \mathcal{M}\,(x^\star - y^\star), \tag{2.46}$$

we conclude

$$x^\star = y^\star.$$

That completes the proof of Theorem 2.2. □

We shall now provide more error estimates.

Proposition 2.3. *Under the hypotheses of Theorem 2.2, the following estimates hold*

$$\| y_n - x_n \| \leq \| x_n - x^\star \| + \frac{L \ \| x_n - x^\star \|^2}{2\,(1 - L_0 \ \| x_n - x_0 \|)}, \tag{2.47}$$

and

$$\| x_{n+1} - x^\star \| \leq \mu_n, \tag{2.48}$$

where,

$$\mu_n = \frac{1}{2}\,\frac{L \ \| x_{n+1} - y_n \|^2 + L \ \| y_n - x_n \|^2 + 2\,(1 + L_0 \ \| y_n - x_0 \|) \ \| x_{n+1} - y_n \|}{1 - L_0 \displaystyle\int_0^1 ((1-\theta) \ \| x^\star - x_0 \| + \theta \ \| x_{n+1} - x_0 \|)\, d\theta}.$$

Proof. Using (1.2), we obtain the identities:

$$y_n - x_n = x^\star - x_n + F'(x_n)^{-1}\, F'(x_0) \int_0^1 F'(x_0)^{-1}\,(F'(x_n + \theta\,(x^\star - x_n)) - F'(x_n))\,(x^\star - x_n)\, d\theta, \tag{2.49}$$

and

$$x_{n+1} - x^\star = (\mathcal{M}_{n+1}^{-1}\, F'(x_0))\ (F'(x_0)^{-1}\, F(x_{n+1})), \tag{2.50}$$

where,

$$\mathcal{M}_{n+1} = \int_0^1 F'(x^\star + \theta\,(x_{n+1} - x^\star))\, d\theta. \tag{2.51}$$

Using (2.22), (2.36), and (2.49), we obtain:

$$\begin{aligned} \| y_n - x_n \| &\leq \| x^\star - x_n \| \\ &\quad + \| F'(x_n)^{-1}\, F'(x_0) \| \int_0^1 \| F'(x_0)^{-1}\,(F'(x_n + \theta\,(x^\star - x_n)) \\ &\qquad - F'(x_n) \| \ \| x^\star - x_n \| \ d\theta \\ &\leq \| x_n - x^\star \| + \frac{L \ \| x_n - x^\star \|^2}{2\,(1 - L_0 \ \| x_n - x_0 \|)}, \end{aligned}$$

which shows (2.47).

As in (2.45), we have

$$\begin{aligned} & \| F'(x_0)^{-1}\,(\mathcal{M}_{n+1} - F'(x_0)) \| \\ \leq\ & L_0 \int_0^1 \| x^\star + \theta\,(x_{n+1} - x^\star) - x_0 \| \ d\theta \\ \leq\ & L_0 \int_0^1 (\theta \ \| x_{n+1} - x_0 \| + (1-\theta) \ \| x^\star - x_0 \|)\, d\theta \\ \leq\ & L_0\, s^\star \\ <\ & 1. \end{aligned} \tag{2.52}$$

It follows from (2.52), and the Banach Lemma on invertible operators that $\mathcal{M}_{n+1}^{-1}$ exists, and

$$\| \mathcal{M}_{n+1}^{-1} F'(x_0) \| \leq \frac{1}{1 - L_0 \displaystyle\int_0^1 ((1-\theta) \, \| x^\star - x_0 \| + \theta \, \| x_{n+1} - x_0 \|) \, d\theta}. \tag{2.53}$$

Finally, using (2.39), (2.50), and (2.53), we get

$$\| x_{n+1} - x^\star \| \leq \| \mathcal{M}_{n+1}^{-1} F'(x_0) \| \, \| F'(x_0)^{-1} F(x_{n+1}) \| \leq \mu_n,$$

which shows (2.48).

That completes the proof of Proposition 2.3. □

Remark 2.4. (a) Note that $s^{\star\star}$ given in closed form by (2.7) can replace $s^\star$ in condition (2.24).

(b) If we assume

$$\| F'(x_0)^{-1} \, F'(x) \| \leq c_0, \quad \text{for all} \quad x \in \mathcal{D} \tag{2.54}$$

then, in view of (2.38), c_0 can replace c in all the results above.

(c) It follows from (2.39) that tighter than $\{s_n\}$ majorizing sequence $\{\overline{s}_n\}$ given by

$$\overline{t}_0 = 0, \; \overline{s}_0 = \eta, \; \overline{t}_{n+1} = \overline{s}_n + a \, (\overline{s}_n - \overline{t}_n)^{1+b},$$

$$\overline{s}_{n+1} = \overline{t}_{n+1} + \frac{L \, (\overline{t}_{n+1} - \overline{s}_n)^2 + L \, (\overline{s}_n - \overline{t}_n)^2 + 2 \, (1 + L_0 \, \overline{s}_n) \, (\overline{t}_{n+1} - \overline{s}_n)}{2 \, (1 - L_0 \, \overline{t}_{n+1})} \tag{2.55}$$

can be used in Theorem 2.2.

(d) The sufficient convergence conditions (see e.g. (2.6)) introduced here are based on our new idea of recurrent functions, and they differ from by the corresponding ones given us in [2], [3], where a Kantorovich–type analysis was used. In practice, we will test these conditions, and apply the ones that are satisfied (if any). In the case that both set of conditions are satisfied, we shall use the error bounds of this paper, since they are always at least as tight, since (1.7) holds.

(e) Note that in case (for special choices of sequence $\{z_n\}$), (see also the introduction, Lemma 3.4, Theorem 3.5, and Remark 3.6), our method (ITSNTM) reduces to earlier ones, then we proceed as in (d) above.

(f) According to the proof of Theorem 2.2, sequence $\{z_n\}$ does not have to be included in $\mathcal{D}$ or $\overline{U}(x_0, s^\star)$. An interesting choice for z_n seems to be

$$z_n = \epsilon \, (y_n - x_n), \quad \epsilon \geq 0.$$

3. Special cases and applications

We provide numerical examples and special cases.

Example 3.1. Case $z_n \neq 0$. Let $\mathcal{X} = \mathcal{Y} = \mathcal{C}[0,1]$, $\mathcal{D} = U(1,1)$, and define operator $\mathcal{P}$ on $\mathcal{D}$ by

$$\mathcal{P}(x)(s) = \lambda\, x(s) \int_0^1 \mathcal{K}(s,t)\, x(t)\, dt - x(s) + y(s). \tag{3.1}$$

Note that every zero of $\mathcal{P}$ satisfies the equation

$$x(s) = y(s) + \lambda\, x(s) \int_0^1 \mathcal{K}(s,t)\, x(t)\, dt. \tag{3.2}$$

Nonlinear integral equations of the form (3.2) are considered Chandrasekhar–type equations [1], [6], [16], [19]–[21], and they arise in the theories of radiative transfer, neutron transport, and in the kinetic theory of gasses [6], [16].

Here, we assume that λ is a real number called the "albedo" for scattering, and the kernel $\mathcal{K}(s,t)$ is a continuous function in two variables s, t, satisfying

(i) $0 < \mathcal{K}(s,t) < 1$,
(ii) $\mathcal{K}(s,t) + \mathcal{K}(t,s) = 1$

for all $(s,t) \in [0,1]^2$.

The space $\mathcal{X}$ is equipped with the max–norm. That is,

$$\| x \| = \max_{0 \le s \le 1} |x(s)|.$$

Let us assume for simplicity that

$$\mathcal{K}(s,t) = \frac{s}{s+t} \quad \text{for all} \quad (s,t) \in [0,1]^2. \tag{3.3}$$

Choose $x_0(s) = y(s) = 1$ for all $s \in [0,1]$, $\lambda = .25$, and

$$z_n = \frac{1}{100}\, F''(x_n)\, (y_n - x_n)^2, \tag{3.4}$$

where F'' is the second Fréchet–derivative of operator F [6].

Note that function $\mathcal{K}$ given by (3.3) satisfies conditions (i) and (ii).

Then, using (2.19)–(2.25), (2.1)–(2.4), and (2.6), we obtain

$$\| \mathcal{P}'(x_0(s))^{-1} \| \le 1.53039421,$$

$$\begin{aligned} L_0 = L &= 2\,|\lambda| \max_{0 \le s \le 1} \left| \int_0^1 \frac{s}{s+t}\, dt \right| \; \| \mathcal{P}'(x_0(s))^{-1} \| \\ &= 2\,|\lambda| \ln 2 \; \| \mathcal{P}'(x_0(s))^{-1} \| \\ &= .530394215, \end{aligned}$$

$$\eta = \| \mathcal{P}'(x_0(s))^{-1}\, \mathcal{P}(x_0(s)) \| \ge |\lambda| \ln 2 \; \| \mathcal{P}'(x_0(s))^{-1} \| = .265197107,$$

$$b = 1, \quad a = \frac{1}{100} \; \| F''(x) \| = \frac{2 \ln 2\,|\lambda|}{100} = .0034657359, \quad \text{for all } x \in \mathcal{D},$$

$$\alpha = .499423497, \quad s^{\star\star} = .663453567, \quad c = 1.351891934,$$
$$\beta = .283591402, \quad \gamma = \delta = .283770148,$$

and

$$\delta = .283770148 \leq 2\ \alpha = .998846994.$$

Moreover, with $s^{\star\star}$ replacing $s^\star$ in (2.33), we get

$$s^{\star\star} \leq R < \frac{2}{L_0} - s^{\star\star} = 3.107326625. \tag{3.5}$$

That is all hypotheses of Theorem 2.2 are satisfied. Hence, sequence $\{x_n\}$ converges to a unique solution $x^\star$ in $\mathcal{D}$ (by (3.5)) of equation (3.2), so that error estimates (2.26)–(2.31) hold with $\{s_n\}$, $s^\star$ or $\{\overline{s}_n\}$, $\overline{s}^\star = \lim_{n\to\infty} \overline{s}_n$, respectively.

Example 3.2. Let $\mathcal{X} = \mathcal{Y} = \mathcal{C}[0,1]$, equipped with the same norm as Example 3.1. Consider the following nonlinear boundary value problem [6]

$$\begin{cases} u'' = -u^3 - \gamma\ u^2 \\ u(0) = 0, \quad u(1) = 1. \end{cases}$$

It is well known that this problem can be formulated as the integral equation

$$u(s) = s + \int_0^1 \mathcal{Q}(s,t)\ (u^3(t) + \gamma\ u^2(t))\ dt \tag{3.6}$$

where, $\mathcal{Q}$ is the Green function:

$$\mathcal{Q}(s,t) = \begin{cases} t\ (1-s), & t \leq s \\ s\ (1-t), & s < t. \end{cases}$$

We observe that

$$\max_{0\leq s\leq 1} \int_0^1 |\mathcal{Q}(s,t)|\, dt = \frac{1}{8}.$$

Then problem (3.6) is in the form (1.1), where, $F : \mathcal{D} \longrightarrow \mathcal{Y}$ is defined as

$$[F(x)]\,(s) = x(s) - s - \int_0^1 \mathcal{Q}(s,t)\ (x^3(t) + \gamma\ x^2(t))\ dt.$$

It is easy to verify that the Fréchet derivative of F is defined in the form

$$[F'(x)v]\,(s) = v(s) - \int_0^1 \mathcal{Q}(s,t)\ (3\ x^2(t) + 2\ \gamma\ x(t))\ v(t)\ dt.$$

If we set $u_0(s) = s$, and $\mathcal{D} = U(u_0, R)$, then since $\| u_0 \| = 1$, it is easy to verify that $U(u_0, R) \subset U(0, R+1)$. It follows that $2\ \gamma < 5$, then

$$\| I - F'(u_0) \| \leq \frac{3\ \| u_0 \|^2 + 2\ \gamma\ \| u_0 \|}{8} = \frac{3 + 2\ \gamma}{8},$$

$$\| F'(u_0)^{-1} \| \leq \frac{1}{1 - \dfrac{3 + 2\ \gamma}{8}} = \frac{8}{5 - 2\ \gamma},$$

$$\| F(u_0) \| \leq \frac{\| u_0 \|^3 + \gamma\ \| u_0 \|^2}{8} = \frac{1 + \gamma}{8},$$

and

$$\| F(u_0)^{-1} F(u_0) \| \leq \frac{1+\gamma}{5-2\gamma}.$$

On the other hand, for $x, y \in \mathcal{D}$, we have

$$[(F'(x)-F'(y))v]\,(s) = -\int_0^1 \mathcal{Q}(s,t)\ (3\ x^2(t) - 3\ y^2(t) + 2\ \gamma\ (x(t)-y(t)))\ v(t)\ dt.$$

Consequently (see [6])

$$\begin{aligned} \| F'(x) - F'(y) \| &\leq \frac{\gamma + 6\ R + 3}{4}\ \| x - y \|, \\ \| F'(x) - F'(u_0) \| &\leq \frac{2\ \gamma + 3\ R + 6}{8}\ \| x - u_0 \|. \end{aligned}$$

Therefore, conditions of Theorem 2.2 hold with

$$\eta = \frac{1+\gamma}{5-2\gamma}, \quad L = \frac{\gamma + 6\ R + 3}{4}, \quad L_0 = \frac{2\ \gamma + 3\ R + 6}{8}.$$

Note that $L_0 < L$.

Application 3.3. Case $z_n = 0$ (Newton's method). *In this case, we set $a = 0$ to obtain*

$$x_{n+1} = x_n - F'(x_n)^{-1}\ F(x_n) \quad (n \geq 0), \quad (x_0 \in \mathcal{D}), \tag{3.7}$$

and

$$t_0 = 0, \quad t_1 = \eta, \quad t_{n+1} = t_n + \frac{L\,(t_{n+1} - t_n)^2}{2\,(1 - L_0\, t_{n+1})}, \quad (n \geq 0). \tag{3.8}$$

Lemma 2.1, and Theorem 2.2 reduce to Lemma 3.4 and Theorem 3.5 respectively:

Lemma 3.4. [9] *Assume there exist constants $L_0 \geq 0$, $L \geq 0$, with $L_0 \leq L$, and $\eta \geq 0$, such that:*

$$h_A = \overline{L}\ \eta \begin{cases} \leq \dfrac{1}{2} & if \quad L_0 \neq 0 \\ < \dfrac{1}{2} & if \quad L_0 = 0, \end{cases} \tag{3.9}$$

where,

$$\overline{L} = \frac{1}{8}\ \left(L + 4\ L_0 + \sqrt{L^2 + 8\ L_0\ L} \right).$$

Then, sequence $\{t_k\}$ ($k \geq 0$) given by (3.8) is well defined, nondecreasing, bounded above by $t^{\star\star}$, and converges to its unique least upper bound $t^\star \in [0, t^{\star\star}]$, where

$$t^{\star\star} = \frac{2\ \eta}{2 - \delta},$$

$$1 \leq \delta = \frac{4\ L}{L + \sqrt{L^2 + 8\ L_0\ L}} < 2 \quad \text{for } L_0 \neq 0.$$

Moreover, the following estimates hold:

$$L_0\, t^\star \leq 1,$$

$$0 \leq t_{k+1} - t_k \leq \frac{\delta}{2}\,(t_k - t_{k-1}) \leq \cdots \leq \left(\frac{\delta}{2}\right)^k \eta, \quad (k \geq 1),$$

$$t_{k+1} - t_k \leq \left(\frac{\delta}{2}\right)^k (2\, h_A)^{2^k - 1}\, \eta, \quad (k \geq 0),$$

$$0 \leq t^\star - t_k \leq \left(\frac{\delta}{2}\right)^k \frac{(2\, h_A)^{2^k-1}\, \eta}{1 - (2\, h_A)^{2^k}}, \quad (2\, h_A < 1), \quad (k \geq 0).$$

Theorem 3.5. ([13]) *Let $F : \mathcal{D} \subseteq \mathcal{X} \longrightarrow \mathcal{Y}$ be a Fréchet–differentiable operator. Assume there exist $x_0 \in \mathcal{D}$, and constants $L_0 > 0$, $L > 0$, $\eta \geq 0$, such that for all $x, y \in \mathcal{D}$:*

hypotheses (2.19)–(2.22) hold,

$$\overline{U}(x_0, t^\star) \subseteq \mathcal{D},$$

and

hypothesis (3.9) of Lemma 3.4 holds.

Then, sequence $\{x_n\}$ $(n \geq 0)$ generated by (3.7) is well defined, remains in $\overline{U}(x_0, t^\star)$ for all $n \geq 0$, and converges to a solution $x^\star \in \overline{U}(x_0, t^\star)$ of equation $F(x) = 0$.

Moreover, the following estimates hold:

$$\| x_{n+1} - x_n \| \leq t_{n+1} - t_n,$$

and

$$\| x_n - x^\star \| \leq t^\star - t_n,$$

where, $\{t_n\}$, and $t^\star$ are given in Lemma 3.4.

Furthemore, if there exists $R \geq t^\star$ such that

$$\overline{U}(x_0, R) \subseteq \mathcal{D},$$

and

$$L_0\,(t^\star + R) < 2,$$

then $x^\star$ is the unique solution of equation (1.1) in $\overline{U}(x_0, R)$.

Remark 3.6. If $L_0 = L$, Lemma 3.4, and Theorem 3.5 reduce to the corresponding ones given by Kantorovich and others [26]. Otherwise (i.e. $L_0 < L$), the sufficient convergence conditions are always weaker, since

$$h_K = L\,\eta \leq \frac{1}{2} \Longrightarrow h_A \leq \frac{1}{2},$$

and the error estimates are tighter [4]–[13].

Example 3.7. Define the scalar function F by $F(x) = c_0\ x + c_1 + c_2\ \sin e^{c_3\ x}$, $x_0 = 0$, where c_i, $i = 0, 1, 2, 3$ are given parameters. Then it can easily be seen that for c_3 large and c_2 sufficiently small, $\dfrac{L}{L_0}$ can be arbitrarily large. That is (3.9) may be satisfied but not the Kantorovich hypothesis.

Example 3.8. ([6]) Consider the same notations as Example 3.1. Let $\theta \in [0, 1]$ be a given parameter. Consider the "Cubic" integral equation

$$u(s) = u^3(s) + \lambda\, u(s) \int_0^1 \mathcal{K}(s, t)\, u(t)\, dt + y(s) - \theta. \tag{3.10}$$

Choose $u_0(s) = y(s) = 1$ for all $s \in [0, 1]$. If we let $\mathcal{D} = U(u_0, 1 - \theta)$, and define the operator F on $\mathcal{D}$ by

$$F(x)(s) = x^3(s) - x(s) + \lambda\, x(s) \int_0^1 \mathcal{K}(s, t)\, x(t)\, dt + y(s) - \theta, \tag{3.11}$$

for all $s \in [0, 1]$, then every zero of F satisfies equation (3.10). Therefore, if we set $\xi = \| F'(u_0)^{-1} \|$, then it follows from hypotheses of Theorem 2.2 that

$$\eta = \xi\ (|\lambda|\ \ln 2 + 1 - \theta),$$

$$L = 2\ \xi\ (|\lambda|\ \ln 2 + 3\,(2 - \theta)) \quad \text{and} \quad L_0 = \xi\ (2\,|\lambda|\ \ln 2 + 3\,(3 - \theta)).$$

It follows from Theorem 3.5 that if condition (3.9) holds, then problem (3.10) has a unique solution near u_0. This assumption is weaker than the one given before using the Newton–Kantorovich hypothesis. Note also that $L_0 < L$ for all $\theta \in [0, 1]$.

Example 3.9. ([6], [12]) Let $\mathcal{X} = \mathcal{Y} = \mathbb{R}^2$, be equipped with the max–norm, $x_0 = (1, 1)^T$, $U_0 = \{x\ : \| x - x_0 \| \leq 1 - \varrho\}$, $\varrho \in \left[0, \dfrac{1}{2}\right)$, and define function F on U_0 by

$$F(x) = (\xi_1^3 - \varrho, \xi_2^3 - \varrho)^T, \qquad x = (\xi_1, \xi_2)^T. \tag{3.12}$$

The Fréchet–derivative of operator F is given by

$$F'(x) = \begin{bmatrix} 3\ \xi_1^2 & 0 \\ 0 & 3\ \xi_2^2 \end{bmatrix}.$$

Using hypotheses of Theorem 3.5, we get:

$$\eta = \frac{1}{3}\ (1 - \varrho), \quad L_0 = 3 - \varrho, \quad \text{and} \quad L = 2\ (2 - \varrho).$$

The Kantorovich condition is violated, since

$$2\, h_K = \frac{4}{3}\ (1 - \varrho)\ (2 - \varrho) > 1 \quad \text{for all} \quad \varrho \in \left[0, \frac{1}{2}\right).$$

Hence, there is no guarantee that Newton's method (1.2) converges to $x^\star = (\sqrt[3]{\varrho}, \sqrt[3]{\varrho})^T$, starting at x_0.

However, our condition (3.9) is true for all $\varrho \in I = \left[.450339002, \frac{1}{2}\right)$. Hence, the conclusions of our Theorem 3.5 can apply to solve equation (3.12) for all $\varrho \in I$.

Remark 3.10. The results obtained in this study extend in the case

$$F(x) + G(x) = 0, \tag{3.13}$$

where F is as in the introduction, and $G : \mathcal{D} \longrightarrow \mathcal{Y}$ is a continuous operator, satisfying

$$\| F(x_0)^{-1} \; (G(x) - G(y)) \| \leq N \;\; \| x - y \|, \quad \text{for all } \; (x, y) \in \mathcal{D}^2. \tag{3.14}$$

Condition (3.14) implies the continuity but not necessarily the differentiability of operator G. The iteration corresponding to (3.13) is given by

$$\begin{aligned} & y_n = x_n - F'(x_n)^{-1} \; (F(x_n) + G(x_n)) \quad (n \geq 0), \quad (x_0 \in \mathcal{D}), \\ & x_{n+1} = y_n - z_n. \end{aligned} \tag{3.15}$$

The identity corresponding to (2.37) is given by

$$\begin{aligned} F(x_{k+1}) + G(x_{k+1}) &= (F(x_{k+1}) - F(y_k) - F'(y_k)\,(x_{k+1} - y_k)) \\ &\quad + F(y_k) + F'(y_k)\,(x_{k+1} - y_k) + G(x_{k+1}) \\ &= (F(x_{k+1}) - F(y_k) - F'(y_k)\,(x_{k+1} - y_k)) \\ &\quad + (F(y_k) - F(x_k) - F'(x_k)\,(y_k - x_k)) - G(x_k) \\ &\quad + G(x_{k+1}) + F'(y_k)\,(x_{k+1} - y_k) \\ &= \int_0^1 (F'(y_k + \theta\;(x_{k+1} - y_k)) - F'(y_k))\;(x_{k+1} - y_k)\; d\theta \\ &\quad + \int_0^1 (F'(x_k + \theta\;(y_k - x_k)) - F'(x_k))\;(y_k - x_k)\; d\theta \\ &\quad + F'(y_k)\,(x_{k+1} - y_k) + G(x_{k+1}) - G(x_k), \end{aligned}$$

leading to

$$\| y_{k+1} - x_{k+1} \| \leq s_{k+1} - t_{k+1}.$$

We have the following estimate

$$\begin{aligned} & \| F'(x_0)^{-1}(F(x_{k+1}) + G(x_{k+1})) \| \\ & \leq \frac{L}{2}\,(t_{k+1} - s_k)^2 + \frac{L}{2}\,(s_k - t_k)^2 + (1 + L_0\, s_k)\,(t_{k+1} - s_k) + N\,(t_{k+1} - t_k). \end{aligned}$$

But since

$$\begin{aligned} \| x_{k+1} - x_k \| &= \| y_k - x_k + z_k \| \\ &\leq \| y_k - x_k \| + \| z_k \| \\ &\leq s_k - t_k + a\,(s_k - t_k)^{1+b} = (1 + a\,(s_k - t_k)^b)\,(s_k - t_k), \end{aligned}$$

the majorizing sequence should given by

$$t_0 = 0,\ s_0 = \eta,\ t_{n+1} = s_n + a\,(s_n - t_n)^{1+b},$$

$$s_{n+1} = t_{n+1} + \frac{L\,(t_{n+1} - s_n)^2 + L\,(s_n - t_n)^2 + 2\,c\,(t_{n+1} - s_n) + 2\,N\,(t_{n+1} - t_n)}{2\,(1 - L_0\,t_{n+1})},$$

whereas the term $2\,a\,c\,\eta^b$ in (2.2) and (2.3) should be

$$2\,a\left(1 + L_0\,s^{\star\star} + \frac{N}{a\,\eta^b}\,(1 + a\,\eta^b)\right)\eta^b$$

if $a \neq 0$, and $\eta \neq 0$, and $2\,N$ if $a = 0$.

(similar changes for majorizing sequence $\{\overline{s}_n\}$). Then, with the above changes, the conclusions of all the results obtained here hold with equation (1.1) replaced by (3.13) (with the exception of the uniqueness part in Theorems 2.2 and 3.5).

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