East Asian Mathematical Journal
Vol. 27 (2011), No. 3, pp. 309–317

SOME PROPERTIES OF MAGIC CUBES

Yangkok Kim and Jaechil Yoo*

Abstract. In this paper we introduce an algorithm to construct an infinite family of magic cubes and investigate properties of magic cubes, which extend the results in [6].

1. Introduction

The study of magic squares has a long history([4]). Some remarkable properties of magic squares have been shown and some generalizations have drawn on interesting mathematics([1] and [3]). In [2] an operation on the set of all magic squares was introduced which makes the set of magic squares into a free monoid. In [6] a product on the set of all matrices was introduced and an algorithm was given to construct an infinite family of magic squares. Under the product operation, it was shown that the set of all magic squares formed a free monoid as in [2] and moreover the product preserved the symmetrical property introduced in [5] to give infinitely many square-palindromic magic squares. In this article, we extend the results in [6] to construct an infinite family of magic cubes and investigate properties of magic cubes.

2. Cube algebra

An n-cube is an n-dimensional array of real numbers. When $n = 2$, we have usual matrices. For an n-cube A, every component can be written by an n-ary expression $a_{(i_1,i_2,\ldots,i_n)}$ or $A_{(i_1,i_2,\ldots,i_n)}$, and we denote $\mathbf{M}_A = \max\{a_{(i_1,i_2,\ldots,i_n)}\}$ and $\mathbf{m}_A = \min\{a_{(i_1,i_2,\ldots,i_n)}\}$. When $1 \le i_\ell \le s_\ell$ for $1 \le \ell \le n$, we call A an $s_1 \times s_2 \times \cdots \times s_n$ n-cube and denote it by $A = \left[a_{(i_1,i_2,\ldots,i_n)}\right]_{1\le i_\ell \le s_\ell}$ or simply $A = \left[a_{(i_1,i_2,\ldots,i_n)}\right]$. In particular, if $s_1 = \cdots = s_n = 1$, we simply write $[r]$ for a $1 \times \cdots \times 1$ n-cube whose unique component is r. If $s_1 = s_2 = \cdots = s_n = s$, an $s_1 \times s_2 \times \cdots \times s_n$ n-cube A is called an n-cube of degree $\deg(A) = s$. We now define the equality of two n-cubes in a natural way: two n-cubes $\left[a_{(i_1,i_2,\ldots,i_n)}\right]_{1\le i_\ell \le s_\ell}$ and $\left[b_{(i_1,i_2,\ldots,i_n)}\right]_{1\le i_\ell \le t_\ell}$ are equal if $s_\ell = t_\ell$ for all $1 \le \ell \le n$ and $a_{(i_1,i_2,\ldots,i_n)} = b_{(i_1,i_2,\ldots,i_n)}$ for all $i_1, i_2, \ldots, i_n$. An n-cube obtained from

Received September 17, 2010; Revised October 25, 2010; Accepted April 5, 2011.
2000 *Mathematics Subject Classification.* 00A05, 00A08, 15A30.
Key words and phrases. magic cube, free monoid, symmetrical magic cube.
*Corresponding author: yoo@deu.ac.kr. This work was supported by Dongeui University.

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an n-cube A by fixing some variables of $i_1, i_2, \ldots, i_n$ is called an n-subcube of A. With this notation we also express an n-cube $A = \left[a_{(i_1,i_2,\ldots,i_n)}\right]$ by writing s_n n-subcubes of A side by side, namely,

$$A : \left[a_{(i_1,i_2,\ldots,i_{n-1},1)}\right] \ \left[a_{(i_1,i_2,\ldots,i_{n-1},2)}\right] \ \cdots \ \left[a_{(i_1,i_2,\ldots,i_{n-1},s_n)}\right] .$$

By $\mathcal{M}_{s_1\times\cdots\times s_n}$ we denote the set of all $s_1 \times s_2 \times \cdots \times s_n$ n-cubes. It then can be easily shown that $\mathcal{M}_{s_1\times\cdots\times s_n}$ is a real vector space with vector addition and scalar multiplication defined below; for $A = \left[a_{(i_1,i_2,\ldots,i_n)}\right], B = \left[b_{(j_1,j_2,\ldots,j_n)}\right] \in \mathcal{M}_{s_1\times\cdots\times s_n}$, and $k \in \mathbb{R}$,

$$kA = [ka_{(i_1,i_2,\ldots,i_n)}]$$

and

$$A + B = [\gamma_{(i_1,i_2,\ldots,i_n)}]$$

where

$$\gamma_{(i_1,i_2,\ldots,i_n)} = a_{(i_1,i_2,\ldots,i_n)} + b_{(i_1,i_2,\ldots,i_n)}.$$

We now define a product on the set of all n-cubes as follows: for two n-cubes $A = [a_{(i_1,i_2,\ldots,i_n)}] \in \mathcal{M}_{s_1\times\cdots\times s_n}$ and $B = [b_{(i_1,i_2,\ldots,i_n)}] \in \mathcal{M}_{t_1\times\cdots\times t_n}$, the product $A * B$ of A and B is an $s_1t_1 \times \cdots \times s_nt_n$ n-cube defined by

$$A * B = [C_{(i_1,i_2,\ldots,i_{n-1},1)}] \ \left[C_{(i_1,i_2,\ldots,i_{n-1},2)}\right] \ \cdots \ \left[C_{(i_1,i_2,\ldots,i_{n-1},t_n)}\right]$$

where $1 \le i_\ell \le t_\ell$ for all $\ell = 1, 2, \ldots, n-1$ and $C_{(i_1,i_2,\ldots,i_{n-1},k)}$ is an $s_1 \times \cdots \times s_n$ n-cube defined by

$$C_{(i_1,i_2,\ldots,i_{n-1},k)} = (t_1t_2\cdots t_n)A + b_{(i_1,i_2,\ldots,i_{n-1},k)}\mathbf{1}_{s_1\times\cdots\times s_n} \tag{1}$$

and $\mathbf{1}_{s_1\times\cdots\times s_n}$ is the $s_1 \times s_2 \times \cdots \times s_n$ n-cube of all ones. Moreover under modulo $t_1t_2\cdots t_n$,

$$A * B = [b_{(i_1,i_2,\ldots,i_{n-1},1)}\mathbf{1}_{s_1\times\cdots\times s_n}] \ \left[b_{(i_1,i_2,\ldots,i_{n-1},2)}\mathbf{1}_{s_1\times\cdots\times s_n}\right] \ \cdots \ \left[b_{(i_1,i_2,\ldots,i_{n-1},t_n)}\mathbf{1}_{s_1\times\cdots\times s_n}\right].$$

For example, we consider two 3-cubes A and B of order 3 and 4

A :

1	15	26	23	7	12	18	20	4
17	19	6	3	14	25	22	9	11
24	8	10	16	21	5	2	13	27

and

B :

1	63	62	4	48	18	19	45	32	34	35	29	49	15	14	52
60	6	7	57	21	43	42	24	37	27	26	40	12	54	55	9
56	10	11	53	25	39	38	28	41	23	22	44	8	58	59	5
13	51	50	16	36	30	31	33	20	46	47	17	61	3	2	64

.

Then

$$A * B \quad : \quad \begin{array}{|c|c|c|}\hline C_{(1,1,1)} & \cdots & C_{(1,4,1)} \\ \hline \vdots & \ddots & \vdots \\ \hline C_{(4,1,1)} & \cdots & C_{(4,4,1)} \\ \hline \end{array}\; \begin{array}{|c|c|c|}\hline C_{(1,1,2)} & \cdots & C_{(1,4,2)} \\ \hline \vdots & \ddots & \vdots \\ \hline C_{(4,1,2)} & \cdots & C_{(4,4,2)} \\ \hline \end{array}$$
$$\begin{array}{|c|c|c|}\hline C_{(1,1,3)} & \cdots & C_{(1,4,3)} \\ \hline \vdots & \ddots & \vdots \\ \hline C_{(4,1,3)} & \cdots & C_{(4,4,3)} \\ \hline \end{array}\; \begin{array}{|c|c|c|}\hline C_{(1,1,4)} & \cdots & C_{(1,4,4)} \\ \hline \vdots & \ddots & \vdots \\ \hline C_{(4,1,4)} & \cdots & C_{(4,4,4)} \\ \hline \end{array}.$$

We do find $C_{(4,1,1)} = 4 \cdot 4 \cdot 4A + b_{(4,1,1)}\mathbf{1}_{3\times3\times3}$. We note that

$64A$:

64	960	1664	1472	448	768	1152	1280	256
1088	1216	384	192	896	1600	1408	576	704
1536	512	640	1024	1344	320	128	832	1728

and

$b_{(4,1,1)}\mathbf{1}_{3\times3\times3} = 13 \cdot \mathbf{1}_{3\times3\times3}$:

13	13	13	13	13	13	13	13	13
13	13	13	13	13	13	13	13	13
13	13	13	13	13	13	13	13	13

.

Hence

$C_{(4,1,1)}$:

77	973	1677	1485	461	781	1165	1293	269
1101	1229	397	205	909	1613	1421	589	717
1549	525	653	1037	1357	333	141	845	1741

.

We note that under modulo 64,

$A * B$:

$1 \cdot \mathbf{1}_3$	$63 \cdot \mathbf{1}_3$	$62 \cdot \mathbf{1}_3$	$4 \cdot \mathbf{1}_3$	$48 \cdot \mathbf{1}_3$	$18 \cdot \mathbf{1}_3$	$19 \cdot \mathbf{1}_3$	$45 \cdot \mathbf{1}_3$
$60 \cdot \mathbf{1}_3$	$6 \cdot \mathbf{1}_3$	$7 \cdot \mathbf{1}_3$	$57 \cdot \mathbf{1}_3$	$21 \cdot \mathbf{1}_3$	$43 \cdot \mathbf{1}_3$	$42 \cdot \mathbf{1}_3$	$24 \cdot \mathbf{1}_3$
$56 \cdot \mathbf{1}_3$	$10 \cdot \mathbf{1}_3$	$11 \cdot \mathbf{1}_3$	$53 \cdot \mathbf{1}_3$	$25 \cdot \mathbf{1}_3$	$39 \cdot \mathbf{1}_3$	$38 \cdot \mathbf{1}_3$	$28 \cdot \mathbf{1}_3$
$13 \cdot \mathbf{1}_3$	$51 \cdot \mathbf{1}_3$	$50 \cdot \mathbf{1}_3$	$16 \cdot \mathbf{1}_3$	$36 \cdot \mathbf{1}_3$	$30 \cdot \mathbf{1}_3$	$31 \cdot \mathbf{1}_3$	$33 \cdot \mathbf{1}_3$

$32 \cdot \mathbf{1}_3$	$34 \cdot \mathbf{1}_3$	$35 \cdot \mathbf{1}_3$	$29 \cdot \mathbf{1}_3$	$49 \cdot \mathbf{1}_3$	$15 \cdot \mathbf{1}_3$	$14 \cdot \mathbf{1}_3$	$52 \cdot \mathbf{1}_3$
$37 \cdot \mathbf{1}_3$	$27 \cdot \mathbf{1}_3$	$26 \cdot \mathbf{1}_3$	$40 \cdot \mathbf{1}_3$	$12 \cdot \mathbf{1}_3$	$54 \cdot \mathbf{1}_3$	$55 \cdot \mathbf{1}_3$	$9 \cdot \mathbf{1}_3$
$41 \cdot \mathbf{1}_3$	$23 \cdot \mathbf{1}_3$	$22 \cdot \mathbf{1}_3$	$44 \cdot \mathbf{1}_3$	$8 \cdot \mathbf{1}_3$	$58 \cdot \mathbf{1}_3$	$59 \cdot \mathbf{1}_3$	$5 \cdot \mathbf{1}_3$
$20 \cdot \mathbf{1}_3$	$46 \cdot \mathbf{1}_3$	$47 \cdot \mathbf{1}_3$	$17 \cdot \mathbf{1}_3$	$61 \cdot \mathbf{1}_3$	$3 \cdot \mathbf{1}_3$	$2 \cdot \mathbf{1}_3$	$64 \cdot \mathbf{1}_3$

where $\mathbf{1}_3 = \mathbf{1}_{3\times3\times3}$.

3. Results

Lemma 3.1. *For two n-cubes* $A \in \mathcal{M}_{s_1\times\cdots\times s_n}$ *and* $B \in \mathcal{M}_{t_1\times\cdots\times t_n}$, *let* $\alpha_\ell = (j_\ell - 1)s_\ell + i_\ell$ *with* $1 \le \ell \le n$, $1 \le i_\ell \le s_\ell$, $1 \le j_\ell \le t_\ell$. *Then*

$$(A * B)_{(\alpha_1,\alpha_2,\ldots,\alpha_n)} = t_1 t_2 \cdots t_n A_{(i_1,i_2,\ldots,i_n)} + B_{(j_1,j_2,\ldots,j_n)}.$$

Proof. From the definition of a product, it is clear. □

Corollary 3.2. *$\mathcal{M}$ is left and right cancellative with respect to $*$. That is, $A*B = A*C$ implies $B = C$, and $B*A = C*A$ implies $B = C$ for n-cubes A, B and C.*

Proof. Clear from Lemma 3.1. □

For n-cubes $A \in \mathcal{M}_{s_1\times\cdots\times s_n}$, $B \in \mathcal{M}_{t_1\times\cdots\times t_n}$ and $C \in \mathcal{M}_{u_1\times\cdots\times u_n}$, let

$$\alpha_\ell = (k_\ell - 1)s_\ell t_\ell + (j_\ell - 1)s_\ell + i_\ell$$

where $1 \le \ell \le n$, $1 \le i_\ell \le s_\ell$, $1 \le j_\ell \le t_\ell$, $1 \le k_\ell \le u_\ell$. Then

$$\begin{aligned} & (A*(B*C))_{(\alpha_1,\alpha_2,\ldots,\alpha_n)} \\ = \ & t_1t_2\cdots t_n u_1u_2\cdots u_n A_{(i_1,i_2,\ldots,i_n)} \\ & +(B*C)_{((k_1-1)t_1+j_1,(k_2-1)t_2+j_2,\ldots,(k_n-1)t_n+j_n)} \\ = \ & t_1t_2\cdots t_n u_1u_2\cdots u_n A_{(i_1,i_2,\ldots,i_n)} + u_1u_2\cdots u_n B_{(j_1,j_2,\ldots,j_n)} + C_{(k_1,k_2,\ldots,k_n)} \end{aligned}$$

and

$$\begin{aligned} & ((A*B)*C)_{(\alpha_1,\alpha_2,\ldots,\alpha_n)} \\ = \ & u_1u_2\cdots u_n (A*B)_{((j_1-1)s_1+i_1,(j_2-1)s_2+i_2,\ldots,(j_n-1)s_n+i_n)} + C_{(k_1,k_2,\ldots,k_n)} \\ = \ & u_1u_2\cdots u_n \left(t_1t_2\cdots t_n A_{(i_1,i_2,\ldots,i_n)} + B_{(j_1,j_2,\ldots,j_n)}\right) + C_{(k_1,k_2,\ldots,k_n)} \end{aligned}$$

Lemma 3.3. *For n-cubes A, B and C, $A*(B*C) = (A*B)*C$. That is, $*$ is associative.*

We note that for an n-cube $A \in \mathcal{M}_{s_1\times\cdots\times s_n}$, $[r]*A = s_1s_2\cdots s_n r\mathbf{1}_{s_1\times\cdots\times s_n} + A$ and $A*[r] = A + r\mathbf{1}_{s_1\times\cdots\times s_n}$ for all r. In particular, $[0]$ is the identity. For an n-cube A and a positive integer m, we write $A^0 = [0]$ and $A^m = A^{m-1} * A$.

Corollary 3.4. *The set $\mathcal{M}$ of all cubes of integers is a monoid with respect to the operation $*$.*

We recall that for a, b in a monoid $\mathcal{M}$ with an operation $\circ$, we say that a divides b if there exists an element x in $\mathcal{M}$ such that $a \circ x = b$. The elements which divide the identity are called the units of $\mathcal{M}$. For a, b in $\mathcal{M}$, a and b are associated if and only if there exists a unit u such that $a \circ u = b$. If q is a non-unit, we say that q is an irreducible element if q cannot be written as a product of two non-units. We note that for every r, $[r]$ is a unit and conversely every unit in $\mathcal{M}$ is of the form $[r]$. In fact $[r]*[-r] = [-r]*[r] = [0]$. Moreover if A and B in $\mathcal{M}$ are associated, then $A = B + k\mathbf{1}_{m_1\times\cdots\times m_n}$ for some integer k and m_i.

Definition 1. An n-cube of integers is called magic if all components are distinct, and the average of the numbers in each orthogonal or in each of the 2^{n-1} great diagonals is equal to the average of the numbers in the whole n-cube.

From the definition a magic n-cube of degree s is an n-cube $A = \left[a_{(i_1,i_2,\ldots,i_n)}\right]$ of integers such that $\mathbf{M}_A - \mathbf{m}_A = s^n - 1$. When A is a magic n-cube, we define $||A||$ by the sum of the numbers in each orthogonal. For example,

A :

1	15	26	23	7	12	18	20	4
17	19	6	3	14	25	22	9	11
24	8	10	16	21	5	2	13	27

is a magic 3-cube of degree 3 with $||A|| = 42, \mathbf{m}_A = 1$, and $\mathbf{M}_A = 27$. We also note that for two magic n-cubes A and B of degree s and t respectively, $\mathbf{m}_{A*B} = t^n \mathbf{m}_A + \mathbf{m}_B$, $\mathbf{M}_{A*B} = t^n \mathbf{M}_A + \mathbf{M}_B$, and $\mathbf{m}_{A^k} = (\frac{s^{nk}-1}{s^n-1})\mathbf{m}_A$, $\mathbf{M}_{A^k} = (\frac{s^{nk}-1}{s^n-1})\mathbf{M}_A$ for a positive integer k.

Theorem 3.5. *If A and B are magic n-cubes of degree s and t respectively, then $A*B$ is a magic n-cube of degree st. Moreover $||A*B|| = t^{n+1}||A|| + s||B||$.*

Proof. Let $A = \left[a_{(i_1,i_2,\ldots,i_n)}\right]$ and $B = \left[b_{(j_1,j_2,\ldots,j_n)}\right]$ be magic n-cubes of degree s, t respectively. Then by definition $A*B = [\gamma_{(i_1,i_2,\ldots,i_n)}]$ is an n-cube of degree st. We first note that for each fixed $i_2, \ldots, i_n$, the value $\sum_{k=1}^{st} \gamma_{(k,i_2,\ldots,i_n)}$ is independent of each fixed values $i_2, \ldots, i_n$ as follows:

$$\begin{aligned} \sum_{k=1}^{st} \gamma_{(k,i_2,\ldots,i_n)} &= t \cdot t^n \sum_{\ell=1}^{s} a_{(\ell,i_2,\ldots,i_n)} + s \sum_{\ell=1}^{t} b_{(\ell,j_2,\ldots,j_n)} \\ &= t^{n+1}||A|| + s||B||. \end{aligned}$$

Similarly we can get the same value for orthogonals and great-diagonals of $A * B$. Hence $A * B$ is a magic n-cube and

$$||A * B|| = t^{n+1}||A|| + s||B||.$$

□

We note that if A and B are magic cubes, then each $C_{(i_1,i_2,\ldots,i_n)}$ in (1) is a magic cube. Let $\mathcal{MC}$ be the set of all magic cubes. Then $\mathcal{MC}$ forms a submonoid of $\mathcal{M}$ We now show that $\mathcal{MC}$ is free. We first note that the easy induction on the degree of the magic n-cube shows that every magic n-cube can be written as a product of irreducible magic n-cubes. Equivalently, the set of all irreducible magic n-cubes generates the monoid. Hence in order to show that $\mathcal{MC}$ is free, we only have to prove that it is freely generated by its irreducible elements.

Theorem 3.6. *If A and $A * B$ are magic n-cubes, then B is a magic n-cube.*

Proof. Let $\deg(A) = s$, $\deg(B) = t$, and $A * B = [\gamma_{(i_1,i_2,\ldots,i_n)}]$. Suppose that $B = \left[b_{(i_1,i_2,\ldots,i_n)}\right]$ is not a magic n-cube. Then without loss of generality we can assume $\sum_{i=1}^{t} b_{(k,i,1,\ldots,1)} \neq \sum_{i=1}^{t} b_{(\ell,i,1,\ldots,1)}$ for some $k \neq \ell$. We take α, β

such that $(k-1)s < \alpha \le ks$ and $(\ell-1)s < \beta \le \ell s$. Then for $\alpha \ne \beta$,

$$\sum_{i=1}^{st} \gamma_{(\alpha,i,1,\ldots,1)} = t^{n+1}||A|| + s\sum_{i=1}^{t} b_{(k,i,1,\ldots,1)}$$

and

$$\sum_{i=1}^{st} \gamma_{(\beta,i,1,\ldots,1)} = t^{n+1}||A|| + s\sum_{i=1}^{t} b_{(\ell,i,1,\ldots,1)}$$

which are distinct, a contradiction. □

We also note that if B and $A*B$ are magic n-cubes, then A is a magic n-cube. In the following lemmas, we let $\deg(A) = p, \deg(B) = q, \deg(X) = s$, and $\deg(Y) = t$ and write $B = [b_{(i_1,i_2,\ldots,i_n)}]$, $Y = [y_{(i_1,i_2,\ldots,i_n)}]$.

Lemma 3.7. *If $A*B = X*Y$ for magic n-cubes A, B, X and Y and $\deg(A) < \deg(X)$, then $\deg(A)$ divides $\deg(X)$.*

Proof. Suppose not. We consider the following two n-subcubes:

$$(A*B)_{(i_1,i_2,1,\ldots,1)} = \begin{bmatrix} C_{(1,1,1,\ldots,1)} & \cdots & C_{(1,q,1,\ldots,1)} \\ \vdots & \ddots & \vdots \\ C_{(q,1,1,\ldots,1)} & \cdots & C_{(q,q,1,\ldots,1)} \end{bmatrix}$$

and

$$(X*Y)_{(i_1,i_2,1,\ldots,1)} = \begin{bmatrix} Z_{(1,1,1,\ldots,1)} & \cdots & Z_{(1,t,1,\ldots,1)} \\ \vdots & \ddots & \vdots \\ Z_{(t,1,1,\ldots,1)} & \cdots & Z_{(t,t,1,\ldots,1)} \end{bmatrix}$$

where $C_{(i,j,1,\ldots,1)}$ and $Z_{(i,j,1,\ldots,1)}$ are n-cubes of degree p and s respectively. Then for $1 \le i \le p$,

$$(Z_{(1,2,1,\ldots,1)})_{(i,1,1,\ldots,1)} = (Z_{(1,1,1,\ldots,1)})_{(i,1,1,\ldots,1)} + \alpha = (C_{(1,1,1,\ldots,1)})_{(i,1,1,\ldots,1)} + \alpha$$

for some α and

$$(Z_{(1,2,1,\ldots,1)})_{(i,1,1,\ldots,1))} = (C_{(1,1,1,\ldots,1)})_{(i,\ell,1,\ldots,1))} + \beta$$

for some $1 < \ell < p$ and β. Since $C_{(1,1,1,\ldots,1)}$ is a magic cube, we have

$$\sum_{i=1}^{p}(C_{(1,1,1,\ldots,1)})_{(i,1,1,\ldots,1)} = \sum_{i=1}^{p}(C_{(1,1,1,\ldots,1)})_{(i,\ell,1,\ldots,1)}$$

and so $\alpha = \beta$. This means $(C_{(1,1,1,\ldots,1)})_{(i,1,1,\ldots,1)} = (C_{(1,1,1,\ldots,1)})_{(i,\ell,1,\ldots,1)}$, a contradiction. □

Lemma 3.8. *If for magic n-cubes A, B, X and Y, $A*B = X*Y$, then $X = A*E$ for some magic cube E.*

Proof. By Lemma 3.7 we write $pq = st$, $pr = s$ and $q = rt$. Then we have

$$Z_{(1,1,1,\ldots,1)} = t^n X + y_{(1,1,1,\ldots,1)}\mathbf{1}_{s\times s\times\cdots\times s}.$$

and

$$Z_{(1,1,1,\ldots,1)} = \begin{bmatrix} C_{(1,1,1,\ldots,1)} & \cdots & C_{(1,r,1,\ldots,1)} \\ \vdots & \ddots & \vdots \\ C_{(r,1,1,\ldots,1)} & \cdots & C_{(r,r,1,\ldots,1)} \end{bmatrix}$$

where $C_{(i,j,1,\ldots,1)} = q^n A + b_{(i,j,1,\ldots,1)}\mathbf{1}_{p\times p\times\cdots\times p}$. Hence $t^n X + y_{(1,1,1,\ldots,1)}\mathbf{1}_{s\times s\times\cdots\times s}$ $= [C_{(i,j,1,\ldots,1)}]$ and so

$$\begin{aligned} t^n X &= [C_{(i,j,1,\ldots,1)} - y_{(1,1,1,\ldots,1)}\mathbf{1}_{s\times s\times\cdots\times s}] \\ &= [q^n A + (b_{(i,j,1,\ldots,1)} - y_{(1,1,1,\ldots,1)})\mathbf{1}_{p\times p\times\cdots\times p}] \\ &= [r^n t^n A + (b_{(i,j,1,\ldots,1)} - y_{(1,1,1,\ldots,1)})\mathbf{1}_{p\times p\times\cdots\times p}] \end{aligned}$$

and

$$\begin{aligned} X &= \left[t^n A + \left(\frac{b_{(i,j,1,\ldots,1)} - y_{(1,1,1,\ldots,1)}}{t^n} \right) \mathbf{1}_{p\times p\times\cdots\times p} \right] \\ &= A * E \end{aligned}$$

where $E = [e_{(i,j,1,\ldots,1)}]$ is an n-cube of degree r and

$$e_{(i,j,1,\ldots,1)} = \frac{b_{(i,j,1,\ldots,1)} - y_{(1,1,1,\ldots,1)}}{t^n}.$$

By Lemma 3.6, E is a required magic n-cube. □

From Lemma 3.8, Corollary 3.2 and the remarks above Theorem 3.6 we have the following theorem.

Theorem 3.9. *The monoid $\mathcal{MC}$ of all magic n-cubes is a free monoid.*

4. Examples of symmetrical magic n-cubes

We recall that a magic square $A = [a_{ij}]$ of degree n is called *symmetrical* if the sum of each pair of two opposite entries $a_{ij}, a_{n+1-i,n+1-j}$ with respect to the center is $\frac{2}{n}\left(\sum_{i=1}^{n} a_{i1}\right)$. Similarly we define the same concept for the magic n-cubes.

Definition 2. A magic n-cube A of degree s is called *symmetrical* if the sum of each pair of two opposite entries $A_{(i_1,i_2,\ldots,i_n)}$ and $A^*_{(i_1,i_2,\ldots,i_n)} = A_{(s+1-i_1,s+1-i_2,\ldots,s+1-i_n)}$ with respect to the center is $\frac{2||A||}{s}$.

We note that a magic 3-cube

A :

1	15	26	23	7	12	18	20	4
17	19	6	3	14	25	22	9	11
24	8	10	16	21	5	2	13	27

is symmetrical and $A^n (n \geq 2)$ is also symmetrical. Indeed we have the following.

Theorem 4.1. *The symmetrical property is preserved by the operation $*$ in $\mathcal{MC}$. That is, for two symmetrical magic cubes A and B, $A*B$ is a symmetrical magic cube.*

Proof. Let A and B be symmetrical magic cubes of degree s and t respectively, and let $\alpha_\ell = (j_\ell - 1)s + i_\ell$ with $1 \leq \ell \leq n$, $1 \leq i_\ell \leq s$, $1 \leq j_\ell \leq t$. Then

$$(A * B)_{(\alpha_1,\alpha_2,\dots,\alpha_n)} = t^n A_{(i_1,i_2,\dots,i_n)} + B_{(j_1,j_2,\dots,j_n)}.$$

We note that for a pair of two opposite entries $(A * B)_{(\alpha_1,\alpha_2,\dots,\alpha_n)}$ and $(A * B)^*_{(\alpha_1,\alpha_2,\dots,\alpha_n)}$ in $A * B$, there are two pairs of opposite entries $A_{(i_1,i_2,\dots,i_n)}$, $A^*_{(i_1,i_2,\dots,i_n)}$ and $B_{(j_1,j_2,\dots,j_n)}, B^*_{(j_1,j_2,\dots,j_n)}$ in A and B respectively such that

$$\begin{aligned} & (A * B)_{(\alpha_1,\alpha_2,\dots,\alpha_n)} + (A * B)^*_{(\alpha_1,\alpha_2,\dots,\alpha_n)} \\ = \ & t^n A_{(i_1,i_2,\dots,i_n)} + B_{(j_1,j_2,\dots,j_n)} + t^n A^*_{(i_1,i_2,\dots,i_n)} + B^*_{(j_1,j_2,\dots,j_n)} \\ = \ & t^n (A_{(i_1,i_2,\dots,i_n)} + A^*_{(i_1,i_2,\dots,i_n)}) + (B_{(j_1,j_2,\dots,j_n)} + B^*_{(j_1,j_2,\dots,j_n)}). \end{aligned}$$

Since A and B are symmetrical, we have

$$\begin{aligned} & (A * B)_{(\alpha_1,\alpha_2,\dots,\alpha_n)} + (A * B)^*_{(\alpha_1,\alpha_2,\dots,\alpha_n)} \\ = \ & \frac{t^n 2||A||}{s} + \frac{2||B||}{t} = \frac{t^{n+1} 2||A|| + 2s||B||}{st} \\ = \ & 2\left(\frac{t^{n+1}||A|| + s||B||}{st}\right) = \frac{2||A * B||}{st} \end{aligned}$$

which completes the proof. □

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Yangkok Kim
Department of Mathematics
Dongeui University
Busan 614-714, Korea
E-mail address: ykkim@deu.ac.kr

Jaechil Yoo
Department of Mathematics
Dongeui University
Busan 614-714, Korea
E-mail address: yoo@deu.ac.kr